

Projective Maximal Principles In General Vector Spaces

Mihai TURINICI

Abstract. A projective counterpart of the vector maximality principle established by Goepfert, Tammer and Zălinescu [Nonl. Anal., 39 (2000), 909-922] is proposed. The basic instrument for this is a lot of ordering principles comparable with the one in Turinici [Libertas Math., 20 (2000), 161-172].

Keywords: Convex cone, quasi-order, complete metric, maximal element, sequential inductivity, decreasing proper function, Cauchy and asymptotic sequence, self-closeness, projective anti-metric, Archimedean property.

Mathematics Subject Classification (2000): Primary 46A40. Secondary 54E40.

1 Introduction

Let Y be a (real) separated *locally convex space*; and K , some (convex) cone of it ($\alpha K + \beta K \subseteq K, \forall \alpha, \beta \geq 0$). The relation (\leq_K) on Y defined as

(a1) $(y_1, y_2 \in Y)$: $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$

is reflexive and transitive; hence a *quasi-order*; also denoted as (\leq) , when K is understood. [Moreover, it is *compatible* with the linear structure of Y]. Let H be another cone of Y with $K \subseteq H$; and pick some $k^0 \in K \setminus (-H)$. Further, take some complete metric space (X, d) . The relation (\succeq) over $X \times Y$ introduced as

(b1) $(x_1, y_1) \succeq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \leq y_1 - y_2$

is again reflexive and transitive; hence a quasi-order. Finally, take some nonempty part A of $X \times Y$. For a number of both practical and theoretical reasons, it would be useful to determine sufficient conditions under which the quasi-ordered structure (A, \succeq) has points with certain maximal properties. The basic 2000 result in this direction, obtained by Goepfert, Tammer and Zălinescu [11], deals with the case $H = \text{cl}(K)$ (=the *closure* of K). Technically speaking, it extends the one in Phelps [18, Ch 3, Sect 3.12]; which may be viewed as a "product" version of Ekeland's variational principle [8] (in short: EVP). Further aspects of historical nature may be found in Goepfert, Riahi, Tammer and Zălinescu [10, Ch 3, Sect 10].

A relevant application of this result refers to vector EVP. For technical reasons, we complete Y with an element $\infty \notin Y$ subject to

$$(c1) \quad \infty = b + \infty = \infty + b, \forall b \in Y; \quad \infty = \lambda \infty, \forall \lambda > 0 \\ b \leq \infty \text{ and } \infty \not\leq b, \quad \text{for each } b \in Y.$$

Take a function $F : X \rightarrow Y \cup \{\infty\}$. The precise corollary consists in applying this result to $A = \text{gr}(F) \cap (X \times Y)$; it includes the maximality statements in Isac [13] and Nemeth [17]; which, in turn, extend (in a direct way) Ekeland's variational principle we already quoted.

Technically speaking, the authors' argument is based on the Cantor intersection theorem. Further, in 2002, Turinici [25] proposed a different approach, via ordering principles related to Brezis-Browder's [4] (cf. Section 2); and stressed that conclusions like before are extendable beyond the locally convex setting, to the case of $H = \text{arch}(K)$ (the *Archimedean closure* of K). It is our aim in the following to get a "projective" counterpart of the "product" maximal principle above; details will be given in Section 4. The specific instrument of it is the concept of *projective anti-metric* (developed in Section 3), which may be related to some constructions in Thompson [21]. Finally, in Section 5, a projective counterpart of the precise vector EVP is proposed. Further aspects will be discussed elsewhere.

2 Brezis-Browder principles

(A) Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive and transitive relation) (\leq) over M ; as well as a function $x \mapsto \psi(x)$ from M to $R_+ := [0, \infty[$. Call the point $z \in M$, (\leq, ψ) -*maximal* when: $w \in M$ and $z \leq w$ imply $\psi(z) = \psi(w)$. A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [4]:

Proposition 1 *Suppose that*

(2a) (M, \leq) *is sequentially inductive:*
each ascending sequence has an upper bound (modulo (\leq))

(2b) ψ *is (\leq) -decreasing ($x \leq y \implies \psi(x) \geq \psi(y)$).*

Then, for each $u \in M$ there exists a (\leq, ψ) -maximal $v \in M$ with $u \leq v$.

Note that $\text{Codom}(\psi) \subseteq R_+$ is not essential for the conclusion above; cf. Cârjă, Necula and Vrabie [6, Ch 2, Sect 2.1]. In addition (by the argument developed there) Proposition 1 is reducible to the *Principle of Dependent Choices* (stated in Wolk [27]). Finally, (R_+, \geq) may be substituted by a separable ordering structure (P, \leq) without altering the conclusion above; see Turinici [26] for details.

This principle, including Ekeland's [8], found some useful applications to convex and nonconvex analysis (cf. the above references). So, a discussion about its key condition (2a) would be not without profit. Let (Z, \leq) be some quasi-ordered structure. Take a function $z \mapsto \varphi(z)$ from Z to $R \cup \{-\infty, \infty\}$; and let M be some nonempty part of Z . For simplicity reasons, we let again φ stand for the restriction of φ to M . The following "relative" form of Proposition 1 will be useful for us.

Proposition 2 Suppose (2b) holds (with φ in place of ψ) as well as

(2c) φ is inf-proper over M : $\text{Dom}(\varphi) := \{x \in M; \varphi(x) < \infty\}$
is nonempty and $\varphi_* := \inf\{\varphi(M)\} > -\infty$

(2d) each ascending sequence in $\text{Dom}(\varphi)$
is bounded above in M (modulo (\leq)).

Then, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with i) $u \leq v$ and ii) $x \in M, v \leq x$ imply $\varphi(v) = \varphi(x)$.

Proof Let $u \in \text{Dom}(\varphi)$ be arbitrary fixed. Put $M(u, \leq) := \{x \in M; u \leq x\}$; and introduce the function (from M to R_+) $\psi(x) = \varphi(x) - \varphi_*, x \in M$. By the imposed conditions, Proposition 1 applies to $M(u, \leq)$ and (\leq, ψ) ; wherefrom the conclusion is clear. ■

For the moment, Proposition 2 is a logical consequence of Proposition 1. The reciprocal is also true, by simply taking $Z = M, \varphi = \psi$. Hence, these two results are logically equivalent. As already precised, the inf-properness condition (2c) is not essential for the conclusion above; but, it will suffice for our purposes. Further enlargements of such facts were obtained in Altman [1] and Anisiu [2]; see also Kang and Park [15].

(B) A semi-metric version of these developments may be given along the following lines. Let (M, \leq) be taken as before. By a *pseudometric* over M we shall mean any map $e : M \times M \rightarrow R_+$. If, in addition, e is *reflexive* [$e(x, x) = 0, \forall x \in M$], *triangular* [$e(x, z) \leq e(x, y) + e(y, z), \forall x, y, z \in M$] and *symmetric* [$e(x, y) = e(y, x), \forall x, y \in M$], we say that it is a *semi-metric* (on M). Suppose that we fixed such an object. Call the point $z \in M, (\leq, e)$ -*maximal*, when: $w \in M$ and $z \leq w$ imply $e(z, w) = 0$. Note that, if (in addition) e is *sufficient* [$e(x, y) = 0$ implies $x = y$], the (\leq, e) -maximal property becomes: $w \in M, z \leq w \implies z = w$ (and reads: z is (strongly) (\leq) -maximal). So, existence results involving such points may be viewed as "metrical" versions of the Zorn-Bourbaki principle (cf. Moore [16, Ch 4, Sect 4]). To get appropriate conditions for these (in the non-sufficient setting), one may proceed as below. Call the (ascending) sequence (x_n) in M, e -*Cauchy* when: $\forall \delta > 0, \exists n(\delta)$ such that $n(\delta) \leq p \leq q \implies e(x_p, x_q) \leq \delta$; and *d-asymptotic*, provided: $e(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Clearly, each (ascending) e -Cauchy sequence is e -asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent each other:

(2e) each ascending sequence is e -Cauchy

(2f) each ascending sequence is e -asymptotic.

By definition, either of these will be referred to as (M, \leq) is *regular* (modulo e). The following maximality result in Turinici [23] is available.

Proposition 3 Assume that (M, \leq) is sequentially inductive and regular (modulo e). Then, for each $u \in M$ there exists an (\leq, e) -maximal $v \in M$ with $u \leq v$.

This result extends the Brezis-Browder ordering principle (Proposition 1); to which it reduces when $e(x, y) = |\psi(x) - \psi(y)|$ (where ψ is the above one). The reciprocal inclusion

is also true [hence, these results are (mutually) equivalent]; we refer to the quoted paper by Turinici for details.

(C) Let us now return to the setting of Proposition 2. The basic question to be solved is that of establishing concrete circumstances under which (2d) be fulfilled. Let $(Z, \leq; d)$ be a quasi-ordered semi-metric structure. Take a function $z \mapsto \varphi(z)$ from Z to $R \cup \{-\infty, \infty\}$; and let M be some nonempty part of Z . Denote again by φ the restriction to M of φ ; and assume that (2b) (with φ in place of ψ) holds, as well as (2c).

Proposition 4 *Suppose that (in addition)*

(2g) $x, y \in \text{Dom}(\varphi)$, $x \leq y \implies d(x, y) \leq \varphi(x) - \varphi(y)$

(2h) *each ascending d -Cauchy sequence in $\text{Dom}(\varphi)$ is bounded above in M (modulo (\leq)).*

Then, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with the properties j) $u \leq v$ and jj) $x \in M$, $v \leq x \implies d(v, x) = 0$, $\varphi(v) = \varphi(x)$.

Proof We show that (2g)+(2h) \implies (2d); wherefrom, Proposition 2 applies to our data. Let (x_n) be an ascending sequence in $\text{Dom}(\varphi)$. The (real) sequence $(\varphi(x_n))$ is, by (2b)+(2c), descending and bounded; hence a Cauchy one. This, along with (2g), shows that (x_n) is (in addition) d -Cauchy in $\text{Dom}(\varphi)$; wherefrom (by (2h)) it is bounded above in M ; hence the claim. By Proposition 2 it then follows that, for each $u \in \text{Dom}(\varphi)$, there exists $v \in \text{Dom}(\varphi)$ with the properties i) and ii) written there. The former of these is just j) above. And the latter one gives jj) if one takes (2g) into account. ■

As a matter of fact, Proposition 4 may be viewed as a common extension of both Proposition 1 and Proposition 3; we do not give details.

(D) A useful application of these facts is to monotone variational principles. Let (M, \leq) be a quasi-ordered structure; and $d : M \times M \rightarrow R_+$ be some (standard) metric over it with

(2i) d is (\leq) -complete: each ascending d -Cauchy sequence converges.

Further, let $\varphi : M \rightarrow R \cup \{-\infty, \infty\}$ be some function with the properties (2b) (with φ in place of ψ) and (2c). The following metrical counterpart of Proposition 2 is available.

Proposition 5 *Suppose that (in addition)*

(2j) *each ascending sequence in $\text{Dom}(\varphi)$ is d -Cauchy*

(2k) (\leq) *is relatively self-closed over $\text{Dom}(\varphi)$: the limit of each ascending sequence in $\text{Dom}(\varphi)$ is an upper bound of it (in M).*

Then, conclusions of Proposition 2 are retainable.

Proof We claim that (2d) is fulfilled by our data; and, from this, all is clear. Let (x_n) be an ascending sequence in $\text{Dom}(\varphi)$. By (2i) and (2j), $x_n \rightarrow x$ as $n \rightarrow \infty$, for some $x \in M$. This, along with (2k), yields $x_n \leq x$, for all n ; and the claim follows. ●

(E) Now, a natural question to be solved is that of determining concrete circumstances under which (2j) and/or (2k) be fulfilled. Concerning the former of these, the following answer is to be noted. [The conditions (2b), (2c) and (2i) prevail].

Proposition 6 *Suppose that (in addition) (2k) holds, as well as (2g). Then, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with p) $u \leq v$ and pp) $x \in M, v \leq x \implies v = x$.*

Proof We show that (2k)+(2g) \implies (2j); wherefrom, Proposition 5 applies to these data. Let (x_n) be an ascending sequence in $\text{Dom}(\varphi)$. By (2b)+(2c), $(\varphi(x_n))$ is descending and bounded; hence a Cauchy one. This, added to (2g), shows that (x_n) is d -Cauchy; and the claim follows. By Proposition 5 it results that, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with the properties i) and ii) written there. The former of these is just p) above. And the latter gives pp), by (2g) and the sufficiency of d . ■

It remains now to discuss the possibility of realizing (2k). Let (M, d) be a metric space; and $\varphi : M \rightarrow R \cup \{-\infty, \infty\}$ be a function as in (2c). The relation over M defined as

$$(a2) \quad (x, y \in M): x \leq y \text{ if and only if } d(x, y) + \varphi(y) \leq \varphi(x)$$

is reflexive and transitive; hence a quasi-order on M .

Proposition 7 *Assume that (2i) is true for this quasi-order and*

(2m) φ is descending d -lsc: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever

$$(x_n) \subseteq \text{Dom}(\varphi) \text{ satisfies } x_n \rightarrow x \text{ and } (\varphi(x_n)) \text{ is descending.}$$

Then, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with q) $u \leq v$ and qq) $x \in M, d(v, x) \leq \varphi(v) - \varphi(x)$ imply $v = x$.

Proof By the convention (a2), it is clear that (2b)+(2g) hold for our data. We claim that (2k) also holds (under (2m)); and this will complete the reasoning (via Proposition 6). Let (x_n) be an ascending (modulo (\leq)) convergent sequence in $\text{Dom}(\varphi)$; i.e.,

$$(2n) \quad d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \quad \text{whenever } n \leq m;$$

and $x_n \rightarrow x$ as $n \rightarrow \infty$. The sequence $(\varphi(x_n))$ is descending in such a case. Combining with (2m) yields (passing to limit as $m \rightarrow \infty$ in (2n)) $d(x_n, x) \leq \varphi(x_n) - \varphi(x)$ (i.e.: $x_n \leq x$), for each n ; and the claim follows. ■

In particular, (2m) holds under

$$(2p) \quad \varphi \text{ is } d\text{-lsc: } \liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \rightarrow x.$$

Moreover, (2i) is true if d is complete (in the usual sense). Note that, in such a case, Proposition 7 is nothing but Ekeland's variational principle (EVP). Further aspects may be found in Hamel [12, ch 4]; see also Turinici [24]

3 Projective anti-metrics

Let Y be a (real) vector space. Take a convex cone L of Y (cf. Section 1); which, in addition, is non-degenerate ($L \neq \{0\}$) and proper ($L \neq Y$). Denote by (\leq_L) its induced quasi-order (cf. (a1)); when L is understood, we indicate this as (\leq) , for simplicity. Further, let $\alpha > 1$ be arbitrary fixed; and put $\exp_\alpha(t) = \alpha^t, t \in R$; when no confusion can arise, we simply write \exp_α as \exp . Denote, for each $y, v \in L$,

$$(a3) \Theta(L, \alpha; y, v) = \{t \in R_+; v \exp(t) \leq y\}, \theta(L, \alpha; y, v) = \sup \Theta(L, \alpha; y, v)$$

(where, by convention, $\sup(\emptyset) = -\infty$). We therefore defined a couple of functions $\Theta(\cdot, \cdot) := \Theta(L, \alpha; \cdot, \cdot)$ and $\theta(\cdot, \cdot) := \theta(L, \alpha; \cdot, \cdot)$ from $L \times L$ to $\mathcal{P}(R_+)$ and $R \cup \{-\infty, \infty\}$ respectively; the latter of these will be referred to as the *projective anti-metric* attached to (L, α) . The proposed construction may be viewed as a counterpart of the one in Gerth (Tammer) and Weidner [9]. A technical motivation for our terminology will be offered later. For the moment, we shall be interested in giving some basic properties of these maps.

(A) We start by noting that, for each $(y, v) \in L \times L$,

$$\Theta(y, v) \text{ is hereditary: } s \in \Theta(y, v) \implies [0, s] \subseteq \Theta(y, v); \quad (3.1)$$

so, it is an (R_+) -initial interval $[0, \lambda[$ (where $0 \leq \lambda \leq \infty$) or $[0, \lambda]$ (where $0 \leq \lambda < \infty$). In addition, we have (by definition)

$$\begin{aligned} y \geq v &\iff \Theta(y, v) \neq \emptyset \text{ (hence } \theta(y, v) \in [0, \infty]) \\ y \not\geq v &\iff \Theta(y, v) = \emptyset \text{ (hence } \theta(y, v) = -\infty). \end{aligned} \quad (3.2)$$

(B) An interesting question to be solved refers to the ambient cone L being *directionally closed*:

$$(b3) \Theta(y, v) \text{ is (nonempty) closed, for each } y, v \in L, y \geq v.$$

Note that it will suffice considering those pairs (y, v) with

$$(3a) (y \geq v \text{ and } v \in L \setminus (-L) \text{ (} v \text{ is strictly positive)}); \text{ hence } y \in L \setminus (-L).$$

For, if $v \in (-L)$ (or, equivalently, $-v \in L$) one has $v\tau \leq y$, for each $\tau \geq 0$; and this gives $\Theta(y, v) = R_+$ (hence $\theta(y, v) = \infty$). After a convention in Cristescu [7, Ch 5, Sect 1], let us say that L is *Archimedean* if

$$(c3) h \in Y, w \in L \text{ and } [h \leq_L w\lambda, \forall \lambda > 0] \text{ imply } h \in -L.$$

Lemma 1 Assume that L is Archimedean and let $y, v \in L$ be as in (3a). Then: i) $0 \leq \theta(y, v) < \infty$ and ii) $\Theta(y, v)$ is (nonempty) compact (hence $\theta(y, v) \in \Theta(y, v)$). In particular, L is directionally closed (see above).

Proof i) Assume by contradiction that $\theta(y, v) = \infty$. By definition, we have $v\lambda \leq y, \forall \lambda \geq 1$; wherefrom $v \leq y\mu, \forall \mu > 0$. This, along with the imposed hypotheses gives $v \in (-L)$; contradiction. ii) Without loss of generality, assume that $\theta := \theta(y, v) > 0$. By the hereditary property of $\Theta(\cdot, \cdot)$, we have $v \exp(\theta) - y \leq v\tau, \forall \tau > 0$. So (again by the Archimedean condition) $v \exp(\theta) \leq y$; and conclusion is clear. ■

(C) Returning to the general case, note that

$$\theta(\delta u, u) = \log(\delta), \forall u \in L \setminus (-L), \forall \delta \geq 1. \quad (3.3)$$

(Here, $\log = \log_\alpha$ is the inverse of $\exp = \exp_\alpha$). Further information may be obtained from the homogeneous property

$$\Theta(\lambda y, \lambda v) = \Theta(y, v) \text{ (hence } \theta(\lambda y, \lambda v) = \theta(y, v)), \forall \lambda > 0; \quad (3.4)$$

and the (order) monotonic one

$$y_1 \geq y_2 \geq v_2 \geq v_1 \implies \theta(y_1, v_1) \geq \theta(y_2, v_2). \quad (3.5)$$

In fact, as a consequence of this, we have (for each (u, δ) as in (3.3))

$$\theta(y, v) \leq \log(\delta), \quad \text{whenever } u \leq v \leq y \leq \delta u; \quad (3.6)$$

i.e., $(y, v) \vdash \theta(y, v)$ has "many" finite values, so as to work with it efficiently.

(D) Another property to be discussed is the "anti-triangular" one.

Lemma 2 *Under these conventions,*

$$y_1 \geq y_2 \geq y_3 \implies \theta(y_1, y_3) \geq \theta(y_1, y_2) + \theta(y_2, y_3). \quad (3.7)$$

Proof Without loss, one may assume $\theta(y_1, y_2) > 0$, $\theta(y_2, y_3) > 0$ (cf. (3.5)). Let $\lambda < \theta(y_1, y_2)$, $\mu < \theta(y_2, y_3)$ be arbitrary fixed. By definition, $y_2 \exp(\lambda) \leq y_1$, $y_3 \exp(\mu) \leq y_2$; wherefrom

$$y_3 \exp(\lambda + \mu) = [y_3 \exp(\mu)] \exp(\lambda) \leq y_2 \exp(\lambda) \leq y_1;$$

so, $\lambda + \mu \leq \theta(y_1, y_3)$. This, by the arbitrariness of λ, μ ends the argument. \bullet

This property motivates our terminology; because for a (semi-)metric, the inequality of (3.7) is with the dual order (cf. Section 2). On the other hand, the projective character of $(y, v) \vdash \theta(y, v)$ is motivated by the formal similarity between this construction and the one due to Thompson [21]. Further aspects may be found in Birkhoff [3] and Bushell [5]; see also Turinici [22].

4 Main result

With these preliminaries, we may now return to the question of the introductory part. Let Y be a (real) vector space; and K , some (convex) cone of it. Denote by (\leq_K) the induced quasi-order (cf. (a1)); also written as (\leq) , for simplicity. Further, let H be another (convex) cone of Y with

(4a) $K \subseteq H$ and H is Archimedean (see above).

Finally, given $\alpha > 1$, let $\exp = \exp_\alpha$ be the associated exponential function in Section 3. We may now introduce the maps $\{\Theta, \theta\}$ attached to (H, α) , under the model of (a3); note that, by (4a) above, the projective anti-metric θ has the extra properties of Lemma 1. In addition, let (X, d) be a metric space. The relation (\succeq) over $X \times K$ introduced as

(a4) $(x_1, y_1) \succeq (x_2, y_2)$ iff $y_2 \exp(d(x_1, x_2)) \leq y_1$

is reflexive and transitive (by the properties of d); hence a quasi-order on it. Finally, take some (nonempty) part A of $X \times K$. As in Section 1, we are interested to determine sufficient conditions under which (A, \succeq) has points with certain maximality properties. The basic one may be written as

- (4b) $P_Y(A)$ is strongly bounded below (modulo H):
there exists $v \in H \setminus (-H)$ with $P_Y(A) \subseteq v + H$.

[Here, P_X, P_Y are the projection operators from $X \times Y$ to X and Y respectively]. And the specific one is

- (4c) each (\succeq) -ascending ϵ -Cauchy sequence in A
is bounded above in A (modulo (\succeq)).

Here, ϵ stands for the semi-metric on $X \times K$ introduced as

$$(b4) \quad \epsilon((x_1, y_1), (x_2, y_2)) = d(x_1, x_2), (x_1, y_1), (x_2, y_2) \in X \times K.$$

The (first) main result of our exposition is

Theorem 1 *Let the precise assumptions be in force. Then, for each $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$ with the properties*

$$(x_0, y_0) \succeq (\bar{x}, \bar{y}) \text{ [hence } y_0 \geq \bar{y}] \quad (4.1)$$

$$\begin{aligned} \text{if } (x', y') \in A \text{ fulfills } (\bar{x}, \bar{y}) \succeq (x', y') \text{ then} \\ \bar{x} = x' \text{ and } \theta(\bar{y}, v) = \theta(y', v); \text{ hence } \theta(\bar{y}, y') = 0. \end{aligned} \quad (4.2)$$

Proof Define the function φ from $Z = X \times K$ to $R \cup \{-\infty, \infty\}$ according to: $\varphi(x, y) = \theta(y, v)$, for $(x, y) \in X \times K$; moreover, denote its restriction to A in the same way. We claim that Proposition 4 is applicable to $(Z, \succeq; \epsilon)$, $M = A$ and φ . First, (2c) holds in the sense

$$0 \leq \varphi(x, y) < \infty, \forall (x, y) \in A \text{ (hence } \text{Dom}(\varphi) = A);$$

this results at once from Lemma 1 and (4b). Secondly, take some couple $(x_1, y_1), (x_2, y_2)$ in A with $(x_1, y_1) \succeq (x_2, y_2)$. By the very definition of the anti-metric θ , this yields (cf. Lemma 2)

$$d(x_1, x_2) \leq \theta(y_1, y_2) \leq \theta(y_1, v) - \theta(y_2, v);$$

wherefrom, (2b) and (2g) are true. Finally, (2h) is nothing else than (4c) (with respect to our data); hence the claim. By Proposition 4 it then follows that, for each $a_0 = (x_0, y_0)$ in A there exists $\bar{a} = (\bar{x}, \bar{y})$ in A with $j) a_0 \succeq \bar{a}$ and $jj) a' = (x', y') \in A, \bar{a} \succeq a' \implies \epsilon(\bar{a}, a') = 0, \varphi(\bar{a}) = \varphi(a')$. But, from this, (4.1) and the first half of (4.2) are clear. Combining with $0 \leq \theta(\bar{y}, y') \leq \theta(\bar{y}, v) - \theta(y', v) = 0$ gives the second half of (4.2); and completes the argument. ■

An interesting question refers to the concrete possibilities of deriving (4c). For example, this condition holds whenever d is complete and

- (4d) if $((x_n, y_n)) \subseteq A$ is (\succeq) -ascending and $x_n \rightarrow x$ then $x \in P_X(A)$
and there exists $y \in A(x)$ such that $y_n \geq y$, for all n
- (4e) K is directionally closed [in the sense of (b3)].

[Here, for each $(x, y) \in A$, $A(x)$ (respectively, $A(y)$) stands for the x -section (respectively, y -section) of (the relation) A]. In fact, let $((x_n, y_n))$ be some (\succeq) -ascending e -Cauchy sequence in A . By completeness, $x_n \rightarrow x$ as $n \rightarrow \infty$ for some $x \in X$; and this, along with (4d), tells us that $x \in P_X(A)$ and there exists $y \in A(x) (\subseteq K)$ with $y_n \succeq y$, for all n . In addition, $y \succeq_H v$ (by (4b)); hence $y \in K \setminus (-H) \subseteq K \setminus (-K)$. On the other hand, the imposed hypothesis reads $y_m \exp(d(x_n, x_m)) \leq y_n$, if $m \geq n$; wherefrom $y \exp(d(x_n, x_m)) \leq y_n$, for the same couples (m, n) . Passing to limit as $m \rightarrow \infty$ yields (via (4e))

$$y \exp(d(x_n, x)) \leq y_n \text{ (i.e.: } (x_n, y_n) \succeq (x, y)), \text{ for all } n;$$

and the conclusion follows.

In particular, Theorem 1 includes the result in Phelps [18, Ch 3, Sect 3.12] when $Y = \mathbb{R}$, $K = H = \mathbb{R}_+$. So, we may ask whether it includes as well the related statement in Turinici [25]. Further aspects may be found in Rozoveanu [19] and Tammer [20].

5 A completion

Let Y be a (real) vector space; and K , some (convex) cone of it. Denote by (\leq_K) the induced quasi-order (cf. (a1)); also written as (\leq) , when K is understood. For technical reasons, we shall complete Y with an element $\infty \notin Y$ subject to the conditions (1c). Further, let H be another cone of Y with the property (4a); and, for the arbitrary fixed $\alpha > 1$, let $\exp = \exp_\alpha$ be the associated exponential function in Section 3. Define the couple of maps $\{\Theta, \theta\}$ attached to (H, α) (under the model of Section 3); note that, by the admitted condition (4a), the anti-metric θ has the properties of Lemma 1. Finally, take a complete metric space (X, d) ; as well as a function $F : X \rightarrow K \cup \{\infty\}$. It is our aim in the following to formulate a vector type Ekeland variational principle involving these data. To this end, let (\succeq) stand for the relation over X given as

$$(a5) \quad (x_1, x_2 \in X): x_1 \succeq x_2 \text{ iff } F(x_2) \exp(d(x_1, x_2)) \leq F(x_1).$$

This is reflexive and transitive; hence a quasi-order on X . Now, the posed question amounts to finding the maximal elements of (X, \succeq) . The natural approach for this is related to the monotone variational principles in Section 2. Precisely, the key condition to be used here may be written as

$$(5a) \quad F \text{ is proper and strongly bounded below (modulo } H): \\ \text{there exists } v \in H \setminus (-H) \text{ with } F(X) \subseteq (v + H) \cup \{\infty\}.$$

And the specific assumption reads

$$(5b) \quad (\succeq) \text{ is relatively self-closed over } \text{Dom}(F) \text{ (cf. Section 2)}.$$

We are now in position to state the second main result in this exposition.

Theorem 2 *Let the precise conditions hold. Then, for each $x_0 \in \text{Dom}(F)$ there exists $\bar{x} \in \text{Dom}(F)$ with the properties*

$$F(\bar{x}) \exp(d(x_0, \bar{x})) \leq F(x_0) \text{ (hence } F(\bar{x}) \leq F(x_0)) \quad (5.1)$$

$$x' \in X, F(x') \exp(d(\bar{x}, x')) \leq F(\bar{x}) \text{ imply } \bar{x} = x'. \quad (5.2)$$

Proof Define the function η from H to $R \cup \{-\infty, \infty\}$ as: $\eta(y) = \theta(y, v)$, $y \in H$. By the properties of the anti-metric θ (cf. Section 3) it results that

$$y_1, y_2 \in H, y_1 \leq_H y_2 \implies \eta(y_1) \leq \eta(y_2) \quad [\eta \text{ is increasing}].$$

Putting $\eta(\infty) = \infty$, we completed this up to some function $\eta : H \cup \{\infty\} \rightarrow R \cup \{-\infty, \infty\}$ which is still increasing over $H \cup \{\infty\}$. Further, let φ stand for the composed function (from X to $R \cup \{-\infty, \infty\}$): $\varphi(x) = \eta(F(x))$, $x \in X$ (i.e.: $\varphi = \eta \circ F$). We show that conditions of Proposition 6 are fulfilled for $(X, \succeq; d)$ and φ . In fact, (2b) is clear by the remarks above; because

$$x_1 \succeq x_2 \implies F(x_1) \geq F(x_2) \implies \eta(F(x_1)) \geq \eta(F(x_2)).$$

Further, (2c) is evident via $\text{Dom}(\varphi) = \text{Dom}(F)$; and (2g) is deductible in the same way as in Theorem 1. Finally, (2k) is just the regularity condition (5b); hence the claim. By Proposition 6 it then follows that, for the arbitrary fixed $x_0 \in \text{Dom}(F)$ there exists $\bar{x} \in \text{Dom}(F)$ with the properties p) and pp) stated there. This, along with the definition of (\succeq) , yields (5.1)+(5.2); and the proof is complete. ●

In particular, sufficient conditions for (5b) are (4e) and

- (5c) F is descending lsc: $F(x_n) \geq F(x)$, $\forall n$, whenever $(x_n) \subseteq \text{Dom}(F)$ satisfies $x_n \rightarrow x$ and $(F(x_n))$ is (\leq) -descending.

In fact, let (x_n) be some (\succeq) -ascending sequence in $\text{Dom}(F)$ with $x_n \rightarrow x$ as $n \rightarrow \infty$, for some $x \in X$. Clearly, $(F(x_n))$ is (\leq) -descending; and this, along with (5c), tells us that $F(x_n) \geq F(x)$, for all n (hence, in particular, $x \in \text{Dom}(F)$). On the other hand, the imposed hypothesis reads $F(x_m) \exp(d(x_n, x_m)) \leq F(x_n)$, if $m \geq n$; wherefrom $F(x) \exp(d(x_n, x_m)) \leq F(x_n)$, for the same couples (m, n) . Passing to limit as $m \rightarrow \infty$ yields (via (4e)) $F(x) \exp(d(x_n, x)) \leq F(x_n)$ (i.e.: $x_n \succeq x$), for all n ; and the claim is proved. Note that the corresponding version of Theorem 2 includes the standard Ekeland's variational principle [8] when $Y = R$, $K = H = R_+$. Further aspects may be found in Isac and Tammer [14].

Acknowledgement. This research was supported by Grant PN II PCE ID.387, from the National Authority for Scientific Research, Romania.

References

- [1] M. Altman, *A generalization of the Brezis-Browder principle on ordered sets*, Nonlinear Analysis, 6 (1981), 157-165.
- [2] M. C. Anisiu, *On maximality principles related to Ekeland's theorem*, Seminar Funct. Analysis Numer. Meth. (Faculty of Math. Research Seminars), Preprint No. 1 (8 pp), "Babeş-Bolyai" Univ., Cluj-Napoca (România), 1987.
- [3] G. Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc., 85 (1957), 219-227.

- [4] H. Brezis and F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, *Advances Math.*, 21 (1976), 355-364.
- [5] P. J. Bushell, *Hilbert's metric and positive contraction mappings in a Banach space*, *Arch. Rational Mech. Anal.*, 52 (1973), 330-338.
- [6] O. Cârjă, M. Necula and I. I. Vrabie, *Viability, Invariance and Applications*, North Holland Mathematics Studies vol. 207, Elsevier B. V., Amsterdam, 2007.
- [7] R. Cristescu, *Topological Vector Spaces*, Noordhoff Intl. Publishers, Leyden (The Netherlands), 1977.
- [8] I. Ekeland, *Nonconvex minimization problems*, *Bull. Amer. Math. Soc. (New Series)*, 1 (1979), 443-474.
- [9] C. Gerth (Tammer) and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, *J. Optim. Theory Appl.*, 67 (1990), 297-320.
- [10] A. Goepfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, [Canad. Math. Soc. Books in Math. vol. 17], Springer, New York, 2003.
- [11] A. Goepfert, C. Tammer and C. Zălinescu, *On the vectorial Ekeland's variational principle and minimal points in product spaces*, *Nonlinear Analysis*, 39 (2000), 909-922.
- [12] A. Hamel, *Variational Principles on Metric and Uniform Spaces*, (Habilitation Thesis), Martin-Luther University, Halle-Wittenberg (Germany), 2005.
- [13] G. Isac, *The Ekeland's principle and Pareto ε -efficiency*, in "Multi-Objective Programming and Goal Programming" (M. Tamiz ed.), pp. 148-163, *L. Notes Econ. Math. Systems* vol. 432, Springer, Berlin, 1996.
- [14] G. Isac and C. Tammer, *Nuclear and full nuclear cones in product spaces: Pareto efficiency and an Ekeland type variational principle*, *Positivity*, 14 (2004), 1-28.
- [15] B. G. Kang and S. Park, *On generalized ordering principles in nonlinear analysis*, *Nonlinear Analysis*, 14 (1990), 159-165.
- [16] G. H. Moore, *Zermelo's Axiom of Choice: its Origin, Development and Influence*, Springer, New York, 1982.
- [17] A. B. Nemeth, *A nonconvex vector minimization problem*, *Nonlinear Analysis*, 10 (1986), 669-678.
- [18] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Springer, Berlin, 1989.
- [19] P. Rozoveanu, *Ekeland's variational principle for vector valued functions*, *Math. Reports [St. Cerc. Mat.]*, 2(52) (2000), 351-366.
- [20] C. Tammer, *A generalization of Ekeland's variational principle*, *Optimization*, 25 (1992), 129-141.

- [21] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc., 14 (1963), 438-443.
- [22] M. Turinici, *Volterra functional equations via projective techniques*, J. Math. Anal. Appl., 103 (1984), 211-229.
- [23] M. Turinici, *Metric variants of the Brezis-Browder ordering principle*, Demonstr. Math., 22 (1989), 213-228.
- [24] M. Turinici, *Variational principles on semi-metric structures*, Libertas Math., 20 (2000), 161-171.
- [25] M. Turinici, *Minimal points in product spaces*, An. Șt. Univ. "Ovidius" Constanța (Ser. Math.), 10 (2002), 109-122.
- [26] M. Turinici, *Brezis-Browder principles in separable ordered sets*, Libertas Math., 26 (2006), 15-30.
- [27] E. S. Wolk, *On the principle of dependent choices and some forms of Zorn's lemma*, Canad. Math. Bull., 26 (1983), 365-367.