LIBERTAS MATHEMATICA, vol XXVIII (2008)

Applications Of Fixed Point Method In Nonlinear Analysis[1]

Cristian VLADIMIRESCU

Abstract. The present paper is devoted to certain applications of the fixed point method to some concrete existence problems. More exactly, the Miranda's Theorem is used to a general boundary value problem for an ordinary differential equation, and the Schauder's Fixed Point Theorem is used for a general result looking the existence of the zeros of a nonlinear operator.

Keywords: Fixed point theorems, nonlinear operators, nonlinear boundary value problems, boundary value problems on infinite intervals, set-valued mappings.

Mathematics Subject Classification (2000): 24B16, 34B15, 34B40, 34C25, 34C37, 45G10, 46C50, 47H10, 47H95, 47J05, 54C60.

1. Introduction

Undoubtedly, the fixed point method is one of the strongest and most used methods in the existence problems, but its effective application in concrete problems in not too easy sometimes.

Moreover, there are necessary intermediate results following from a fixed point theorem.

In the second section of the present work we will use the Miranda's Theorem (see [4]) to prove the existence of solutions for the boundary value problem

$$\dot{x} = f(t, x) \tag{1.1}$$

$$G(x) = 0, \tag{1.2}$$

where $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function and $G : C([0, T], \mathbb{R}^N) \to \mathbb{R}^N$ is a continuous operator. In particular, one obtains an existence result for the periodic solutions of the equation (1.1).

The Miranda's Theorem is not a fixed point theorem, it is a result assuring the existence of a zero for a continuous application from $\mathbb{R}^N$ into $\mathbb{R}^N$ but the Miranda's result is equivalent to the Brouwer's result.

[1] *This paper is dedicated to the memory of Professor Cezar Avramescu*

The idea to use the Brouwer's Theorem to prove the existence of periodic solutions for the equation (1.1) belongs to Poincaré. By attaching to equation (1.1) the initial condition

$$x(0) = c \tag{1.3}$$

and admitting that $(1.1) + (1.3)$ has an unique solution on $[0, T]$, let it $x(t; c)$, then (1.1) has solutions fulfilling

$$x(0) = x(T) \tag{1.4}$$

if and only if the application $c \to x(c, T)$ admits a fixed point.

Equivalently, $(1.1) + (1.4)$ has solutions if and only if the application

$$\Phi(c) := \int_0^T f(s, x(s, c))\, ds$$

admits a zero. Clearly, the condition (1.4) is a particular case of the condition (1.2).

In the third section of this article we will present a generalization of the Miranda's Theorem in infinite dimensional Banach spaces. More precisely, one gives sufficient conditions for a compact operator defined in a Banach space to admit zeros. The proof of this result uses the Schauder's Fixed Point Theorem.

2. Applications of the Miranda's Theorem

Set

$$\overline{\Sigma_l} := \left\{ c \in \mathbb{R}^N,\ |c| \leq l \right\},$$

where $l > 0$ and

$$|c| = \max\left\{ |c_i|,\ i \in \overline{1, N} \right\},\ c = (c_i)_{i \in \overline{1,N}}.$$

Denote

$$S_i^+ := \left\{ c \in \overline{\Sigma_l},\ c_i = l \right\},\ S_i^- := \left\{ c \in \overline{\Sigma_l},\ c_i = -l \right\},\ c = (c_i)_{i \in \overline{1,N}}.$$

Let $F : \overline{\Sigma_l} \to \mathbb{R}^N$, $F = (F_i)_{i \in \overline{1,N}}$ be a continuous operator; consider the equation

$$F(c) = 0. \tag{2.1}$$

The Miranda's Theorem asserts that if the following conditions

$$(\forall)\ i \in \overline{1, N},\ (\forall)\ c \in S_i^+,\ F_i(c) \leq 0, \tag{2.2}$$

$$(\forall)\ i \in \overline{1, N},\ (\forall)\ c \in S_i^-,\ F_i(c) \geq 0 \tag{2.3}$$

are fulfilled, then the equation (2.1) admits solutions.

If we denote by $\langle \cdot, \cdot \rangle$ the inner product of $\mathbb{R}^N$, then (2.2), (2.3) are fulfilled in particular if

$$\langle F(c), c \rangle \leq 0,\ |c| = l. \tag{2.4}$$

The condition (2.4) is more elegant, while (2.2), (2.3) are more lucrative.

As we mentioned, we will give an application of this theorem. Start with the presentation of notations and hypotheses (we refer the reader to [2], [3], [5], [6]).

Set

$$X = C\left([0,T], \mathbb{R}^N\right),$$

endowed with the usual norm $\|\cdot\|$. Denote, as usually, for $\rho > 0$,

$$\overline{B_\rho} := \{x \in X,\ \|x\| \le \rho\}.$$

Let $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function, $f = (f_i)_{i \in \overline{1,N}}$; set

$$M_\rho := \sup\left\{|f(t,x)|,\ t \in [0,T],\ x \in \overline{B_\rho}\right\}.$$

In addition, let $G : X \to \mathbb{R}^N$, $G = (G_i)_{i \in \overline{1,N}}$ be a continuous operator.

Consider the boundary value problem

$$\dot{x} = f(t,x), \tag{2.5}$$

$$G(x) = 0. \tag{2.6}$$

We will prove the following result.

Theorem 2.1 *Admit the following hypotheses:*

i) the next inequality holds

$$\limsup_{\rho \to +\infty} \frac{M_\rho}{\rho} < k, \tag{2.7}$$

where

$$k < \frac{1}{2T}; \tag{2.8}$$

ii) there exists a number $r > 0$ such that for every $i \in \overline{1,N}$ the following implications

$$(x_i(t) > r,\ (\forall)\ t \in [0,T]) \Longrightarrow (G_i(x)(t) < 0,\ (\forall)\ t \in [0,T]), \tag{2.9}$$

$$(x_i(t) < r,\ (\forall)\ t \in [0,T]) \Longrightarrow (G_i(x)(t) > 0,\ (\forall)\ t \in [0,T]) \tag{2.10}$$

hold.

Then, the problem (2.5) + (2.6) *admits solutions.*

Proof. Choose a number $\rho > 0$, such that

$$\rho > \max\left\{\frac{r}{1 - 2kT}, \frac{M_\rho}{k}\right\}. \tag{2.11}$$

Firstly, suppose that there exists a number $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x - y|,\ t \in [0,T],\ x \in \overline{B_\rho}. \tag{2.12}$$

Attaching to equation (2.5) the initial condition

$$x(0) = c, \tag{2.13}$$

it follows that the problem $(2.5)+(2.13)$ admits an unique solution, denoted by $x(t;c)$.

Define an operator $H:\overline{B_\rho}\to X$ through

$$(Hx)(t)=c+\int_0^t f(s,x(s))\,ds, \tag{2.14}$$

where

$$c\in\overline{\Sigma_a},\ a=\rho(1-kT). \tag{2.15}$$

Obviously $a>0$, from (2.8) and (2.11).

But, for $x\in\overline{B_\rho}$ one has

$$|(Hx)(t)|\le|c|+M_\rho t\le\rho(1-kT)+M_\rho T<\rho(1-kT)+\rho kT=\rho,$$

and so

$$\|Hx\|\le\rho. \tag{2.16}$$

Since $x(t;c)$ is the unique fixed point of H it results that $x(t;c)$ is defined on $[0,T]$ and hence

$$|x(t;c)|\le\rho,\ t\in[0,T]. \tag{2.17}$$

Consequently, from a well known result of the qualitative theory of differential equations, it follows that the application $c\to x(\cdot;c)$ is continuous from $\overline{\Sigma_a}$ to X.

By (2.11) it results that $\rho>\frac{r}{1-2kT}$ and so

$$\rho kT+r<\rho(1-kT). \tag{2.18}$$

Let

$$l=\rho kT+r. \tag{2.19}$$

For $c\in S_i^+$ we have

$$x_i(t;c)=l+\int_0^t f(s,x(s;c))\,ds>l-TM_\rho>l-k\rho T=r.$$

Similarly, for $c\in S_i^-$ we have

$$x_i(t;c)<-r.$$

Define on $\overline{\Sigma_l}$ the operator F by setting

$$F(c)=G(x(\cdot;c)).$$

One remarks immediately that F satisfies the hypotheses of the Miranda's Theorem and therefore it admits zeros. If c is such a zero, then $x(t;c)$ is a solution of the problem $(2.5)+(2.6)$.

The treatment of the case when f is only continuous is done following a classical reason, taking into account that f can be approximated uniformly through a polynomial sequence. □

Set in particular

$$Gx := \int_0^T f(s, x(s))\, ds.$$

Admit for f some conditions of Landesman-Lazer type, i.e.

$$\begin{array}{l} f_i(t,x) < \delta_-(t), \ t \in [0,T], \ x_i > r \\ f_i(t,x) > \delta_+(t), \ t \in [0,T], \ x_i < -r \end{array}, \tag{2.20}$$

$$\int_0^T \delta_-(t)\, dt < 0, \ \int_0^T \delta_+(t)\, dt > 0. \tag{2.21}$$

Whence, the Theorem 2.1 is applicable. If $x(t;c)$ is a solution for $(2.5)+(2.6)$, then

$$x(T;c) - x(0;c) = \int_0^T \dot{x}(s;c)\, ds = G(x(\cdot;c)) = 0,$$

and therefore (2.5) admits at least one solution satisfying the condition $x(0) = x(T)$.

3. A generalization of the Miranda's Theorem

Let $(X, \|\cdot\|)$ be a Banach space. Set for $\rho > 0$

$$\overline{B_\rho} := \{x \in X, \ \|x\| \le \rho\}, \ S_\rho := \{x \in X, \ \|x\| = \rho\}.$$

Let $F : \overline{B_\rho} \to X$ be a compact operator (i.e. F is continuous and $F\left(\overline{B_\rho}\right)$ is a relatively compact set).

Consider the equation

$$F(x) = 0. \tag{3.1}$$

We will give sufficient conditions for the equation (3.1) to admit solutions (see [1]).

Let $Q : X \times X \to \mathbb{R}$ be a function satisfying the conditions

(Q_1) $Q(\lambda x, y) = \lambda Q(x,y)$, $\lambda \ge 0$, $x, y \in X$;

(Q_2) $(x \ne 0) \Longrightarrow (Q(x,x) \ne 0)$.

Q may be in particular an inner product or a semi-inner product.

An intermediate result is contained into the following proposition.

Proposition 3.1 *Suppose that*

$$Q(x,x)\, Q(F(x), x) < 0, \ x \in S_\rho. \tag{3.2}$$

Then

$$\inf\left\{\|F(x)\|, \ x \in \overline{B_\rho}\right\} = 0. \tag{3.3}$$

Proof. If

$$0 \in F\left(\overline{B_\rho}\right), \tag{3.4}$$

then (3.3) yields. So, admit that

$$0 \notin F\left(\overline{B_\rho}\right). \tag{3.5}$$

Define then on $\overline{B_\rho}$ an operator Φ through

$$\Phi(x) = \frac{\rho}{\|F(x)\|} \cdot F(x). \tag{3.6}$$

Φ is continuous and we have $\Phi\left(\overline{B_\rho}\right) \subset S_\rho \subset \overline{B_\rho}$. If

$$0 \notin \overline{F\left(\overline{B_\rho}\right)}, \tag{3.7}$$

then the compactness of F implies the compactness of Φ. Therefore, from the Schauder's Theorem it follows that Φ admits a fixed point. If x is such a fixed point, it results that

$$x \in S_\rho,\ x\|F(x)\| = F(x), \tag{3.8}$$

and so

$$Q(F(x), x) = \|F(x)\| Q(x, x),$$

which contradicts (3.2). Hence, (3.7) can not hold, and therefore it results (3.3). □

It is interesting to know when (3.4) holds. From the above reason it follows that this thing happens in particular when

$$(0 \notin F(B_\rho)) \Longrightarrow \left(0 \notin \overline{F(B_\rho)}\right) \tag{3.9}$$

or, equivalently, when

$$(0 \notin F(B_\rho)) \Longrightarrow (0 \notin \partial F(B_\rho)). \tag{3.10}$$

A first case when (3.5) holds is the one when F is expansive, i.e.

$$\|F(x) - F(y)\| \geq L\|x - y\|,\ L > 0,\ x,\ y \in \overline{B_\rho}. \tag{3.11}$$

Indeed, if (3.5) doesn't hold, it results $0 \in \overline{F\left(\overline{B_\rho}\right)}$. If

$$F(x_m) \to 0,\ x_m \in \overline{B_\rho},$$

then by

$$\|x_m - x_n\| \leq \frac{1}{L}\|F(x_n) - F(x_m)\|$$

it follows that $x_m \to x \in \overline{B_\rho}$, hence $F(x) = 0$, which contradicts (3.5).

Another important particular case is when for a number $L > 0$

$$\|F(x) - F(y)\| \leq L\|x - y\|,\ x,\ y \in \overline{B_\rho}, \tag{3.12}$$

and Q is an inner product such that

$$Q(F(x), x) < -a^2,\ a \neq 0,\ x \in S_\rho. \tag{3.13}$$

Setting

$$q(x) := Q(F(x), x)$$

it results that q is uniformly continuous on $\overline{B_\rho}$. Since

$$q(x) < -a^2,\ x \in S_\rho \tag{3.14}$$

one can inductively construct balls $\overline{B_n}$, $n \in \mathbb{N}$, such that

$$\overline{B_{n+1}} \subset \overline{B_n},\ q(x) < -\frac{a^2}{2^n},\ x \in \partial\overline{B_n},\ \cup_{n\in\mathbb{N}} \overline{B_n} = \overline{B_\rho}. \tag{3.15}$$

Admitting that

$$0 \notin F\left(\overline{B_\rho}\right),\ 0 \in \overline{F\left(\overline{B_\rho}\right)}, \tag{3.16}$$

then, from $F(x_m) \to 0$, it follows

$$q(x_m) \to 0. \tag{3.17}$$

The sequence of balls $\overline{B_n}$ can not be finite, since if $\overline{B_k}$ would be the last of it, then we would have

$$q(x_m) < -\frac{a^2}{2^k},\ (\forall)\ m,$$

which contradicts (3.17).

If the sequence $\overline{B_n}$ is infinite, since anyone of it can not contain infinitely many terms of the sequence x_n, it follows that $x_m \to 0$, which, by the continuity of F, implies $F(0) = 0$.□

References

[1] C. Avramescu, A generalization of Miranda's theorem, *Seminar on Fixed Point Theory*, Cluj-Napoca, **III**, 121-128 (2002).

[2] C. Avramescu, An existence result for periodic solutions via Miranda's theorem, *Anal. Univ. A.I. Cuza*, Iaşi, Tome **XLIX**, s. 1a, Mathematics, Fasc. 1, 129-136 (2003).

[3] C. Avramescu, C. Vladimirescu, On the existence of zeros of continuous functions defined in $\mathbb{R}^n$, *Revue Roumaine Math. Pures Appl.*, **50**(5-6), 431-436 (2005).

[4] C. Miranda, Un'osservazione su un teorema di Brouwer, *Bull. U.M.I.* 2, Vol. **3**, 82-87 (1940-1941).

[5] C. Vladimirescu, C. Avramescu, *Applications of the Fixed Point Method to Ordinary Differential and Integral Equations on Noncompact Intervals*, Universitaria Press, Craiova, 325 pp., 2006, ISBN 973-742-278-3, ISBN 978-973742-278-1.

[6] E. Zeidler, *Nonlinear Analysis and its Applications, Vol. I, Fixed Point Theory*, Springer-Verlag, 1993.