

# Continuous Selections for Multifunctions Satisfying the Carathéodory Type Conditions. The Darboux Problem for Third Order Hyperbolic Inclusions

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**Abstract.** In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form  $u_{xyz} \in F(x, y, z, u)$ , where  $F$  is a multifunction satisfying the Carathéodory type conditions defined on a compact subset of  $\mathbb{R}^{n+3}$  and whose values are non-empty compact and not necessarily convex subsets in  $\mathbb{R}^n$ . We prove a theorem which establishes the existence of a continuous selection for each of the functions  $(x, y, z) \rightarrow F(x, y, z, u(x, y, z))$  with respect to a given family of continuous functions  $(x, y, z) \rightarrow u(x, y, z)$ . Using this result and the Schauder's Fixed Point Theorem, it is obtained an existence theorem of an absolutely continuous solution for the considered Darboux Problem.

**Keywords:** multifunction, upper and lower semi-continuity of multifunctions, measurable multifunction, hyperbolic inclusion, initial values, absolutely continuous in Carathéodory's sense function.

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## 1. Introduction

In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form

$$\frac{\partial^3 u(x, y, z)}{\partial x \partial y \partial z} \in F(x, y, z, u), \quad (x, y, z) \in D = [0, a] \times [0, b] \times [0, c], \quad u \in B \subset \mathbb{R}^n, \quad (1.1)$$

with the initial values

$$\begin{cases} u(x, y, 0) = \varphi(x, y), & (x, y) \in D_1 = [0, a] \times [0, b], \\ u(0, y, z) = \psi(y, z), & (y, z) \in D_2 = [0, b] \times [0, c], \\ u(x, 0, z) = \chi(x, z), & (x, z) \in D_3 = [0, a] \times [0, c], \end{cases} \quad (1.2)$$

where  $\varphi, \psi, \chi$  are absolutely continuous in Carathéodory's sense functions [4, §565-§570],  $\varphi \in C^*(D_1; \mathbb{R}^n)$ ,  $\psi \in C^*(D_2; \mathbb{R}^n)$ ,  $\chi \in C^*(D_3; \mathbb{R}^n)$  and they satisfy the conditions

$$\begin{cases} u(x, 0, 0) = \varphi(x, 0) = \chi(x, 0) = v^1(x), & x \in [0, a], \\ u(0, y, 0) = \varphi(0, y) = \psi(y, 0) = v^2(y), & y \in [0, b], \\ u(0, 0, z) = \psi(0, z) = \chi(0, z) = v^3(z), & z \in [0, c], \\ u(0, 0, 0) = v^1(0) = v^2(0) = v^3(0) = v^0. \end{cases} \quad (1.3)$$

$F : D \times B \rightarrow \text{comp } A$  is a multifunction satisfying the Carathéodory type conditions whose values are non-empty compact and not necessarily convex subsets of  $\mathbb{R}^n$ ,  $A$  is the closed ball centered at the origin of  $\mathbb{R}^n$  with radius  $M$  and  $B$  is the closed ball centered at the origin of  $\mathbb{R}^n$  with radius  $r = M_1 + Mabc$ .

We prove a theorem which establishes the existence of a continuous selection for each of the functions  $(x, y, z) \rightarrow F(x, y, z, u(x, y, z))$  with respect to a given family of continuous functions  $(x, y, z) \rightarrow u(x, y, z)$ . Using this result and the Schauder's Fixed Point Theorem it is obtained an existence theorem of an absolutely continuous solution for Darboux Problem (1.1) + (1.2).

In [36], the same Darboux Problem was considered under similar conditions to the ones in this paper, with the difference that  $F$  satisfied a stronger (a more restrictive) condition, i.e. it was a continuous multifunction. In that reference, results of the same type as the ones in this paper were obtained.

In [32] we considered the Darboux Problem (1.1) + (1.2) where  $F : D \times \Omega \rightarrow 2^{\mathbb{R}^n}$  was a multifunction with compact convex and non-empty values and  $\Omega \subset \mathbb{R}^n$  was an open subset.

Under suitable assumptions, we proved an existence theorem for a local solution of the Darboux Problem (1.1) + (1.2) using the Kakutani Ky Fan Fixed Point Theorem, and that the set of its solutions is compact in Banach space  $C(D_0; \mathbb{R}^n)$ ,  $D_0 = [0, x_0] \times [0, y_0] \times [0, z_0] \subseteq D$ ; moreover, as a function of the initial values this set defines an upper semi-continuous multifunction.

In [33] we proved a theorem of prolongation for the solutions of the considered Darboux problem (1.1) + (1.2) and also an existence theorem for a saturated solution.

In [34] we proved a characterization theorem for the solutions of Darboux Problem (1.1) + (1.2) using the Aumann integral [2] defined for multifunctions.

In [35], using the notion of uniform convergence on compact sets as defined by Arrigo Cellina [7], [8] for a sequence of single-valued functions  $f_k : \Lambda \rightarrow \mathbb{R}^n$  such that  $f_k \rightarrow F$ , where  $F$  is a multifunction, we considered a sequence of approximating univalued equations of the form  $u_{xyx} = f_k(x, y, z, u)$  and we proved that they have a unique solution, using Schauder's Fixed Point Theorem. Using a characterization theorem for the solutions of the Darboux Problem (1.1) + (1.2) for the specified inclusion [34], we proved that the sequence of solutions to the approximating univalued equations uniformly converges, on compact sets, to a solution of the Darboux Problem (1.1) + (1.2) for the considered inclusion.

This paper has been suggested by [1], [3] and [29] and it provides an extension of the results in those articles.

## 2. Preliminaries

The definitions and Theorems 2.1 – 2.6 in this section are recalled from [4], [7]-[28].

**Definition 2.1.** Let  $X$  and  $Y$  be two non-empty sets. A *multifunction*  $\Phi : X \rightarrow 2^Y$  is a function from  $X$  into the family of all non-empty subsets of  $Y$ .

To each  $x \in X$ , a subset  $\Phi(x)$  of  $Y$  is associated by the multifunction  $\Phi$ . The set

$$\bigcup_{x \in X} \Phi(x) \text{ is the range of } \Phi. \quad \Phi(X) = \left\{ \bigcup_{x \in X} \Phi(x) \mid x \in X \right\}.$$

**Definition 2.2.** Let us consider  $\Phi : X \rightarrow 2^Y$ .

a) If  $A \subset X$ , the *image* of  $A$  by  $\Phi$  is  $\Phi(A) = \bigcup_{x \in A} \Phi(x)$ ;

b) If  $B \subset Y$ , the *counterimage* of  $B$  by  $\Phi$  is

$$\Phi^-(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The *graph* of  $\Phi$ , denoted  $\text{graph } \Phi$ , is the set

$$\text{graph } \Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}.$$

**Definition 2.3.** Let us now take  $\Phi : X \rightarrow 2^Y$ . An element  $x \in X$  with the property that  $x \in \Phi(x)$  is called a *fixed point* of the multifunction  $\Phi$ .

**Definition 2.4.** A univalued function  $\varphi : X \rightarrow Y$  is said to be a *selection* of  $\Phi : X \rightarrow 2^Y$  if  $\varphi(x) \in \Phi(x)$  for all  $x \in X$ .

**Definition 2.5.** Let  $X$  and  $Y$  be two topological spaces. The multifunction  $\Phi : X \rightarrow 2^Y$  is *upper semi-continuous* if, for any closed  $B \subseteq Y$ ,  $\Phi^-(B)$  is closed in  $X$ .

**Definition 2.6.** If  $X$  and  $Y$  are two topological spaces, the multifunction  $\Phi : X \rightarrow 2^Y$  is *lower semi-continuous* if, for every open subset  $\Omega \subseteq Y$ , the set  $\Phi^-(\Omega)$  is open in  $X$ .

**Definition 2.7.** The multifunction  $\Phi : X \rightarrow 2^Y$  is *continuous* if it is both upper and lower semi-continuous.

**Definition 2.8.** If  $(X, \mathcal{F})$  is a measurable space and  $Y$  is a topological space, the multifunction  $\Phi : X \rightarrow 2^Y$  is *measurable (weakly measurable)*, if  $\Phi^-(B) \in \mathcal{F}$  for every closed (open) subset  $B \subseteq Y$ ,  $\mathcal{F}$  being the  $\sigma$ -algebra of the measurable sets of  $X$ , i.e.  $\Phi^-(B)$  is measurable.

**Theorem 2.1** [24]. *Let  $X$  and  $Y$  be two metric spaces,  $Y$  being compact, and  $\Phi : X \rightarrow 2^Y$  a multifunction with the property that  $\Phi(x)$  is a closed subset of  $Y$  for any  $x \in X$ . The following assertions are equivalent:*

- i) *the multifunction  $\Phi$  is upper semi-continuous;*
- ii) *the graph of  $\Phi$  is a closed subset of  $X \times Y$ ;*
- iii) *any would be the sequences  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$ , from  $x_n \rightarrow x$ ,  $y_n \in \Phi(x_n)$  and  $y_n \rightarrow y$  it follows that  $y \in \Phi(x)$ .*

**Theorem 2.2** [16]. *When  $X$  is first countable and  $Y$  is a metric space,  $\Phi : X \rightarrow 2^Y$  is lower semi-continuous if and only if, for every  $\bar{x} \in X$ , every sequence  $\{x_n\}$  in  $X$  converging*

to  $\bar{x}$  and every  $\bar{y} \in \Phi(\bar{x})$ , there exists a sequence  $\{y_n\}$  in  $Y$  converging to  $\bar{y}$ , such that  $y_n \in \Phi(x_n)$  for all  $n \in \mathbb{N}$ .

**Definition 2.9.** [4], [9], [13] The function  $u : \Delta \rightarrow \mathbb{R}^n$ ,  $\Delta \subset \mathbb{R}^2$ , is *absolutely continuous in Carathéodory's sense* [4, §565 - §570] if and only if it is continuous on  $\Delta$ , absolutely continuous in  $x$  (for any  $y$ ), absolutely continuous in  $y$  (for any  $x$ ),  $u_x(x, y)$  is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in  $y$  (for any  $x$ ) and  $u_{xy}$  is Lebesgue-integrable on  $\Delta$ .

**Theorem 2.3.** [4], [9], [28] The function  $u : \Delta \rightarrow \mathbb{R}^n$ ,  $\Delta = [0, a] \times [0, b] \subset \mathbb{R}^2$ , is *absolutely continuous in Carathéodory's sense on  $\Delta$*  if and only if there exist  $f \in L^1(\Delta; \mathbb{R}^n)$ ,  $g \in L^1([0, a]; \mathbb{R}^n)$ ,  $h \in L^1([0, b]; \mathbb{R}^n)$  such that

$$u(x, y) = \int_0^x \int_0^y f(s, t) ds dt + \int_0^x g(s) ds + \int_0^y h(t) dt + u(0, 0).$$

We denote the class of absolutely continuous functions in Carathéodory's sense on  $\Delta$  by  $C^*(\Delta; \mathbb{R}^n)$  [13]. In [9], this space is denoted by  $AC(\Delta; \mathbb{R}^n)$ .

**Theorem 2.4.** [9] The space  $C^*(\Delta; \mathbb{R}^n)$  endowed with the norm

$$\|u(\cdot, \cdot)\| = \int_0^a \int_0^b \|u_{xy}(s, t)\| ds dt + \int_0^a \|u_x(s, 0)\| ds + \int_0^b \|u_y(0, t)\| dt + \|u(0, 0)\|,$$

where  $\Delta = [0, a] \times [0, b] \subset \mathbb{R}^2$ , and  $\|\cdot\|$  is the Euclidean norm, is a Banach space.

**Definition 2.10.** [14] The function  $u : D \rightarrow \mathbb{R}^n$ ,  $D \subset \mathbb{R}^3$ , is *absolutely continuous in Carathéodory's sense* [4, §565 - §570] if and only if  $u(x, y, z)$  is continuous on  $D$ , absolutely continuous in each variable (for any pair of the other two variables) and similarly for  $u_x(x, y, z)$ ,  $u_y(x, y, z)$ ,  $u_z(x, y, z)$ ,  $u_{xy}(x, y, z)$ ,  $u_{yz}(x, y, z)$ ,  $u_{zx}(x, y, z)$ , and  $u_{xyz}$  is Lebesgue-integrable on  $D$ .

**Theorem 2.5.** [9] The function  $u : D \rightarrow \mathbb{R}^n$ ,  $D = [0, a] \times [0, b] \times [0, c] \subset \mathbb{R}^3$ , is *absolutely continuous in Carathéodory's sense on  $D$*  if and only if there exist  $f \in L^1(D; \mathbb{R}^n)$ ,  $g_1 \in L^1(D_1; \mathbb{R}^n)$ ,  $g_2 \in L^1(D_2; \mathbb{R}^n)$ ,  $g_3 \in L^1(D_3; \mathbb{R}^n)$ ,  $h_1 \in L^1([0, a]; \mathbb{R}^n)$ ,  $h_2 \in L^1([0, b]; \mathbb{R}^n)$ ,  $h_3 \in L^1([0, c]; \mathbb{R}^n)$ , such that

$$\begin{aligned} u(x, y, z) = & \int_0^x \int_0^y \int_0^z f(r, s, t) dr ds dt + \int_0^x \int_0^y g_1(r, s) dr ds + \\ & + \int_0^y \int_0^z g_2(s, t) ds dt + \int_0^x \int_0^z g_3(r, t) dr dt + \\ & + \int_0^x h_1(r) dr + \int_0^y h_2(s) ds + \int_0^z h_3(t) dt + u(0, 0, 0). \end{aligned}$$

We denote the class of absolutely continuous functions in Carathéodory's sense on  $D$  by  $C^*(D; \mathbb{R}^n)$  [14].

**Theorem 2.6.** [9] *The space  $C^*(D; \mathbb{R}^n)$  endowed with the norm*

$$\begin{aligned} \|u(\cdot, \cdot, \cdot)\| = & \int_0^a \int_0^b \int_0^c \|u_{xyz}(r, s, t)\| dr ds dt + \int_0^a \int_0^b \|u_{xy}(r, s, 0)\| dr ds + \\ & + \int_0^b \int_0^c \|u_{yz}(0, s, t)\| ds dt + \int_0^a \int_0^c \|u_{xz}(r, 0, t)\| dr dt + \\ & + \int_0^a \|u_x(r, 0, 0)\| dr + \int_0^b \|u_y(0, s, 0)\| ds + \\ & + \int_0^c \|u_z(0, 0, t)\| dt + \|u(0, 0, 0)\|, \end{aligned}$$

where  $D = [0, a] \times [0, b] \times [0, c] \subset \mathbb{R}^3$ , and  $\|\cdot\|$  is the Euclidean norm, is a Banach space.

We denote by  $d(x, y)$  the Euclidean distance from  $x$  to  $y$ ,  $x, y \in \mathbb{R}^n$ , where  $\mathbb{R}^n$  is the Euclidean space.  $B[x, r]$  is the closed ball of radius  $r > 0$  centered at  $x \in \mathbb{R}^n$ . If  $A, B \subset \mathbb{R}^n$ ,  $d(x, A) = \inf \{d(x, y) \mid y \in A\}$ ,  $d^*(A, B) = \sup \{d(y, B) \mid y \in A\}$ ,  $d(x, \emptyset) = \infty$ .

Let  $(X, d)$  be a metric space and  $\mathcal{P}(X)$  the set of subsets of  $X$ . For  $A, B \subset X$  we have

**Definition 2.11.** [9], [24] The function  $d_H : \mathcal{P}(X) \rightarrow [0, +\infty]$

$$d_H(A, B) = \max \{d^*(A, B), d^*(B, A)\} = \max \left\{ \sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A) \right\}$$

is the Hausdorff-Pompeiu pseudometric.

The function  $d_H$  defines a metric on the space  $\mathcal{F}(X)$  of the non-empty and closed subsets of  $X$ , called the Hausdorff-Pompeiu metric.

### 3. Continuous selections. The measurable case

We recalled the main notations and results given in [36].

Let be the multifunction  $F : D \times B \rightarrow \text{comp } A$  whose values are non-empty compact and not necessarily convex subsets of  $\mathbb{R}^n$ ,  $D = [0, a] \times [0, b] \times [0, c] \subset \mathbb{R}^3$ ,  $A$  is the closed ball centered at the origin of  $\mathbb{R}^n$  with radius  $M$ , where  $M$  is given by (3.3), and  $B$  is the closed ball centered at the origin of  $\mathbb{R}^n$  with radius  $r = M_1 + Mabc$ , where  $M_1$  is given by (3.2).

**Proposition 3.1.**  *$A$  is a compact space for the metric (Euclidean distance)  $d$  induced on  $A$  by the Euclidean norm on  $\mathbb{R}^n$ .*

**Proposition 3.2.** *The set  $\text{comp } A$  is a compact metric space under  $d_H$ , where  $d_H$  is the Hausdorff-Pompeiu metric on  $\text{comp } A$  induced by  $d$ .*

Let  $C(D; \mathbb{R}^n)$  be the Banach space of continuous functions from  $D$  into  $\mathbb{R}^n$  and  $\mathcal{L}^1(D; \mathbb{R}^n)$  the Banach space of equivalence classes of Lebesgue-integrable functions on  $D$  taking values in  $\mathbb{R}^n$ .

Let the following hypotheses be satisfied:

(H<sub>1</sub>)  $F : D \times B \rightarrow \text{comp } A$  satisfies:

- a) For each  $u \in B$ ,  $(x, y, z) \rightarrow F(x, y, z, u)$  is measurable in  $D$ ;  
 b) For each  $(x, y, z) \in D$ ,  $u \rightarrow F(x, y, z, u)$  is continuous in  $B$ ;  
 (H<sub>2</sub>) The functions  $\varphi \in C^*(D_1; \mathbb{R}^n)$ ,  $\psi \in C^*(D_2; \mathbb{R}^n)$ ,  $\chi \in C^*(D_3; \mathbb{R}^n)$  given by (1.2) are absolutely continuous in Carathéodory's sense functions and satisfy conditions (1.3).

**Remark 3.1.** The function  $\alpha : D \rightarrow \mathbb{R}^n$  defined by

$$\begin{aligned} \alpha(x, y, z) &= \varphi(x, y) + \psi(y, z) + \chi(x, z) - \varphi(x, 0) - \varphi(0, y) - \psi(0, z) + \psi(0, 0) = \\ &= \varphi(x, y) + \psi(y, z) + \chi(x, z) - v^1(x) - v^2(y) - v^3(z) + v^0, \end{aligned} \quad (3.1)$$

is an absolutely continuous in Carathéodory's sense function on  $D$ ,  $\alpha \in C^*(D; \mathbb{R}^n)$  [4, §565-§570].

Suppose that the following hypothesis holds:

- (H<sub>3</sub>) The function  $\alpha : D \rightarrow \mathbb{R}^n$  defined by (3.1) is bounded, that is

$$\|\alpha(x, y, z)\| \leq M_1, \quad (x, y, z) \in D. \quad (3.2)$$

Define  $\mathcal{K}$  to be the set of absolutely continuous in Carathéodory's sense functions  $u : D \rightarrow \mathbb{R}^n$  satisfying

$$\left\| \frac{\partial^3 u(x, y, z)}{\partial x \partial y \partial z} \right\| \leq M, \quad \text{for a.e. } (x, y, z) \in D, \quad (3.3)$$

and the conditions (1.2).

**Proposition 3.3.** *The set  $\mathcal{K}$  is a non-empty compact and convex subset of  $C(D; \mathbb{R}^n)$ .*

**Proof.** The membership  $u \in \mathcal{K}$  implies  $u \in C(D; \mathbb{R}^n)$ . Integrating  $\frac{\partial^3 u(x, y, z)}{\partial x \partial y \partial z}$  on  $D$  and using the conditions (1.2) we obtain

$$\begin{aligned} u(x, y, z) &= u(x, y, 0) + u(x, 0, z) - u(x, 0, 0) + u(0, y, z) - \\ &\quad - u(0, y, 0) - u(0, 0, z) + u(0, 0, 0) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r, s, t)}{\partial r \partial s \partial t} dr ds dt = \\ &= \varphi(x, y) + \psi(y, z) + \chi(x, z) - \varphi(x, 0) - \varphi(0, y) - \\ &\quad - \psi(0, z) + u(0, 0, 0) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r, s, t)}{\partial r \partial s \partial t} dr ds dt = \\ &= \varphi(x, y) + \psi(y, z) + \chi(x, z) - v^1(x) - v^2(y) - v^3(z) + v^0 + \\ &\quad + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r, s, t)}{\partial r \partial s \partial t} dr ds dt = \\ &= \alpha(x, y, z) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r, s, t)}{\partial r \partial s \partial t} dr ds dt, \quad (x, y, z) \in D. \end{aligned} \quad (3.4)$$

The compactness of the set  $\mathcal{K}$  results using the theorem of Arzelà-Ascoli. The set  $\mathcal{K}$  is equibounded. From (3.2), (3.3) and (3.4) we have

$$\begin{aligned} \|u(x, y, z)\| &\leq \|\alpha(x, y, z)\| + \int_0^x \int_0^y \int_0^z \left\| \frac{\partial^3 u(r, s, t)}{\partial r \partial s \partial t} \right\| dr ds dt \leq \\ &\leq M_1 + \int_0^x \int_0^y \int_0^z M dr ds dt = \\ &= M_1 + Mxyz \leq M_1 + Mabc = r, \quad r > 0, \quad (x, y, z) \in D. \end{aligned}$$

The set  $\mathcal{K}$  is equicontinuous. Using the absolute continuity of the integral it follows that

$$\begin{aligned} \|u(x+h, y+k, z+l) - u(x, y, z)\| &\leq \varepsilon \quad \text{for } h, k, l \in \mathbb{R} \\ &\text{with } |h|, |k|, |l| < \delta(\varepsilon). \end{aligned}$$

The set  $\mathcal{K}$  is convex. Indeed let be  $u_1, u_2 \in \mathcal{K}$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$  with  $0 \leq \lambda_1 \leq 1$ ,  $0 \leq \lambda_2 \leq 1$ ,  $\lambda_1 + \lambda_2 = 1$ . From (3.3) and (1.2) we have

$$\left\| \frac{\partial^3 u_i(x, y, z)}{\partial x \partial y \partial z} \right\| \leq M, \quad (x, y, z) \in D, \quad i = \overline{1, 2},$$

and

$$\begin{cases} u_i(x, y, 0) = \varphi(x, y), & (x, y) \in D_1 = [0, a] \times [0, b], \\ u_i(0, y, z) = \psi(y, z), & (y, z) \in D_2 = [0, b] \times [0, c], \\ u_i(x, 0, z) = \chi(x, z), & (x, z) \in D_3 = [0, a] \times [0, c]. \end{cases} \quad i = \overline{1, 2}$$

Using the properties of absolutely continuous in Carathéodory's sense functions (Theorem 2.5), it results that  $\lambda_1 u_1 + \lambda_2 u_2 \in C^*(D; \mathbb{R}^n)$ .

The relations (3.3) and (1.2) for the function  $\lambda_1 u_1 + \lambda_2 u_2$  hold.

$$\begin{aligned} \left\| \frac{\partial^3 (\lambda_1 u_1 + \lambda_2 u_2)(x, y, z)}{\partial x \partial y \partial z} \right\| &= \left\| \lambda_1 \frac{\partial^3 u_1(x, y, z)}{\partial x \partial y \partial z} + \lambda_2 \frac{\partial^3 u_2(x, y, z)}{\partial x \partial y \partial z} \right\| \leq \\ &\leq \lambda_1 \left\| \frac{\partial^3 u_1(x, y, z)}{\partial x \partial y \partial z} \right\| + \lambda_2 \left\| \frac{\partial^3 u_2(x, y, z)}{\partial x \partial y \partial z} \right\| \leq \\ &\leq \lambda_1 M + \lambda_2 M = (\lambda_1 + \lambda_2) M = M, \quad (x, y, z) \in D, \end{aligned}$$

and

$$\begin{aligned} (\lambda_1 u_1 + \lambda_2 u_2)(x, y, 0) &= \lambda_1 u_1(x, y, 0) + \lambda_2 u_2(x, y, 0) = \lambda_1 \varphi(x, y) + \lambda_2 \varphi(x, y) = \\ &= (\lambda_1 + \lambda_2) \varphi(x, y) = \varphi(x, y), \quad (x, y) \in D_1, \\ (\lambda_1 u_1 + \lambda_2 u_2)(0, y, z) &= \lambda_1 u_1(0, y, z) + \lambda_2 u_2(0, y, z) = \lambda_1 \psi(y, z) + \lambda_2 \psi(y, z) = \\ &= (\lambda_1 + \lambda_2) \psi(y, z) = \psi(y, z), \quad (y, z) \in D_2, \\ (\lambda_1 u_1 + \lambda_2 u_2)(x, 0, z) &= \lambda_1 u_1(x, 0, z) + \lambda_2 u_2(x, 0, z) = \lambda_1 \chi(x, z) + \lambda_2 \chi(x, z) = \\ &= (\lambda_1 + \lambda_2) \chi(x, z) = \chi(x, z), \quad (x, z) \in D_3. \end{aligned}$$

Hence  $\lambda_1 u_1 + \lambda_2 u_2 \in \mathcal{K}$  and the set  $\mathcal{K}$  is convex.

**Remark 3.2.** The membership  $u \in \mathcal{K}$  implies  $(x, y, z, u(x, y, z)) \in D \times B$  for each  $(x, y, z) \in D$ . In view of the fact that each  $u \in \mathcal{K}$  generates a function  $(x, y, z) \rightarrow F(x, y, z, u(x, y, z))$  from  $D$  into  $\text{comp } A$ , we shall denote this function by  $G(u)$ ,

$$G(u)(x, y, z) = F(x, y, z, u(x, y, z)), \quad (x, y, z) \in D. \quad (3.5)$$

The main theorem in [36] on the existence of a continuous selection is the following:

**Theorem 3.1** (Th. 4.1 in [36]). *If the hypothesis  $(H_1)$  is satisfied, that is  $F : D \times B \rightarrow \text{comp } A$  is a continuous multifunction, then there exists a continuous function  $g : \mathcal{K} \rightarrow \mathcal{L}^1(D; \mathbb{R}^n)$  such that, for every  $u \in \mathcal{K}$ ,  $g(u)(x, y, z) \in G(u)(x, y, z)$  for a.e.  $(x, y, z) \in D$ , that is  $g(u)$  is a continuous selection of  $G(u)$  given by (3.5).*

The Theorem 3.1 can be extended to multifunctions defined on  $D \times B$  and valued in  $\text{comp } A$ , and satisfying the Carathéodory's type conditions, similarly to [1], [29].

**Theorem 3.2.** *Suppose that  $F : D \times B \rightarrow \text{comp } A$  satisfies the hypothesis  $(H_1)$*

a) *For each  $u \in B$ ,  $(x, y, z) \rightarrow F(x, y, z, u)$  is measurable in  $D$ ;*

b) *For each  $(x, y, z) \in D$ ,  $u \rightarrow F(x, y, z, u)$  is continuous in  $B$ ;*

*Then there exists a continuous function  $g : \mathcal{K} \rightarrow \mathcal{L}^1(D; \mathbb{R}^n)$  such that, for each  $u \in \mathcal{K}$ ,  $g(u)(x, y, z) \in G(u)(x, y, z)$  for a.e.  $(x, y, z) \in D$ , that is  $g(u)$  is a continuous selection of  $G(u)$  given by (3.5).*

**Proof.** One uses the proof of Theorem 3.1 (Th. 4.1 in [36]), which is based upon the following two properties of the multifunction  $F : D \times B \rightarrow \text{comp } A$  that are direct consequences of the assumed continuity:

1° The family  $\{F(x, y, z, \cdot)\}$ ,  $(x, y, z) \in D$ , of maps of  $B$  into  $\text{comp } A$  is uniformly equicontinuous;

2° For each  $u \in C(D; \mathbb{R}^n)$  the map  $G(u) : (x, y, z) \rightarrow F(x, y, z, u(x, y, z))$  is measurable in  $D$ .

We shall show that the hypothesis of Theorem 3.2 induces essentially the same properties for  $F$ 's. It is a consequence of the following two lemmas.

The Lemma 3.1 is a similar result to the theorems of Scorza-Drăgăni type for multifunctions, [1], [15], [27].

**Lemma 3.1.** *Suppose  $F : D \times B \rightarrow \text{comp } A$  satisfies the hypothesis  $(H_1)$  of Theorem 3.2. Then, for each  $\varepsilon > 0$ , there exists a closed set  $E \subset D$  with  $\mu(D - E) < \varepsilon$ , such that the family  $\{F(x, y, z, \cdot)\}$ ,  $(x, y, z) \in E$ , of maps of  $B$  into  $\text{comp } A$  is uniformly equicontinuous.*

**Proof.** Let the functions  $\delta_n : D \rightarrow \mathbb{R}$ ,  $n \geq 1$ , be defined by

$$\delta_n(x, y, z) = \sup \left\{ \rho > 0 \mid \|u - v\| < \rho \Rightarrow d_H(F(x, y, z, u), F(x, y, z, v)) < \frac{1}{n}, u, v \in B \right\}. \quad (3.6)$$

To verify the conclusion of this lemma, one shows that for each  $\varepsilon > 0$  there exists a closed set  $E \subset D$ , with  $\mu(D - E) < \varepsilon$ , such that the restriction  $\delta_n|_E$  is continuous.

Let  $C = \{u_m\}_{m \geq 1}$  be a dense subset of  $B$ . The hypothesis  $(H_1)$  a) implies the existence of a closed set  $L \subset D$  with  $\mu(D - L) < \frac{\varepsilon}{2}$  such that the restriction to  $L$  of every

function  $(x, y, z) \rightarrow F(x, y, z, u_m)$  is continuous. Then, each restriction  $\delta_n|_L$  is upper semi-continuous, hence measurable.

Indeed, suppose, by reductio ad absurdum, that for some (fixed)  $n > 1$  there exists a point  $(x_0, y_0, z_0) \in D$ , a constant  $\nu > 0$ , and a sequence  $\{(x_k, y_k, z_k)\}_{k \geq 1}$  convergent to  $(x_0, y_0, z_0)$  in  $D$ , such that

$$\delta_n(x_k, y_k, z_k) \geq \delta_n(x_0, y_0, z_0) + \nu, \quad \text{for all } k \geq 1. \quad (3.7)$$

Let  $v_1, v_2$  be two points in  $D$  with

$$\delta_n(x_0, y_0, z_0) \leq \|v_1 - v_2\| \leq \delta_n(x_0, y_0, z_0) + \frac{3}{4}\nu, \quad (3.8)$$

such that, for some  $K > 0$ ,

$$d_H(F(x_0, y_0, z_0, v_1), F(x_0, y_0, z_0, v_2)) = \frac{1}{n} + K. \quad (3.9)$$

Then, there exist points  $u_{m_1}, u_{m_2}$  in  $C$  and a point  $(x_k, y_k, z_k) \in L$  such that

$$\|v_1 - u_{m_1}\| < \frac{\nu}{8}, \quad \|v_2 - u_{m_2}\| < \frac{\nu}{8}. \quad (3.10)$$

By the hypothesis  $(H_1)$  a) we also have

$$\begin{cases} d_H(F(x_0, y_0, z_0, v_1), F(x_0, y_0, z_0, u_{m_1})) < \frac{K}{8}, \\ d_H(F(x_0, y_0, z_0, v_2), F(x_0, y_0, z_0, u_{m_2})) < \frac{K}{8}, \end{cases} \quad (3.11)$$

and

$$\begin{cases} d_H(F(x_0, y_0, z_0, u_{m_1}), F(x_k, y_k, z_k, u_{m_1})) < \frac{K}{8}, \\ d_H(F(x_0, y_0, z_0, u_{m_2}), F(x_k, y_k, z_k, u_{m_2})) < \frac{K}{8}. \end{cases} \quad (3.12)$$

According to (3.7), (3.8), (3.10) we obtain

$$\begin{aligned} \|u_{m_1} - u_{m_2}\| &\leq \|u_{m_1} - v_1\| + \|v_1 - v_2\| + \|v_2 - u_{m_2}\| < \frac{\nu}{4} + \|v_1 - v_2\| \leq \\ &\leq \delta_n(x_0, y_0, z_0) + \nu \leq \delta_n(x_k, y_k, z_k). \end{aligned} \quad (3.13)$$

The relations (3.6) and (3.13) imply

$$d_H(F(x_k, y_k, z_k, u_{m_1}), F(x_k, y_k, z_k, u_{m_2})) < \frac{1}{n}. \quad (3.14)$$

Then, from (3.11), (3.12), (3.14) we obtain

$$\begin{aligned} &d_H(F(x_0, y_0, z_0, v_1), F(x_0, y_0, z_0, v_2)) \leq d_H(F(x_0, y_0, z_0, v_1), F(x_0, y_0, z_0, u_{m_1})) + \\ &+ d_H(F(x_0, y_0, z_0, u_{m_1}), F(x_k, y_k, z_k, u_{m_1})) + d_H(F(x_k, y_k, z_k, u_{m_1}), F(x_k, y_k, z_k, u_{m_2})) + \\ &+ d_H(F(x_k, y_k, z_k, u_{m_2}), F(x_0, y_0, z_0, u_{m_2})) + d_H(F(x_0, y_0, z_0, u_{m_2}), F(x_0, y_0, z_0, v_2)) < \\ &< d_H(F(x_k, y_k, z_k, u_{m_1}), F(x_k, y_k, z_k, u_{m_2})) + \frac{1}{2}K < \frac{1}{2}K + \frac{1}{n}, \end{aligned} \quad (3.15)$$

which is in contradiction to (3.9).

It follows that there exists a closed set  $E \subset L$  with  $\mu(L - E) \leq \frac{\varepsilon}{2}$  such that each restriction  $\delta_n|_E$  is continuous. Obviously, by construction,  $\mu(D - E) \leq \varepsilon$ .

**Lemma 3.2.** *Suppose that  $F : D \times B \rightarrow \text{comp } A$  satisfies the hypothesis  $(H_1)$  a), b) of Theorem 3.2. Then, for each  $u \in C(D; \mathbb{R}^n)$ ,  $G(u) : (x, y, z) \rightarrow F(x, y, z, u(x, y, z))$  is measurable in  $D$ .*

**Proof.** Let  $u \in C(D; \mathbb{R}^n)$  be given, and let  $\{u_n\}_{n \geq 1}$  be a sequence of piecewise constant maps in  $D$  that converges to  $u$  uniformly in  $D$ . We only need to show that, for every  $\varepsilon > 0$ , there exists a closed set  $E_0 \subset D$  with  $\mu(D - E_0) < \varepsilon$  such that  $G(u)|_{E_0}$  is continuous.

Taking into account Lemma 3.1, there exists a closed set  $E \subset D$  with  $\mu(D - E) \leq \frac{\varepsilon}{2}$  such that the sequence  $\{G(u_n)\}_{n \geq 1}$  converges to  $G(u)$  uniformly in  $E$ . Since each  $G(u_n)$  is measurable in  $D$  and hence in  $E$ , there exists a closed set  $E_0 \subset E$  with  $\mu(E - E_0) \leq \frac{\varepsilon}{2}$  such that each restriction  $G(u_n)|_{E_0}$  is continuous. This implies that  $G(u)|_{E_0}$  is continuous.

To prove the Theorem 3.2, it has to be selected an increasing sequence  $\{E_n\}_{n \geq 1}$  of closed sets  $E_n \subset D$  with  $\mu(D - E_n) < 2^{-n}$  such that each restriction  $F|_{E_n \times B}$  is continuous and, for each  $n \geq 1$ , it is defined

$$F^n(x, y, z, u) = \begin{cases} F(x, y, z, u), & (x, y, z, u) \in E_n \times B, \\ 0, & (x, y, z, u) \in D - E_n \times B. \end{cases}$$

Then, as a consequence of Theorem 3.1, that there exists, for every  $n \geq 1$ , a continuous map  $g^n : \mathcal{K} \rightarrow \mathcal{L}^1(D; \mathbb{R}^n)$  such that for each  $u \in \mathcal{K}$ ,

$$g^n(u)(x, y, z) \in F(x, y, z, u(x, y, z)) \quad \text{for a.e. } (x, y, z) \in E_n.$$

Let  $A_1 = E_1$  and  $A_{n+1} = E_{n+1} - E_n$  for every  $n \geq 1$  so that  $E_n = \bigcup_{k=1}^n A_k$ , and define  $g(u)$ , for each  $u \in \mathcal{K}$ , by setting

$$g(u)|_{A_n} = g^n(u) \quad \text{and} \quad g(u)|_{D - \bigcup_{n=1}^{\infty} A_n} = 0.$$

Obviously,  $g$  maps  $\mathcal{K}$  into  $\mathcal{L}^1(D; \mathbb{R}^n)$  and, for each  $u \in \mathcal{K}$ ,

$$g(u)(x, y, z) \in F(x, y, z, u(x, y, z)) \quad \text{for a.e. } (x, y, z) \in D.$$

Moreover,  $g : \mathcal{K} \rightarrow \mathcal{L}^1(D; \mathbb{R}^n)$  is continuous since, for any  $u \in \mathcal{K}$  and  $w \in \mathcal{K}$ ,

$$\begin{aligned} \iiint_D \|g(u)(x, y, z) - g(w)(x, y, z)\| dx dy dz &= \iiint_{D - E_n} \|g(u)(x, y, z) - g(w)(x, y, z)\| dx dy dz + \\ &+ \iiint_{E_n} \|g(u)(x, y, z) - g(w)(x, y, z)\| dx dy dz \leq \\ &\leq 2^{-n+1}M + \sum_{k=1}^n \iiint_{A_k} \|g^k(u)(x, y, z) - g^k(w)(x, y, z)\| dx dy dz \end{aligned}$$

whatever  $n \geq 1$ , and each  $g^k$  is continuous by construction.

#### 4. The Darboux Problem for third order hyperbolic inclusions

**Definition 4.1.** The *Darboux Problem* for the hyperbolic inclusion (1.1) means to determine a *solution* of this inclusion which satisfies the initial conditions (1.2).

**Definition 4.2.** A function  $U : D \rightarrow \mathbb{R}^n$  is called a *solution* of the Darboux Problem (1.1) + (1.2) if it is absolutely continuous in Carathéodory's sense on  $D$ ,  $U \in C^*(D; \mathbb{R}^n)$  [4, §565-§570], and it satisfies (1.1) for a.e.  $(x, y, z) \in D$ , and also the initial conditions (1.2) for all  $(x, y) \in D_1$ , all  $(y, z) \in D_2$ , all  $(x, z) \in D_3$ .

**Theorem 4.1.** Assume that the hypotheses  $(H_1)$  a), b) and  $(H_2)$ ,  $(H_3)$  are satisfied. Then, there exists an absolutely continuous function  $\hat{u} : D \rightarrow \mathbb{R}^n$  which is a solution of the Darboux Problem (1.1) + (1.2).

**Proof.** The proof is similar to that given for Theorem 5.1 [36]. Using Theorem 3.2, there exists a continuous selection  $g : \mathcal{K} \rightarrow \mathcal{L}^1(D; \mathbb{R}^n)$  for  $G(u)$  given by (3.5). Let  $h(u)$ , for each  $u \in \mathcal{K}$ , be the continuous function  $h(u) : D \rightarrow \mathbb{R}^n$  defined by

$$h(u)(x, y, z) = \alpha(x, y, z) + \int_0^x \int_0^y \int_0^z g(u)(r, s, t) dr ds dt, \quad (x, y, z) \in D. \quad (4.1)$$

Using (3.1) we have

$$\begin{aligned} h(u)(x, y, z) &= \varphi(x, y) + \psi(y, z) + \chi(x, z) - \varphi(x, 0) - \varphi(0, y) - \psi(0, z) + \psi(0, 0) + \\ &+ \int_0^x \int_0^y \int_0^z g(u)(r, s, t) dr ds dt, \quad (x, y, z) \in D, \end{aligned} \quad (4.2)$$

and which can be written as

$$\begin{aligned} h(u)(x, y, z) &= \int_0^x \int_0^y \int_0^z g(u)(r, s, t) dr ds dt + \int_0^x \int_0^y \frac{\partial^2 \varphi(r, s)}{\partial r \partial s} dr ds + \\ &+ \int_0^y \int_0^z \frac{\partial^2 \psi(s, t)}{\partial s \partial t} ds dt + \int_0^x \int_0^z \frac{\partial^2 \chi(r, t)}{\partial r \partial t} dr dt + \int_0^x \frac{\partial \varphi(r, 0)}{\partial r} dr + \\ &+ \int_0^y \frac{\partial \varphi(0, s)}{\partial s} ds + \int_0^z \frac{\partial \psi(0, t)}{\partial t} dt + u(0, 0, 0), \end{aligned} \quad (4.3)$$

for  $(x, y, z) \in D$ .

Indeed, we have

$$\begin{aligned} \int_0^x \int_0^y \frac{\partial^2 \varphi(r, s)}{\partial r \partial s} dr ds &= \int_0^x \left[ \int_0^y \frac{\partial^2 \varphi(r, s)}{\partial r \partial s} ds \right] dr = \int_0^x \frac{\partial \varphi(r, s)}{\partial r} \Big|_{s=0}^{s=y} dr = \\ &= \int_0^x \left[ \frac{\partial \varphi(r, y)}{\partial r} - \frac{\partial \varphi(r, 0)}{\partial r} \right] dr = \varphi(r, y) \Big|_{r=0}^{r=x} - \varphi(r, 0) \Big|_{r=0}^{r=x} = \\ &= [\varphi(x, y) - \varphi(0, y)] - [\varphi(x, 0) - \varphi(0, 0)] = \\ &= \varphi(x, y) - \varphi(0, y) - \varphi(x, 0) + \varphi(0, 0), \end{aligned} \quad (4.4)$$

$$\begin{aligned}
\int_0^y \int_0^z \frac{\partial^2 \psi(s, t)}{\partial s \partial t} ds dt &= \int_0^y \left[ \int_0^z \frac{\partial^2 \psi(s, t)}{\partial s \partial t} dt \right] ds = \int_0^y \frac{\partial \psi(s, t)}{\partial s} \Big|_{t=0}^{t=z} ds = \\
&= \int_0^y \left[ \frac{\partial \psi(s, z)}{\partial s} - \frac{\partial \psi(s, 0)}{\partial s} \right] ds = \psi(s, z) \Big|_{s=0}^{s=y} - \psi(s, 0) \Big|_{s=0}^{s=y} = \\
&= [\psi(y, z) - \psi(0, z)] - [\psi(y, 0) - \psi(0, 0)] = \\
&= \psi(y, z) - \psi(0, z) - \psi(y, 0) + \psi(0, 0), \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
\int_0^x \int_0^z \frac{\partial^2 \chi(r, t)}{\partial r \partial t} dr dt &= \int_0^x \left[ \int_0^z \frac{\partial^2 \chi(r, t)}{\partial r \partial t} dt \right] dr = \int_0^x \frac{\partial \chi(r, t)}{\partial r} \Big|_{t=0}^{t=z} dr = \\
&= \int_0^x \left[ \frac{\partial \chi(r, z)}{\partial r} - \frac{\partial \chi(r, 0)}{\partial r} \right] dr = \chi(r, z) \Big|_{r=0}^{r=x} - \chi(r, 0) \Big|_{r=0}^{r=x} = \\
&= [\chi(x, z) - \chi(0, z)] - [\chi(x, 0) - \chi(0, 0)] = \\
&= \chi(x, z) - \chi(0, z) - \chi(x, 0) + \chi(0, 0), \tag{4.6}
\end{aligned}$$

$$\int_0^x \frac{\partial \varphi(r, 0)}{\partial r} dr = \varphi(r, 0) \Big|_{r=0}^{r=x} = \varphi(x, 0) - \varphi(0, 0), \tag{4.7}$$

$$\int_0^y \frac{\partial \varphi(0, s)}{\partial s} ds = \varphi(0, s) \Big|_{s=0}^{s=y} = \varphi(0, y) - \varphi(0, 0), \tag{4.8}$$

$$\int_0^z \frac{\partial \psi(0, t)}{\partial t} dt = \psi(0, t) \Big|_{t=0}^{t=z} = \psi(0, z) - \psi(0, 0). \tag{4.9}$$

From (1.3) we have

$$u(0, 0, 0) = \varphi(0, 0) = \psi(0, 0) = \chi(0, 0). \tag{4.10}$$

Replacing (4.4) – (4.10) in (4.3) it results (4.2).

Using Theorem 2.5 from (4.3), it follows that  $h(u) \in C^*(D; \mathbb{R}^n)$  for each  $u \in \mathcal{K}$ , i.e.  $h(u)$  is an absolutely continuous in Carathéodory's sense function. One obtains  $h(u) \in \mathcal{K}$  and  $h(\mathcal{K}) \subseteq \mathcal{K}$ .

From (5.2) it results  $\frac{\partial^3 h(u)(x, y, z)}{\partial x \partial y \partial z} = g(u)(x, y, z)$ ,  $(x, y, z) \in D$ , but  $g(u)(x, y, z) \in G(u)(x, y, z) = F(x, y, z, u(x, y, z))$ ,  $(x, y, z) \in D$ . Hence  $\zeta = g(u)(x, y, z)$  is an element of the ball  $A$ , and consequently  $\|\zeta\| = \|g(u)(x, y, z)\| \leq M$ , i.e.  $\left\| \frac{\partial^3 h(u)(x, y, z)}{\partial x \partial y \partial z} \right\| \leq M$ , and from (4.2)  $h(u)$  satisfies (1.2). Using the definition of the set  $\mathcal{K}$ , this inequality show that  $h(u) \in \mathcal{K}$ . From  $u \in \mathcal{K}$  implies  $h(u) \in \mathcal{K}$ , we conclude that  $h(\mathcal{K}) \subseteq \mathcal{K}$ .

Now, we can apply the Schauder's Fixed Point Theorem and conclude that there exists  $\hat{u} \in \mathcal{K}$  such that  $\hat{u} = h(\hat{u})$ , i.e.  $\hat{u}(x, y, z) = h(\hat{u})(x, y, z)$  at every  $(x, y, z) \in D$ .

This implies that  $\hat{u}$  satisfies (1.1)

$$\frac{\partial^3 \hat{u}(x, y, z)}{\partial x \partial y \partial z} = g(\hat{u}(x, y, z)) \in F(x, y, z, \hat{u}(x, y, z)) \quad \text{for a.e. } (x, y, z) \in D,$$

and the relations (1.2). Therefore  $\hat{u}(x, y, z) = h(\hat{u})(x, y, z)$  is a solution of Darboux Problem (1.1) + (1.2).

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