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Univalence Criteria for Some Integral Operators

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Abstract. In this work certain integral operators are studied and the author determines conditions for the univalence of these integral operators.

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1 Introduction.

Let A be the class of functions f which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$, $f(z) = z + a_2 z^2 + \dots$ with $f(0) = f'(0) - 1 = 0$. We denote S the class of the functions $f \in A$ which are univalent in U.

2 Preliminary results.

We will need the following theorems in this work.

Theorem 2.1. [3] *Let $f \in A$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \ z \in U \tag{2.1}$$

then f is univalent in U.

Theorem 2.2. [4] *Let α be a complex number,* $\operatorname{Re}\alpha > 0$ *and $f(z) = z + a_2 z^2 + \dots$a regular function in U.*

If

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \le 1 \tag{2.2}$$

for all $z \in U$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha - 1} f'(u)\, du \right]^{\frac{1}{\alpha}} \tag{2.3}$$

is regular and univalent in U.

SCHWARZ Lemma [1]. *Let $f(z)$ the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M, M$ fixed. If $f(z)$ has in $z = 0$ one zero with the multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, z \in U_R \tag{2.4}$$

the equality (in the inequality (2.4) for $z \neq 0$) can hold only if $f(z) = e^{i\theta} \frac{M}{R^m} z^m$, where θ is constant.

3 Main Results

Theorem 3.1. *Let the function $g \in A$ verify (2.1), $a+bi$ be a complex number, $a \in (0,1)$, M be a real number such that*

$$0 < M \leq \frac{a\sqrt{a^2+b^2} - 1}{2} \tag{3.1}$$

If

$$|g(z)| \leq M \tag{3.2}$$

for all $z \in U$, then the function

$$T(z) = \left[(a+bi) \int_0^z u^{a+bi-1} \left(\frac{g(u)}{u} \right)^{\frac{1}{a+bi}} du \right]^{\frac{1}{a+bi}} \tag{3.3}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{\frac{1}{a+bi}} du \tag{3.4}$$

The function f is regular in U.

From (3.4) we have

$$f'(z) = \left(\frac{g(z)}{z} \right)^{\frac{1}{a+bi}}, f''(z) = \frac{1}{a+bi} \left(\frac{g(z)}{z} \right)^{\frac{1}{a+bi}-1} \frac{zg'(z) - g(z)}{z^2}$$

and

$$\frac{1 - |z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| = \frac{1 - |z|^{2a}}{a} \frac{1}{\sqrt{a^2+b^2}} \left| \frac{zg'(z)}{g(z)} - 1 \right| \tag{3.5}$$

for all $z \in U$.

From (3.5) we obtain

$$\frac{1 - |z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2a}}{a\sqrt{a^2+b^2}} \left(\left| \frac{zg'(z)}{g(z)} \right| + 1 \right) \tag{3.6}$$

for all $z \in U$, and hence, we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}}\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\left|\frac{g(z)}{z}\right|+1\right) \tag{3.7}$$

for all $z \in U$.

By the Schwarz-Lemma and using (3.7) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}}\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|\cdot M+M+1\right) \tag{3.8}$$

for all $z \in U$.

From (3.8) and because g verifies the condition (2.1) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{(2M+1)\left(1-|z|^{2a}\right)}{a\sqrt{a^2+b^2}} \le \frac{(2M+1)}{a\sqrt{a^2+b^2}} \tag{3.9}$$

for all $z \in U$.

From (3.1) we have

$$\frac{2M+1}{a\sqrt{a^2+b^2}} \le 1 \tag{3.10}$$

Using (3.10) and (3.9) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{3.11}$$

for all $z \in U$.

From (3.4) we obtain $f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{a+bi}}$ and by Theorem 2.2 it results that the function T is in the class S. ■

Theorem 3.2. *Let the function $g \in A$ verify (2.1), $a+bi$ be a complex number, $a \in \left(\frac{1}{2},\infty\right), (a,b) \neq (1,0), M$ be a real number such that*

$$0 < M \le \frac{a-\sqrt{(a-1)^2+b^2}}{2\sqrt{(a-1)^2+b^2}} \tag{3.12}$$

If

$$|g(z)| \le M \tag{3.13}$$

for all $z \in U$, then the function

$$G(z) = \left\{(a+bi)\int_0^z [g(u)]^{a+bi-1}\,du\right\}^{\frac{1}{a+bi}} \tag{3.14}$$

is in the class S.

Proof. From (3.14) we have

$$G(z) = \left[(a+bi) \int_0^z u^{a+bi-1} \left(\frac{g(u)}{u} \right)^{a+bi-1} du \right]^{\frac{1}{a+bi}} \tag{3.15}$$

Let us consider the function

$$p(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{a+bi-1} du \tag{3.16}$$

The function p is regular in U.

From (3.16) we get $p'(z) = \left(\frac{g(z)}{z} \right)^{a+bi-1}$,

$$p''(z) = (a+bi-1) \left(\frac{g(z)}{z} \right)^{a+bi-2} \frac{zg'(z) - g(z)}{z^2}$$

and

$$\frac{1-|z|^{2a}}{a} \left| \frac{zp''(z)}{p'(z)} \right| \leq \frac{1-|z|^{2a}}{a} |a+bi-1| \left(\left| \frac{zg'(z)}{g(z)} \right| + 1 \right) \tag{3.17}$$

for all $z \in U$, and hence, we obtain

$$\frac{1-|z|^{2a}}{a} \left| \frac{zp''(z)}{p'(z)} \right| \leq \sqrt{(a-1)^2 + b^2} \frac{1-|z|^{2a}}{a} \left(\left| \frac{z^2 g'(z)}{g^2(z)} \right| \left| \frac{g(z)}{z} \right| + 1 \right) \tag{3.18}$$

By the Schwarz-Lemma and using (3.18) we have

$$\frac{1-|z|^{2a}}{a} \left| \frac{zp''(z)}{p'(z)} \right| \leq \sqrt{(a-1)^2 + b^2} \frac{1-|z|^{2a}}{a} \left(\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| \cdot M + M + 1 \right) \tag{3.19}$$

From (3.19) and since g satisfies the condition (2.1) we get

$$\frac{1-|z|^{2a}}{a} \left| \frac{zp''(z)}{p'(z)} \right| \leq (2M+1) \sqrt{(a-1)^2 + b^2} \frac{1-|z|^{2a}}{a} \leq \frac{(2M+1)\sqrt{(a-1)^2 + b^2}}{a} \tag{3.20}$$

for all $z \in U$.

From (3.12) we get

$$\frac{(2M+1)\sqrt{(a-1)^2 + b^2}}{a} \leq 1 \tag{3.21}$$

and by (3.20) we have

$$\frac{1-|z|^{2a}}{a} \left| \frac{zp''(z)}{p'(z)} \right| \leq 1 \tag{3.22}$$

for all $z \in U$.

From (3.16) we have $p'(z) = \left(\frac{g(z)}{z} \right)^{a+bi-1}$ and by Theorem 2.2 it results that the function G is in the class S. ■

Remark 1 *For* $0 < M \leq 1$, *Theorem 3.1 and Theorem 3.2 can hold only in the case* $g(z) = kz$, *where* $|k| = 1$.

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