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Nonlinear Vector Maximal Principles In Product Structures

Mihai TURINICI

Abstract. Some nonlinear extensions of the vector maximality statement established by Goepfert, Tammer and Zălinescu [Nonl. Anal., 39 (2000), 909-922] are given. Basic instruments for these are the Brezis-Browder ordering principle [Advances Math., 21 (1976), 355-364] and a (pseudometric) version of it obtained in Turinici [Demonstr. Math., 22 (1989), 213-228].

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1 Introduction

Let Y be a (real) separated *locally convex space*; and K, some *(convex) cone* of it [$\alpha K+\beta K \subseteq K, \forall \alpha, \beta \geq 0$]. The relation ($\leq_K$) on Y defined as

(a1) $(y_1, y_2 \in Y)$: $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$

is reflexive and transitive; hence a *quasi-order*; also denoted as ($\leq$), when K is understood. [In addition, this relation is *compatible* with the linear structure of Y]. Let H be another (convex) cone of Y with $K \subseteq H$; and pick some $k^0 \in K \setminus (-H)$. Further, take some complete metric space (X, d). The relation ($\succeq$) over $X \times Y$ introduced as

(b1) $(x_1, y_1) \succeq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \leq y_1 - y_2$

is again reflexive and transitive; hence a quasi-order. Finally, take some nonempty part A of $X \times Y$. For a number of both practical and theoretical reasons, it would be useful to determine sufficient conditions under which $(A, \succeq)$ has points with certain maximal properties. The basic 2000 result in the area obtained by Goepfert, Tammer and Zălinescu [9], deals with the case $H = \mathrm{cl}(K)$ (=the *closure* of K). Precisely, assume that

(1a) $P_Y(A)$ is K-bounded below [$\exists \widetilde{y} \in Y$ with $P_Y(A) \subseteq \widetilde{y} + K$]

(1b) if $((x_n, y_n)) \subseteq A$ is ($\succeq$)-ascending and $x_n \to x$ then $x \in P_X(A)$
and there exists $y \in A(x)$ such that $(x_n, y_n) \succeq (x, y)$, for all n.

[Here, for each $(x,y) \in A$, $A(x)$ (respectively, $A(y)$) stands for the *x-section* (respectively, *y-section*) of (the relation) A; and P_X, P_Y are the *projection* operators from $X \times Y$ to X and Y respectively].

Theorem 1 *Let the above conditions be in force. Then, for each* $(x_0, y_0) \in A$ *there exists* $(\bar{x}, \bar{y}) \in A$ *with*

$$(x_0, y_0) \succeq (\bar{x}, \bar{y}) \text{ [hence } y_0 \geq \bar{y}\text{]} \tag{1.1}$$

$$\text{if } (x', y') \in A \text{ fulfills } (\bar{x}, \bar{y}) \succeq (x', y') \text{ then } \bar{x} = x'. \tag{1.2}$$

This result includes the ones due to Isac [11] and Nemeth [16] (which, in turn, extend Ekeland's variational principle [6]); and the authors' argument is based on the Cantor intersection theorem. Further, in his 2002 paper, Turinici [20] proposed a different approach, via ordering principles related to Brezis-Browder's [3] (cf. Section 2); and stressed that, conclusions like before are extendable for (non-topological) vector spaces Y to the case of $H = \text{arch}(K)$ (=the *Archimedean closure* of K). It is our aim in this exposition to show that a further enlargement of these facts is possible (by the same techniques). This, essentially, refers to the function

(c1) $\Lambda(t) = k^0 t,\ t \geq 0$ (where k^0 is the above one)

being no longer linear; details will be given in Section 4 (the Archimedean case) and Section 5 (the non-Archimedean case). The specific instrument of our investigations (in this last circumstance) is the concept of *gauge* function (developed in Section 3). Further aspects will be delineated elsewhere.

2 Brezis-Browder principles

(A) Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive and transitive relation) $(\leq)$ over M; as well as a function $x \vdash \psi(x)$ from M to $R_+ := [0, \infty[$. Call the point $z \in M$, *$(\leq, \psi)$-maximal* when: $w \in M$ and $z \leq w$ imply $\psi(z) = \psi(w)$. A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [3]:

Proposition 1 *Suppose that*

(2a) $(M, \leq)$ is sequentially inductive:
each ascending sequence has an upper bound (modulo $(\leq)$)

(2b) ψ is $(\leq)$-decreasing ($x \leq y \Longrightarrow \psi(x) \geq \psi(y)$).

Then, for each $u \in M$ there exists a $(\leq, \psi)$-maximal $v \in M$ with $u \leq v$.

This principle, including Ekeland's [6], found some basic applications to convex and nonconvex analysis (cf. the above references). So, a discussion about its key condition (2a) would be not without profit. Let $(Z, \leq)$ be some quasi-ordered structure. Take a function $z \vdash \varphi(z)$ from Z to $R \cup \{-\infty, \infty\}$; and let M be some nonempty part of Z. For simplicity reasons, we let again φ stand for the restriction of φ to M. The following "relative" form of Proposition 1 will be useful for us.

Proposition 2 *Suppose (2b) holds, as well as*

(2c) φ is inf-proper over M: $\mathrm{Dom}(\varphi) := \{x \in M; \varphi(x) < \infty\}$
is nonempty and $\varphi_* := \inf[\varphi(M)] > -\infty$

(2d) $\mathrm{Dom}(\varphi)$ *is sequentially inductive in M: each ascending sequence in* $\mathrm{Dom}(\varphi)$ *is bounded above in M (modulo $(\leq)$).*

Then, for each $u \in \mathrm{Dom}(\varphi)$ there exists $v \in \mathrm{Dom}(\varphi)$ with i) *$u \leq v$ and* ii) *$x \in M$, $v \leq x$ imply $\varphi(v) = \varphi(x)$.*

Proof Let $u \in \mathrm{Dom}(\varphi)$ be arbitrary fixed. Put $M(u, \leq) := \{x \in M; u \leq x\}$; and introduce the function (from M to R_+) $\psi(x) = \varphi(x) - \varphi_*, x \in M$. By the imposed conditions, Proposition 1 applies to $M(u, \leq)$ and $(\leq, \psi)$; wherefrom the conclusion is clear. ■

For the moment, Proposition 2 is a logical consequence of Proposition 1. The reciprocal is also true, by simply taking $Z = M$, $\varphi = \psi$. Hence, these two results are logically equivalent. Note that the inf-properness condition (2c) is not essential for the conclusion above (cf. Cârjă, Necula and Vrabie [4, Ch 2, Sect 2.1]); but, it will suffice for our purposes. Moreover, $(R, \geq)$ may be substituted by a separable ordering structure $(P, \leq)$ without altering the conclusion above; see Turinici [21] for details. Further enlargements were obtained in Altman [1] and Anisiu [2]; see also Kang and Park [13].

(B) A semi-metric version of these developments may be given along the following lines. Let $(M, \leq)$ be taken as before. By a *pseudometric* over M we shall mean any map $e : M \times M \to R_+$. If, in addition, e is *reflexive* $[e(x,x) = 0, \forall x \in M]$, *triangular* $[e(x,z) \leq e(x,y) + e(y,z), \forall x,y,z \in M]$ and *symmetric* $[e(x,y) = e(y,x), \forall x,y \in M]$, we say that it is a *semimetric* (on M). Suppose that we fixed such an object. Call the point $z \in M$, $(\leq, e)$-*maximal*, in case: $w \in M$ and $z \leq w$ imply $e(z,w) = 0$. [Note that, if (in addition) e is *sufficient* $[e(x,y) = 0$ implies $x = y]$, this property becomes: $w \in M, z \leq w \Longrightarrow z = w$ (and reads: z is strongly $(\leq)$-maximal). So, existence results involving such points may be viewed as "metrical" versions of the Zorn-Bourbaki principle (cf. Moore [15, Ch 4, Sect 4]). But, in the following, sufficiency will not be needed]. To get conditions for such a property, one may proceed as below. Call the (ascending) sequence (x_n) in M, *e-Cauchy* when; $\forall \delta > 0, \exists n(\delta)$ such that $n(\delta) \leq p \leq q \Longrightarrow e(x_p, x_q) \leq \delta$; and *e-asymptotic*, provided: $e(x_n, x_{n+1}) \to 0$, as $n \to \infty$. Clearly, each (ascending) e-Cauchy sequence is e-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent each other:

(2e) each ascending sequence is e-Cauchy

(2f) each ascending sequence is e-asymptotic.

By definition, either of these will be referred to as $(M, \leq)$ is *regular* (modulo e). The following maximality result in Turinici [19] is available.

Proposition 3 *Assume that $(M, \leq)$ is sequentially inductive and regular (modulo e). Then, for each $u \in M$ there exists an $(\leq, e)$-maximal $v \in M$ with $u \leq v$.*

This result includes the Brezis-Browder ordering principle [3] (Proposition 1); to which it reduces when $e(x,y) = |\psi(x) - \psi(y)|$ (where ψ is the above one). The reciprocal inclusion is also true; see the quoted paper for details.

3 Conical gauge functions

Let Y be a (real) *vector space*. Take a convex cone L of Y (cf. Section 1); which, in addition, is *non-degenerate* $[L \neq \{0\}]$ and *proper* $[L \neq Y]$. Denote by $(\leq_L)$ its induced quasi-order (cf. (a1)); when L is understood, we indicate this as $(\leq)$, for simplicity. Further, let the map $\Lambda : R_+ \to L$ be *normal* (modulo L):

(3a) $\Lambda(0) = 0$ and Λ is strictly increasing (modulo L):
$\Lambda(\tau) - \Lambda(t) \in L \setminus (-L)$, whenever $\tau > t$

(3b) Λ is sub-additive: $\Lambda(t_1 + t_2) \leq \Lambda(t_1) + \Lambda(t_2)$, $\forall t_1, t_2 \in R_+$.

Note that, as a consequence of this,

$$\Lambda \text{ is strictly positive (modulo } L\text{)}: \Lambda(t) \in L \setminus (-L), \forall t > 0 \tag{3.1}$$

$$\Lambda \text{ is substractive: } \Lambda(t_1 - t_2) \geq \Lambda(t_1) - \Lambda(t_2),\ \forall t_1, t_2 \in R_+,\ t_1 \geq t_2. \tag{3.2}$$

Having these precise, denote (for $y \in Y$)

(a3) $\Gamma(L; \Lambda; y) = \{s \in R_+; \Lambda(s) \leq y\}$, $\gamma(L; \Lambda; y) = \sup \Gamma(L; \Lambda; y)$.

(By convention, $\sup(\emptyset) = -\infty$). We therefore defined a couple of functions $\Gamma(.) := \Gamma(L; \Lambda; .)$ and $\gamma(.) := \gamma(L; \Lambda; .)$ from Y to $\mathcal{P}(Y)$ and $R \cup \{-\infty, \infty\}$ respectively; the latter of these will be referred to as the *gauge* function attached to $(L; \Lambda)$. For the particular case of *linear* normal functions [i.e., the one of (c1), with $k^0 \in L \setminus (-L)$], such objects were introduced (in the same context) by Turinici [20]; and these, in turn, appear as non-topological extensions of the locally convex ones in Goepfert, Tammer and Zălinescu [9]. The present developments may therefore be viewed as "nonlinear" extensions of the preceding ones. To begin with, note that for each $y \in L$,

$$\Gamma(y) \text{ is hereditary } (s \in \Gamma(y) \Longrightarrow [0, s] \subseteq \Gamma(y));$$

so, it is an (R_+)-initial interval $[0, \alpha[$ (where $0 \leq \alpha \leq \infty$) or $[0, \alpha]$ (where $0 \leq \alpha < \infty$). In addition, we have (by definition)

$$\begin{array}{l} y \in L \Longleftrightarrow \Gamma(y) \neq \emptyset \text{ (hence } \gamma(y) \in [0, \infty]) \\ y \notin L \Longleftrightarrow \Gamma(y) = \emptyset \text{ (hence } \gamma(y) = -\infty). \end{array} \tag{3.3}$$

(A) The list of basic properties for γ is as follows:
i) The gauge function is increasing (over its existence domain)

$$y_1, y_2 \in Y,\ y_1 \leq y_2 \quad \text{implies } \gamma(y_1) \leq \gamma(y_2). \tag{3.4}$$

ii) Further, γ is super-additive and substractive:

$$\gamma(y_1 + y_2) \geq \gamma(y_1) + \gamma(y_2), \text{ whenever the right member exists} \tag{3.5}$$

$$\gamma(y_1 - y_2) \leq \gamma(y_1) - \gamma(y_2), \text{ if } \gamma(y_2)\text{=finite (hence } 0 \leq \gamma(y_2) < \infty). \tag{3.6}$$

Clearly, it will suffice proving the former one. Without loss, we may assume that $\gamma(y_1) > 0$, $\gamma(y_2) > 0$ (cf. (3.3)+(3.4)). By definition (and the hereditary property of $y \vdash \Gamma(y)$) $y_1 \geq \Lambda(t_1)$, $y_2 \geq \Lambda(t_2)$, whenever $0 \leq t_1 < \gamma(y_1)$, $0 \leq t_2 < \gamma(y_2)$; so, combining with (3b) yields $y_1 + y_2 \geq \Lambda(t_1) + \Lambda(t_2) \geq \Lambda(t_1 + t_2)$; that is, $\gamma(y_1 + y_2) \geq t_1 + t_2$. This, and the precise arbitrariness of the couple (t_1, t_2), ends the argument.

iii) An important question to be solved is that of γ being *proper*; i.e., not identically ∞. A positive answer to this is available, via (3a); precisely, we have the so-called *identity relation:*

$$\gamma(\Lambda(t)) = t, \ \forall t \in R_+ \text{ (hence } \gamma(y) \leq t, \quad \text{whenever } y \leq \Lambda(t)). \tag{3.7}$$

In fact, let $t \in R_+$ be arbitrary fixed; it will suffice verifying that $\Gamma(\Lambda(t)) = [0, t]$. Suppose not: there exists $\tau > t$ with $\tau \in \Gamma(\Lambda(t))$. By definition, $\Lambda(\tau) \leq \Lambda(t)$; wherefrom $\Lambda(\tau) - \Lambda(t) \in (-L)$. This yields a contradiction to (3a); and proves the claim.

iv) As a consequence of (3.5), we have the sup-translation property

$$\gamma(y + \Lambda(t)) \geq \gamma(y) + t, \ \text{ for each } y \in Y, \ t \in R_+. \tag{3.8}$$

This relation may be strict; just take $y = -\Lambda(\tau), t = \tau$, for some $\tau > 0$.

(B) Concerning the effectiveness of such a construction, call the function $\psi : R_+ \to R_+$, *normal*, when $\psi(0) = 0$ and ψ is strictly increasing sub-additive on R_+ (see above). Note that such functions exist; such as, e.g.: $\psi(t) = t^\lambda$, $t \in R_+$, for some $\lambda \in]0, 1]$. Suppose that $\{\psi_1, ..., \psi_m\}$ are endowed with such properties; and take some points $\{k^1, ..., k^m\}$ in $L \setminus (-L)$. Then, the function (from R_+ to L)

(b3) $\Lambda(t) = k^1\psi_1(t) + ... + k^m\psi_m(t), \quad t \in R_+$

is a normal one, in the sense of (3a)+(3b). The obtained class of all these covers the linear one (expressed via (c1)); when (as precise) these developments reduce to the ones in Turinici [20]. Further aspects involving the locally convex (modulo Y) case (and the same linear setting) may be found in Goepfert, Riahi, Tammer and Zălinescu [8, Ch 3, Sect 10]; see also Gerth (Tammer) and Weidner [7].

4 Main result

With these preliminaries, we may now return to the question of the introductory part. Let Y be a (real) vector space; and K, some (convex) cone of it. Denote by $(\leq_K)$ the induced quasi-order (cf. (a1)); also written as $(\leq)$, when K is understood. Let H be another (convex) cone of Y with $K \subseteq H$; and the map $\Lambda : R_+ \to K$ be *almost normal* (modulo (K, H)):

(4a) $\Lambda(0) = 0$ and Λ is strictly increasing (modulo (K, H)):
$\Lambda(\tau) - \Lambda(t) \in K \setminus (-H)$, whenever $\tau > t$

(4b) Λ is sub-additive (modulo K):
$\Lambda(t_1 + t_2) \leq \Lambda(t_1) + \Lambda(t_2)$, $\forall t_1, t_2 \in R_+$.

Note that, as a consequence of this (cf. Section 3)

$$\Lambda \text{ is strictly positive (modulo } (K, H)): \ \Lambda(t) \in K \setminus (-H), \ \forall t > 0 \tag{4.1}$$

$$\begin{array}{l}\Lambda \text{ is substractive (modulo } K\text{):}\\ \Lambda(t_1 - t_2) \geq \Lambda(t_1) - \Lambda(t_2),\ \forall t_1, t_2 \in R_+,\ t_1 \geq t_2.\end{array} \tag{4.2}$$

In addition, let (X, d) be a metric space. The relation $(\succeq)$ over $X \times Y$ introduced as

(a4) $(x_1, y_1) \succeq (x_2, y_2)$ iff $\Lambda(d(x_1, x_2)) \leq y_1 - y_2$

is reflexive and transitive (by the properties of Λ); hence a quasi-order on it. Finally, take some (nonempty) part A of $X \times Y$. As in Section 1, we are interested to determine sufficient conditions under which $(A, \succeq)$ should have points with certain maximality properties. Note that, in the linear case of (c1), this problem is just the one in Turinici [20]; which (under d=complete) has a positive answer in the context of ((1b) and)

(4c) $P_Y(A)$ is H-bounded below $[\exists \widetilde{y} \in Y$ with $P_Y(A) \subseteq \widetilde{y} + H]$.

So, it is natural asking whether similar conclusions are retainable in our "nonlinear" setting. Loosely speaking, these depend on the ambient convex cone H being or not *Archimedean*. So, two alternatives are open before us.

In the following, we discuss the former of these, based on H being endowed with such a property (cf. Cristescu [5, Ch 5, Sect 1]):

(4d) $h, v \in H$ and $[h\tau \leq_H v, \forall \tau > 0]$ imply $h \in H \cap (-H)$.

As we shall see, a positive answer is available under

(4e) each $(\succeq)$-ascending e-Cauchy sequence $((x_n, y_n)) \subseteq A$
is bounded above in A (modulo $(\succeq)$).

Here, e stands for the semi-metric on $X \times Y$ introduced as

(b4) $e((x_1, y_1), (x_2, y_2)) = d(x_1, x_2),\ (x_1, y_1), (x_2, y_2) \in X \times Y$.

The (first) main result of our exposition is

Theorem 2 *Let the assumptions (4c)-(4e) be in force. Then, for each starting* $(x_0, y_0) \in A$ *there exists* $(\bar{x}, \bar{y}) \in A$ *with the properties (1.1) and (1.2).*

The latter of the conclusions above reads (under the precise convention)

$$(x', y') \in A,\ (\bar{x}, \bar{y}) \succeq (x', y') \Longrightarrow e((\bar{x}, \bar{y}), (x', y')) = 0.$$

This suggests us a possible deduction of Theorem 2 from Proposition 3. To see the effectiveness of such an approach, we need an auxiliary fact.

Lemma 1 *Let* $((x_n, y_n))$ *be an* $(\succeq)$*-ascending sequence in* A*:*

(4f) $\Lambda(d(x_n, x_m)) \leq y_n - y_m$, *whenever* $n \leq m$.

Then, (x_n) *is* d*-Cauchy in* $P_X(A)$*; hence* $((x_n, y_n))$ *is* e*-Cauchy in* A.

Proof of Lemma 1 Suppose that this would be not valid; i.e. (as d is symmetric), there must be some $\varepsilon > 0$ in such a way that

(4g) for each n, there exists $m > n$ with $d(x_n, x_m) \geq \varepsilon$.

Inductively, we may construct a subsequence $(u_n = x_{i(n)})$ of (x_n) with $d(u_n, u_{n+1}) \geq \varepsilon$, for all n. This in turn yields, for the corresponding subsequence $(v_n = y_{i(n)})$ of (y_n), an evaluation like

$$\Lambda(\varepsilon) \leq \Lambda(d(u_n, u_{n+1})) \leq v_n - v_{n+1}, \text{ for each } n \geq 0.$$

But then, in view of (4c), one derives

$$q\Lambda(\varepsilon) \leq_H v_0 - v_q \leq_H v_0 - \widetilde{y}, \text{ for each } q \geq 1.$$

This, along with (4d), gives $\Lambda(\varepsilon) \in K \cap (-H)$; in contradiction with (4.1). Hence, the working assumption (4g) cannot hold; and the claim follows. ■

Proof of Theorem 2 Let $((x_n, y_n))$ be a $(\succeq)$-ascending sequence in A. By Lemma 1, $((x_n, y_n))$ is an e-Cauchy sequence in A; which tells us that $(A, \succeq)$ is regular (modulo e). Moreover, by (4e), $((x_n, y_n))$ is bounded above (modulo $(\succeq)$) in A; wherefrom, $(A, \succeq)$ is sequentially inductive. Summing up, Proposition 3 is applicable to $(A, \succeq; e)$; so that (from its conclusion) each $a_0 = (x_0, y_0)$ in A is majorized (modulo $(\succeq)$) by some $(\succeq, e)$-maximal $\bar{a} = (\bar{x}, \bar{y})$ in A. This gives the conclusions (1.1)+(1.2) we need. ■

In particular, when Λ is taken as in (c1) (and d is complete) Theorem 2 is just the related statement in Turinici [20]; which, as precise there, incorporates the (locally convex) one in Goepfert, Tammer and Zălinescu [9] (Theorem 1). This inclusion seems to be strict; because the choice (b3) of Λ cannot be reduced to the linear one (appearing in all these papers). Some related aspects may be found in Hamel [10, Ch 4]; see also Tammer [18].

5 A completion

Now, the key regularity assumption used in the result above is (4d). So, it is natural to discuss the alternative of this being avoided. As we shall see below, a positive answer is available; but we must restrict the initial set A in a way imposed by the associated (to H) gauge function.

Let Y be a (real) vector space; and K, some (convex) cone in it. Denote by $(\leq_K)$ its associated quasi-order; also written as $(\leq)$, when K is understood; and let H be another (convex) cone of Y with $K \subseteq H$. We also take a map $\Lambda : R_+ \to K$; which is supposed to be almost normal (modulo (K, H)) in the sense of (4a)+(4b). Clearly, it is also normal (modulo H); so, we may construct the gauge function $\gamma : Y \to R \cup \{-\infty, \infty\}$ attached to (H, Λ), under the model of (a3). Further, letting (X, d) be a metric space, denote (again) by $(\succeq)$ the quasi-order on $X \times Y$ introduced as in (a4); and finally, let A be some (nonempty) part of $X \times Y$. The question to be posed is the same as in Section 4; to solve it, we list the needed conditions. The former of these is (again) (4c); note that it may be also written as

(5a) $P_Y(A) \subseteq H$ (i.e.: $\widetilde{y} = 0$ in that condition).

[For, otherwise, passing to the associated subset $A_0 = \{(x, y) \in X \times Y; (x, y + \widetilde{y}) \in A\}$, this requirement is fulfilled, in view of $P_Y(A_0) = P_Y(A) - \widetilde{y}$]. As a consequence,

$$\inf[\gamma(P_Y(A))] \geq 0 \quad \text{(cf. (3.3) above).} \tag{5.1}$$

However, the alternative $\gamma[P_Y(A)] = \{\infty\}$ cannot be avoided; so, we must accept (as a second condition)

(5b) $P_Y(A) \cap \mathrm{Dom}(\gamma) \neq \emptyset$ ($\gamma(y) < \infty$, for some $y \in P_Y(A)$).

A useful characterization of these is to be realized via the composed function $\Phi(x, y) = \gamma(y), (x, y) \in X \times Y$ (i.e.: $\Phi = \gamma \circ P_Y$). Precisely, let again Φ denote the restriction of this function to A; then, (5a)+(5b) may be written (in a shorter way) as

(5c) $\mathrm{Dom}(\Phi) := \{(x, y) \in A; \Phi(x, y) < \infty\}$ is nonempty; $\inf[\Phi(A)] \geq 0$.

Now, the last condition to be imposed is a variant of (1b) above:

(5d) each $(\succeq)$-ascending e-Cauchy sequence $((x_n, y_n)) \subseteq \mathrm{Dom}(\Phi)$
is bounded above in A (modulo $(\succeq)$).

Here, e stands for the semi-metric over $X \times Y$ introduced as in (b4).

We are now in position to state the second main result in this exposition.

Theorem 3 *Let the precise conditions be in force. Then, for each* $(x_0, y_0) \in \mathrm{Dom}(\Phi)$ *there exists* $(\bar{x}, \bar{y}) \in \mathrm{Dom}(\Phi)$ *with the properties (1.1) and*

$$\text{if } (x', y') \in A \text{ fulfills } (\bar{x}, \bar{y}) \succeq (x', y') \text{ then } \bar{x} = x',\ \gamma(\bar{y}) = \gamma(y'). \tag{5.2}$$

Proof We claim that Proposition 2 is applicable to $(Z = X \times Y, \succeq)$, $M = A$ and $\varphi = \Phi$. In fact, by the remarks above $(x_1, y_1) \succeq (x_2, y_2)$ implies $\Phi(x_1, y_1) \geq \Phi(x_2, y_2)$; i.e., Φ is $(\succeq)$-decreasing. On the other hand, (5c) is just (2c) (with φ substituted by Φ). Finally, (5d) implies (2d) (with $\varphi = \Phi$); and this will establish our claim. In fact, let $((x_n, y_n))$ be a $(\succeq)$-ascending sequence in $\mathrm{Dom}(\Phi)$; i.e., (4f) holds. Combining with the substractivity of the gauge function (cf. Section 3) yields

$$d(x_n, x_m) \leq \gamma(y_n - y_m) \leq \gamma(y_n) - \gamma(y_m), \text{ whenever } n \leq m.$$

The (real) sequence $(\gamma(y_n))$ is descending and bounded (by the choice of our data and (5.1)); hence a Cauchy one. This, added to the above, shows that (x_n) is a d-Cauchy sequence in $P_Y(A)$; or, equivalently, that $((x_n, y_n))$ is e-Cauchy in A; wherefrom (by (5d)) the claim follows. By Proposition 2 we therefore derive that, for $(x_0, y_0) \in \mathrm{Dom}(\Phi)$ there exists $(\bar{x}, \bar{y}) \in \mathrm{Dom}(\Phi)$ with the properties (1.1) and

$$\text{if } (x', y') \in A \text{ fulfills } (\bar{x}, \bar{y}) \succeq (x', y') \text{ then } \Phi(\bar{x}, \bar{y}) = \Phi(x', y').$$

The relation in the left member of this implication yields (see the remarks above): $(x', y') \in \mathrm{Dom}(\Phi), d(\bar{x}, x') \leq \gamma(\bar{y}) - \gamma(y')$. On the other hand, the relation in the right member of the same is just: $\gamma(\bar{y}) = \gamma(y')$; so that (combining these) $d(\bar{x}, x') = 0$; wherefrom $\bar{x} = x'$. This proves (5.2) as well; and concludes the argument. ∎

As before, when Λ is taken as in (c1) (and d is complete) Theorem 3 is nothing but the related statement in Turinici [20], obtained via similar techniques. [Moreover, when H is taken as in (4d), we have (cf. Section 3)

$$\mathrm{Dom}(\gamma) = H; \text{ hence } \mathrm{Dom}(\Phi) = A \text{ (in view of (5a))};$$

and Theorem 3 reduces to Theorem 2 above. But, in the general (nonlinear) setting, this is not true]. Further aspects (of locally convex nature) may be found in Isac and Tammer [12]; see also Rozoveanu [17]. For different structural extensions of these we refer to Hamel [10, Ch 5] and Khanh [14].

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