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New Means and New Refinements of Cesaro's Inequality and of the AM-GM-HM Inequalities (II)

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Abstract. In this paper we present new refinement for the Cesaro's inequality, namely if $a, b, c > 0$, then $(a+b)(b+c)(c+a) \geq 8abc$ and we introduce new means which give new refinements for the AM-GM-HM inequalities.

We extend the method presented in [4].

Keywords: Cesaro's inequality.

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1. Main Results

Theorem 1. *If $x, y, z > 0$, then*

$$(x+y)(y+z)(z+x) \geq 2xyz + 2\sqrt{(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)} \geq$$
$$\geq 2xyz + \sqrt{xy}(xy+z^2) + \sqrt{yz}(yz+x^2) + \sqrt{zx}(zx+y^2) \geq$$
$$\geq 4xyz + 2\sqrt[3]{(x^3+xyz)(y^3+xyz)(z^3+xyz)} \geq 8xyz.$$

Proof. We have

$$\prod(x+y) = 2xyz + \sum x^2y + \sum xy^2 \geq 2xyz + 2\sqrt{\left(\sum x^2y\right)\left(\sum xy^2\right)} =$$
$$= 2xyz + \sqrt{\left(\sum x(xy+z^2)\right)\left(\sum y(xy+z^2)\right)} \geq 2xyz + \sum \sqrt{xy}(xy+z^2) \geq$$
$$\geq 2xyz + 3\sqrt[3]{\prod \sqrt{xy}(xy+z^2)} = 2xyz + \sqrt[3]{\prod \sqrt{xy}(xy+z^2)} + 2\sqrt[3]{\prod \sqrt{xy}(xy+z^2)} \geq$$
$$\geq 2xyz + \sqrt[3]{\prod 2\sqrt{xy}\sqrt{xyz^2}} + 2\sqrt[3]{\prod \sqrt{xy}(xy+z^2)} = 4xyz + 2\sqrt[3]{\prod(x^3+xyz)} \geq 8xyz.$$

Equality holds if and only if $x = y = z$. These offer a chain of new refinements for Cesaro's inequality.

We introduce the following new means:

$$B_1(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)};$$

$$B_2(x_1,x_2,\dots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(2x_1x_2x_3+2\sqrt{(x_1^2x_2+x_2^2x_3+x_3^2x_1)(x_1x_2^2+x_2x_3^2+x_3x_1^2)}\right)^{\frac{1}{3}};$$

$$B_3(x_1,x_2,\dots,x_n)=\frac{1}{2n}\sum_{cyclic}\Big(2x_1x_2x_3+\sqrt{x_1x_2}\,(x_1x_2+x_3^2)+\sqrt{x_2x_3}\,(x_2x_3+x_1^2)+$$
$$+\sqrt{x_3x_1}\,(x_3x_1+x_1^2)\Big)^{\frac{1}{3}};$$

$$B_4(x_1,x_2,\dots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(4x_1x_2x_3+2\sqrt[3]{(x_1^3+x_1x_2x_3)(x_2^3+x_1x_2x_3)(x_3^3+x_1x_2x_3)}\right);$$

$$B_5(x_1,x_2,\dots,x_n)=\frac{1}{n}\sum_{cyclic}\sqrt[3]{x_1x_2x_3}$$

and $\overline{B_k}(x_1,x_2,\dots,x_n)=\frac{1}{B_k\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}\right)}$, where $k\in\{1,2,3,4,5\}$.

If $A(x_1,x_2,\dots,x_n)=\frac{1}{n}\sum_{k=1}^{n}x_k$; $G(x_1,x_2,\dots,x_n)=\sqrt[n]{\prod_{k=1}^{n}x_k}$; $H(x_1,x_2,\dots,x_n)=\frac{n}{\sum_{k=1}^{n}\frac{1}{x_k}}$, then we have the following:

Theorem 2. *If $x_k>0$ $(k=1,2,\dots,n)$, then $A\ge B_1\ge B_2\ge B_3\ge B_4\ge B_5\ge G\ge\overline{B_5}\ge\overline{B_4}\ge\overline{B_3}\ge\overline{B_2}\ge\overline{B_1}\ge H$, which are new refinements for AM-GM-HM inequalities.*

Proof. We have

$$A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{6n}\sum_{cyclic}\left((x_1+x_2)+(x_2+x_3)+(x_3+x_1)\right)\ge$$
$$\ge\frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)}=B_1\ge B_2\ge B_3\ge B_4\ge$$
$$B_5=\frac{1}{n}\sum_{cyclic}\sqrt[3]{x_1x_2x_3}\ge\sqrt[n]{\prod_{cyclic}\sqrt[3]{x_1x_2x_3}}=G.$$

If in this inequalities we take $x_k\to\frac{1}{x_k}$ $(k=1,2,\dots,n)$, the we obtain $G\ge\overline{B_5}\ge\overline{B_4}\ge\overline{B_3}\ge\overline{B_2}\ge\overline{B_1}\ge H$.

Application 2.1. In all triangles ABC holds:

1) $$\frac{2s}{3}\ge B_1(a,b,c)\ge B_2(a,b,c)\ge B_3(a,b,c)\ge B_4(a,b,c)\ge B_5(a,b,c)\ge$$
$$\ge\sqrt[3]{4sRr}\ge\overline{B_5}(a,b,c)\ge\overline{B_4}(a,b,c)\ge\overline{B_3}(a,b,c)\ge\overline{B_2}(a,b,c)\ge\overline{B_1}(a,b,c)\ge\frac{12sRr}{s^2+r^2+4Rr}$$

2) $$\frac{s}{3}\ge B_1(s-a,s-b,s-c)\ge B_2(s-a,s-b,s-c)\ge B_3(s-a,s-b,s-c)\ge$$
$$\ge B_4(s-a,s-b,s-c)\ge B_5(s-a,s-b,s-c)\ge\sqrt[3]{sr^2}\ge\overline{B_5}(s-a,s-b,s-c)\ge$$

$$\geq \overline{B_4}(s-a,s-b,s-c) \geq \overline{B_3}(s-a,s-b,s-c) \geq \overline{B_2}(s-a,s-b,s-c) \geq$$
$$\geq \overline{B_1}s-a,s-b,s-c \geq \frac{3sr}{4R+r}.$$

3) $$\frac{s^2+r^2+4Rr}{6R} \geq B_1(h_a,h_b,h_c) \geq B_2(h_a,h_b,h_c) \geq B_3(h_a,h_b,h_c) \geq$$
$$\geq B_4(h_a,h_b,h_c) \geq B_5(h_a,h_b,h_c) \geq \sqrt[3]{\frac{2s^2r^2}{R}} \geq \overline{B_5}(h_a,h_b,h_c) \geq \overline{B_4}(h_a,h_b,h_c) \geq$$
$$\geq \overline{B_3}(h_a,h_b,h_c) \geq \overline{B_2}(h_a,h_b,h_c) \geq \overline{B_1}(h_a,h_b,h_c) \geq 3r$$

4) $$\frac{4R+r}{3} \geq B_1(r_a,r_b,r_c) \geq B_2(r_a,r_b,r_c) \geq B_3(r_a,r_b,r_c) \geq B_4(r_a,r_b,r_c) \geq$$
$$\geq B_5(r_a,r_b,r_c) \geq \sqrt[3]{s^2r} \geq \overline{B_5}(r_a,r_b,r_c) \geq \overline{B_4}(r_a,r_b,r_c) \geq$$
$$\geq \overline{B_3}(r_a,r_b,r_c) \geq \overline{B_2}(r_a,r_b,r_c) \geq \overline{B_1}(r_a,r_b,r_c) \geq 3r.$$

5) $$\frac{4R+r}{3s} \geq B_1\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq B_2\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq$$
$$\geq B_3\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq B_4\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq B_5\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq \sqrt[3]{\frac{r}{s}} \geq$$
$$\geq \overline{B_5}\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq \overline{B_4}\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq \overline{B_3}\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq$$
$$\geq \overline{B_2}\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq \overline{B_1}\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq \frac{3r}{s}.$$

6) $$\frac{s}{3r} \geq B_1\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq B_2\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq$$
$$\geq B_3\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq B_4\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq B_5\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq \sqrt[3]{\frac{s}{r}} \geq$$
$$\geq \overline{B_5}\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq \overline{B_4}\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq \overline{B_3}\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq$$
$$\geq \overline{B_2}\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq \overline{B_1}\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq \frac{3s}{4R+r}.$$

7) $$\frac{2R-r}{6R} \geq B_1\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq B_2\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq$$
$$\geq B_3\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq B_4\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq B_5\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq$$
$$\geq \sqrt[3]{\frac{r^2}{16R^2}} \geq \overline{B_5}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq \overline{B_4}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq$$
$$\geq \overline{B_3}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq \overline{B_2}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq$$
$$\geq \overline{B_1}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \geq \frac{3r^2}{r^2+(4R+r)^2}.$$

8) $\frac{4R+r}{6R} \geq B_1\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq B_2\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq$

$$\geq B_3\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq B_4\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq B_5\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq$$

$$\geq \sqrt[3]{\frac{s^2}{16R^2}} \geq \overline{B_5}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq \overline{B_4}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq$$

$$\geq \overline{B_3}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq \overline{B_2}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq$$

$$\geq \overline{B_1}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \geq \frac{3r^2}{r^2+(4R+r)^2}.$$

These inequalities offer a chain of refinements for Euler's and Gerretsen's inequalities.

Application 2.2. In all tetrahedrons $ABCD$ holds:

1) $\frac{h_a+h_b+h_c+h_d}{4} \geq B_1(h_a,h_b,h_c,h_d) \geq B_2(h_a,h_b,h_c,h_d) \geq B_3(h_a,h_b,h_c,h_d) \geq$

$$\geq B_4(h_a,h_b,h_c,h_d) \geq B_5(h_a,h_b,h_c,h_d) \geq \sqrt[4]{h_ah_bh_ch_d} \geq \overline{B_5}(h_a,h_b,h_c,h_d) \geq$$

$$\geq \overline{B_4}(h_a,h_b,h_c,h_d) \geq \overline{B_3}(h_a,h_b,h_c,h_d) \geq \overline{B_2}(h_a,h_b,h_c,h_d) \geq \overline{B_1}(h_a,h_b,h_c,h_d) \geq 4r.$$

2) $\frac{r_a+r_b+r_c+r_d}{4} \geq B_1(r_a,r_b,r_c,r_d) \geq B_2(r_a,r_b,r_c,r_d) \geq B_3(r_a,r_b,r_c,r_d) \geq$

$$\geq B_4(r_a,r_b,r_c,r_d) \geq B_5(r_a,r_b,r_c,r_d) \geq \sqrt[4]{r_ar_br_cr_d} \geq \overline{B_5}(r_a,r_b,r_c,r_d) \geq$$

$$\geq \overline{B_4}(r_a,r_b,r_c,r_d) \geq \overline{B_3}(r_a,r_b,r_c,r_d) \geq \overline{B_2}(r_a,r_b,r_c,r_d) \geq \overline{B_1}(r_a,r_b,r_c,r_d) \geq 2r.$$

Theorem 3. *If* $x,y,z>0$, *then* $(x+y)(y+z)(z+x) \geq \frac{8}{3\sqrt{3}}(xy+yz+zx)^{\frac{3}{2}} \geq 8xyz$.

Proof. If $p=x+y+z$; $q=xy+yz+zx$ and $r=xyz$, then $\prod(x+y)=pq-r$ and the first inequality is equivalent with $\left(\frac{pq-r}{8}\right)^2 \geq \left(\frac{q}{3}\right)^3$ but $pq \geq 9r$, therefore we need to show that $\left(\frac{pq-\frac{pq}{9}}{8}\right)^2 \geq \left(\frac{q}{3}\right)^3$ or $p^2 \geq 3q$, but this is true because $p^2-3q=\sum(x-y)^2 \geq 0$. Equality holds if and only if $x=y=z$. This is a new refinement of Cesaro's inequality.

In following we denote $B_6(x_1,x_2,\ldots,x_n)=\frac{1}{\sqrt{3}n}\sum_{cyclic}\sqrt{x_1x_2+x_2x_3+x_3x_1}$ and $\overline{B_6}(x_1,x_2,\ldots,x_n)=\frac{1}{B_6\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$, which are new means and for which we have the following:

Theorem 4. *If* $x_k>0$ $(k=1,2,\ldots,n)$, *then* $A \geq B_1 \geq B_6 \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_6} \geq \overline{B_1} \geq H$.

Proof. We have

$$A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1)) \geq$$

$$\geq \frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)}=B_1 \geq B_6 \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_6} \geq \overline{B_1} \geq H.$$

These are new refinements for AM-GM-HM inequalities.

Theorem 5. *If* $x, y > 0$, *then* $\frac{x+y}{2} \geq B^2(\sqrt{x}, \sqrt{y}) \geq \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \geq \sqrt{xy}$, *where* $B(x,y) = \begin{cases} \frac{x-y}{2arctg\frac{x-y}{x+y}} & if \quad x \neq y \\ x & if \quad x = y \end{cases}$ *is the Bencze's mean* (see [1]).

Proof. If $t \geq 0$, then $\frac{t}{\sqrt{1+t^2}} \leq arctg\ t \leq t$. In this inequalities we take $t = \frac{x-y}{x+y}$, therefore $\sqrt{\frac{x^2+y^2}{2}} \geq B(x,y) \geq \frac{x+y}{2}$. In these we take $x \to \sqrt{x}$ and $y \to \sqrt{y}$, therefore $\frac{x+y}{2} \geq B^2(\sqrt{x}, \sqrt{y}) \geq \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \geq \sqrt{xy}$.

We introduce the following new means: $B_7(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum\limits_{cyclic} B^2(\sqrt{x_1}, \sqrt{x_2})$, $B_8(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum\limits_{cyclic}\left(\frac{\sqrt{x_1}+\sqrt{x_2}}{2}\right)^2$, $B_9(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum\limits_{cyclic}\sqrt{x_1x_2}$ and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{7, 8, 9\}$ for which we have the following:

Theorem 6. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$, *then* $A \geq B_7 \geq B_8 \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_8} \geq \overline{B_7} \geq H$.

Proof. We have $A = \frac{1}{n}\sum\limits_{k=1}^{n} x_k = \frac{1}{n}\sum\limits_{cyclic}\frac{x_1+x_2}{2} \geq B_7 \geq B_8 \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_8} \geq \overline{B_7} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 7. *If* $x, y, z > 0$, *then*

$$(x+y)(y+z)(z+x) \geq 8B^2(\sqrt{x}, \sqrt{y})\, B^2(\sqrt{y}, \sqrt{z})\, B^2(\sqrt{z}, \sqrt{x}) \geq$$
$$\geq \frac{1}{8}\left(\sqrt{x}+\sqrt{y}\right)^2\left(\sqrt{y}+\sqrt{z}\right)^2\left(\sqrt{z}+\sqrt{x}\right)^2 \geq 8xyz.$$

Proof. See the proof of Theorem 5. These are new refinements of Cesaro's inequality:

We introduce the following new means:

$$B_{10}(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{cyclic}\sqrt[3]{B^2(\sqrt{x_1}, \sqrt{x_2})\, B^2(\sqrt{x_2}, \sqrt{x_3})\, B^2(\sqrt{x_3}, \sqrt{x_1})},$$
$$B_{11}(x_1, x_2, \ldots, x_n) = \frac{1}{4n}\sum_{cyclic}\sqrt[3]{\left(\sqrt{x_1}+\sqrt{x_2}\right)^2\left(\sqrt{x_2}+\sqrt{x_3}\right)^2\left(\sqrt{x_3}+\sqrt{x_1}\right)^2}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{10, 11\}$ for which we have

Theorem 8. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$, *then* $A \geq B_1 \geq B_{10} \geq B_{11} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{11}} \geq \overline{B_{10}} \geq \overline{B_1} \geq H$.

Proof. We have $A = \frac{1}{n}\sum\limits_{k=1}^{n} x_k = \frac{1}{6n}\sum\limits_{cyclic}\left((x_1+x_2)+(x_2+x_3)+(x_3+x_1)\right) \geq B_1 \geq B_{10} \geq B_{11} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{11}} \geq \overline{B_{10}} \geq \overline{B_1} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 9. *If* $x, y, z > 0$, *then* $(x+y)(y+z)(z+x) \geq 4\sqrt[3]{x^2y^2z^2}\left(x+y+z-\sqrt[3]{xyz}\right) \geq 8xyz$.

Proof. If $xyz = 1$, then the first inequality is equivalent with $\prod(x+y) \geq 4\left(\sum x - 1\right)$ but using the identity $\prod(x+y) = (\sum x)(\sum xy) - xyz$ we reduce the inequality to the following $\sum xy + \frac{3}{\sum x} \geq 4$ but $\sum xy + \frac{3}{\sum x} \geq 4\sqrt[4]{\frac{(\sum xy)^3}{9\sum x}}$ and so it is enough to prove that $\left(\sum xy\right)^3 \geq 9\sum x$ but this is easy, because we clearly have $\sum xy \geq 3$ and $\left(\sum xy\right)^2 \geq 3xyz\sum x = 3\sum x$. Equality holds if and only if $x = y = z$.

This is a new refinement of Cesaro's inequality.

We introduce the following new means:

$$B_{12}(x_1, x_2, \ldots, x_n) = \frac{\sqrt[3]{4}}{2n} \sum_{cyclic} \left(\sqrt[3]{x_1^2 x_2^2 x_3^2}\left(x_1 + x_2 + x_3 - \sqrt[3]{x_1 x_2 x_3}\right)\right)^{\frac{1}{3}}$$

and $\overline{B_{12}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{12}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ for which we have the following:

Theorem 10. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then* $A \geq B_1 \geq B_{12} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{12}} \geq \overline{B_1} \geq H$.

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}\left((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)\right) \geq$
$\geq \frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)} = B_1 \geq B_{12} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{12}} \geq \overline{B_1} \geq H.$

These are new refinements for AM-GM-HM inequalities.

Theorem 11. *If $x, y, z > 0$, then*

$$(x+y)(y+z)(z+x) \geq \frac{8}{9}(x+y+z)(xy+yz+zx) \geq$$
$$\geq \frac{8}{3}\left\{(xy+yz+zx)\sqrt[3]{xyz}; (x+y+z)\sqrt[3]{x^2y^2z^2}\right\} \geq 8xyz.$$

Proof. We have $\prod(x+y) = \left(\sum x\right)\left(\sum xy\right) - xyz$, therefore the first inequality reduce to $\left(\sum x\right)\left(\sum xy\right) \geq 9xyz$ which is true.

In following we have $\left(\sum x\right)\left(\sum xy\right) \geq 3\sqrt[3]{xyz}\sum xy$ or $\left(\sum x\right)\left(\sum xy\right) \geq 3\sqrt[3]{x^2y^2z^2}\sum x$ etc. Equality holds if and only if $x = y = z$. These are new refinements for Cesaro's inequality.

We introduce the following new means:

$$B_{13}(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt[3]{9}n}\sum_{cyclic}\left((x_1 + x_2 + x_3)(x_1x_2 + x_2x_3 + x_3x_1)\right)^{\frac{1}{3}}$$

$$B_{14}(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt[3]{3}n}\sum_{cyclic}\left(\left\{(x_1x_2 + x_2x_3 + x_3x_1)\sqrt[3]{x_1x_2x_3}; (x_1 + x_2 + x_2)\sqrt[3]{x_1^2x_2^2x_3^2}\right\}\right)^{\frac{1}{3}}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{13, 14\}$, for which we have the following:

Theorem 12. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then* $A \geq B_1 \geq B_{13} \geq B_{14} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{14}} \geq \overline{B_{13}} \geq \overline{B_1} \geq H$.

Proof. We have

$$A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{6n}\sum_{cyclic}\left((x_1+x_2)+(x_2+x_3)+(x_3+x_1)\right)\geq$$
$$\geq\frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)}=B_1\geq$$
$$\geq B_{13}\geq B_{14}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{14}}\geq\overline{B_{13}}\geq\overline{B_1}\geq H.$$

These are new refinements for AM-GM-HM inequalities.

Theorem 13. *If* $x,y,z>0$, *then*

$$(x+y)(y+z)(z+x)\geq 2xyz+\left\{\sum x^2y+xyz\sum\frac{x+y}{x+z};\sum xy^2+xyz\sum\frac{x+z}{x+y}\right\}\geq$$
$$\geq 2xyz+xyz\sum\frac{x+z}{x+y}+xyz\sum\frac{x+y}{x+z}\geq$$
$$\geq 2xyz+xyz\left\{3+\sum\frac{x+y}{x+z};3+\sum\frac{x+z}{x+y};2\sqrt{\left(\sum\frac{x+z}{x+y}\right)\left(\sum\frac{x+y}{x+z}\right)}\right\}\geq 8xyz.$$

Proof. First we show that $\sum\frac{x}{y}\geq\sum\frac{x+y}{x+z}$. Let us take $\frac{x}{y}=\alpha$, $\frac{y}{z}=\beta$, $\frac{z}{x}=\gamma$, then $\alpha\beta\gamma=1$ and $\frac{z+x}{z+y}=\alpha+\frac{1-\alpha}{1+\beta}$. Using similar relations, the inequality reduce to proving $\sum\frac{\alpha-1}{\beta+1}\geq 0$ or $\sum\alpha^2\gamma+\sum\alpha^2\geq\sum\alpha+3$ but $\sum\alpha^2\gamma\geq 3$ and $\sum\alpha^2\geq\frac{1}{3}\left(\sum\alpha\right)^2\geq\sum\alpha$. In same way we prove that $\sum\frac{y}{x}\geq\sum\frac{x+z}{x+y}$. Boots inequalities can be written in the following forms: $\sum xy^2\geq xyz\sum\frac{x+y}{x+z}$ and $\sum x^2y\geq xyz\sum\frac{x+z}{x+y}$. Therefore we have

$$\prod(x+y)=2xyz+\sum x^2y+\sum xy^2\geq$$
$$\geq 2xyz+\left\{\sum x^2y+xyz\sum\frac{x+y}{x+z};\sum xy^2+xyz\sum\frac{x+z}{x+y}\right\}\geq$$
$$\geq 2xyz+xyz\sum\frac{x+y}{x+z}+xyz\sum\frac{x+z}{x+y}\geq$$
$$\geq 2xyz+xyz\left\{3+\sum\frac{x+y}{x+z};3+\sum\frac{x+z}{x+y};2\sqrt{\left(\sum\frac{x+z}{x+y}\right)\left(\sum\frac{x+y}{x+z}\right)}\right\}\geq 8xyz.$$

Equality holds if and only if $x=y=z$. These are new refinements for Cesaro's inequality.

We introduce the following new means:

$$B_{15}(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(2x_1x_2x_3+\left\{\sum x_1^2x_2+x_1x_2x_3\sum\frac{x_1+x_2}{x_1+x_3};\right.\right.$$
$$\left.\left.\sum x_1x_2^2+x_1x_2x_3\sum\frac{x_1+x_3}{x_1+x_2}\right\}\right)^{\frac{1}{3}};$$

$$B_{16}(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(2x_1x_2x_3+x_1x_2x_3\sum\frac{x_1+x_3}{x_1+x_2}+x_1x_2x_3\sum\frac{x_1+x_2}{x_1+x_3}\right)^{\frac{1}{3}};$$

$$B_{17}(x_1, x_2, \ldots, x_n) = \frac{1}{2n} \sum_{cyclic} \left(2x_1x_2x_3 + x_1x_2x_3 \left\{ 3 + \sum \frac{x_1 + x_2}{x_1 + x_3}; 3 + \sum \frac{x_1 + x_3}{x_1 + x_2}; \right. \right.$$

$$\left. \left. 2\sqrt{\left(\sum \frac{x_1 + x_3}{x_1 + x_2} \right) \left(\sum \frac{x_1 + x_2}{x_1 + x_3} \right)} \right\} \right)^{\frac{1}{3}}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{15, 16, 17\}$ for which we have the following:

Theorem 14. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$, *then* $A \geq B_1 \geq B_{15} \geq B_{16} \geq B_{17} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{17}} \geq \overline{B_{16}} \geq \overline{B_{15}} \geq \overline{B_1} \geq H$.

Proof. We have

$$A = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{1}{6n} \sum_{cyclic} ((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq$$

$$\geq \frac{1}{2n} \sum_{cyclic} \sqrt[3]{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)} = B_1 \geq$$

$$\geq B_{15} \geq B_{16} \geq B_{17} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{17}} \geq \overline{B_{16}} \geq \overline{B_{15}} \geq \overline{B_1} \geq H.$$

These are new refinements for AM-GM-HM inequalities.

Theorem 15. *If* $x, y, z > 0$, *then*

$$(x + y)(y + z)(z + x) \geq 2xyz + 2\sqrt{\left(\sum x^2y \right) \left(\sum xy^2 \right)} \geq$$

$$\geq 2xyz + \frac{2}{\sqrt{3}} \sqrt{xyz(x + y + z)^3} \geq 2xyz + 2\sqrt[3]{x^2y^2z^2} \left(\sum x \right) \geq 8xyz.$$

Proof. First we show that $3\left(\sum x^2y\right)\left(\sum xy^2\right) \geq xyz\left(\sum x\right)^3$. Using the AM-GM inequality, we find that $\frac{1}{3} + \frac{y^2z}{\sum x^2y} + \frac{xy^2}{\sum xy^2} \geq \frac{3y\sqrt[3]{xyz}}{\sqrt[3]{3(\sum x^2y)(\sum xy^2)}}$ and two other similar relations $\frac{1}{3} + \frac{z^2x}{\sum x^2y} + \frac{yz^2}{\sum xy^2} \geq \frac{3z\sqrt[3]{xyz}}{\sqrt[3]{3(\sum x^2y)(\sum xy^2)}}$ and $\frac{1}{3} + \frac{x^2y}{\sum x^2y} + \frac{zx^2}{\sum xy^2} \geq \frac{3x\sqrt[3]{xyz}}{\sqrt[3]{3(\sum x^2y)(\sum xy^2)}}$. Then, adding up the three relations, we find exactly the desired inequality. Therefore: $\prod(x + y) = 2xyz + \sum x^2y + \sum xy^2 \geq 2xyz + 2\sqrt{(\sum x^2y)(\sum xy^2)} \geq 2xyz + \frac{2}{\sqrt{3}}\sqrt{xyz\left(\sum x\right)^3} \geq 8xyz$.

Equality holds if and only if $x = y = z$. These are new refinements of Cesaro's inequality.

We introduce the following new means:

$$B_{18}(x_1, x_2, \ldots, x_n) = \frac{1}{2n} \sum_{cyclic} \left(2x_1x_2x_3 + \frac{2}{\sqrt{3}} \sqrt{x_1x_2x_3(x_1 + x_2 + x_3)^3} \right)^{\frac{1}{3}};$$

$$B_{19}(x_1, x_2, \ldots, x_n) = \frac{1}{2n} \sum_{cyclic} \left(2x_1x_2x_3 + 2\sqrt[3]{x_1^2x_2^2x_3^2}(x_1 + x_2 + x_3) \right)^{\frac{1}{3}}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{18, 19\}$ for which we have the following:

Theorem 16. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_1 \geq B_2 \geq B_{18} \geq B_{19} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{19}} \geq \overline{B_{18}} \geq \overline{B_2} \geq \overline{B_1} \geq H$.*

Proof. We have

$$A = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{1}{6n} \sum_{cyclic} ((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq$$
$$\geq \frac{1}{2n} \sum_{cyclic} \sqrt[3]{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)} = B_1 \geq$$
$$\geq B_2 \geq B_{18} \geq B_{19} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{19}} \geq \overline{B_{18}} \geq \overline{B_2} \geq \overline{B_1} \geq H.$$

These are new refinements for AM-GM-HM inequalities.

Theorem 17. *If $x, y, z > 0$, then*

$$(x + y)(y + z)(z + x) \geq \frac{8}{9} \left(\sum x\right) \left(\sum xy\right) \geq \frac{8}{3\sqrt{3}} \sqrt{xyz} \left(\sum x\right)^{\frac{3}{2}} \geq 8xyz.$$

Proof. We have $\prod (x + y) \geq \frac{8}{9} \left(\sum x\right) \left(\sum xy\right) \geq \frac{8}{3\sqrt{3}} \sqrt{xyz} \left(\sum x\right)^{\frac{3}{2}} \geq 8xyz$. Equality holds if and only if $x = y = z$. These are new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{20}(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt[3]{9}n} \sum_{cyclic} (x_1 + x_2 + x_3)^{\frac{1}{3}} (x_1x_2 + x_2x_3 + x_3x_1)^{\frac{1}{3}};$$
$$B_{21}(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt[3]{3\sqrt{3}}n} \sum_{cyclic} (x_1x_2x_3)^{\frac{1}{6}} (x_1 + x_2 + x_3)^{\frac{1}{2}}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{20, 21\}$ for which we have the following:

Theorem 18. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_1 \geq B_{20} \geq B_{21} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{21}} \geq \overline{B_{20}} \geq \overline{B_1} \geq H$.*

Proof. We have

$$A = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{1}{6n} \sum_{cyclic} ((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq$$
$$\geq \frac{1}{2n} \sum_{cyclic} \sqrt[3]{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)} = B_1 \geq$$
$$\geq B_{20} \geq B_{21} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{21}} \geq \overline{B_{20}} \geq \overline{B_1} \geq H.$$

These are new refinements for AM-GM-HM inequalities.

Theorem 19. *If $x, y, z > 0$, then*

$$\sqrt{xy} \le \left(\frac{x^{\frac{3}{4}} + y^{\frac{3}{4}}}{2}\right)^{\frac{4}{3}} \le \left(\frac{(x+y)\left(\sqrt{x}+\sqrt{y}\right)}{4}\right)^{\frac{2}{3}} \le \left(\frac{x\sqrt{x} + \sqrt[4]{x^3y^3} + y\sqrt{y}}{3}\right)^{\frac{2}{3}} \le \frac{x+y}{2}.$$

Proof. With elementary calculus.

We introduce the following new means

$$B_{22}(x_1, x_2, \dots, x_n) = \frac{1}{n}\sum_{cyclic}\left(\frac{x_1\sqrt{x_1} + \sqrt[4]{x_1^3x_2^3} + x_2\sqrt{x_2}}{3}\right)^{\frac{2}{3}},$$

$$B_{23}(x_1, x_2, \dots, x_n) = \frac{1}{n}\sum_{cyclic}\left(\frac{(x_1+x_2)\left(\sqrt{x_1}+\sqrt{x_2}\right)}{4}\right)^{\frac{2}{3}},$$

$$B_{24}(x_1, x_2, \dots, x_n) = \frac{1}{n}\sum_{cyclic}\left(\frac{(x_1+x_2)\left(\sqrt{x_1}+\sqrt{x_2}\right)}{4}\right)^{\frac{2}{3}},$$

and $\overline{B_k}(x_1, x_2, \dots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right)}$, where $k \in \{22, 23, 24\}$ for which we have the following:

Theorem 20. *If $x_k > 0$ $(k = 1, 2, \dots, n)$, then $A \ge B_{22} \ge B_{23} \ge B_{24} \ge B_9 \ge G \ge \overline{B_9} \ge \overline{B_{24}} \ge \overline{B_{23}} \ge \overline{B_{22}} \ge H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{cyclic}\frac{x_1+x_2}{2} \ge B_{22} \ge B_{23} \ge B_{24} \ge B_9 \ge G \ge \overline{B_9} \ge \overline{B_{24}} \ge \overline{B_{23}} \ge \overline{B_{22}} \ge H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 21. *If $x, y, z > 0$, then*

$$(x+y)(y+z)(z+x) \ge \frac{1}{72}\prod_{cyclic}\left(x\sqrt{x} + \sqrt[4]{x^3y^3} + y\sqrt{y}\right)^{\frac{2}{3}} \ge$$

$$\ge \frac{1}{2}\prod_{cyclic}\left(\frac{(x+y)\left(\sqrt{x}+\sqrt{y}\right)}{4}\right)^{\frac{2}{3}} \ge \frac{1}{2}\prod_{cyclic}\left(x^{\frac{3}{4}} + y^{\frac{3}{4}}\right)^{\frac{4}{3}} \ge 8xyz.$$

Proof. See the Theorem 17.

These are new refinements of Cesaro's inequality.

We introduce the following means

$$B_{25}(x_1, x_2, \dots, x_n) = \frac{1}{4\sqrt[3]{9}n}\sum_{cyclic}\left(\left(x_1\sqrt{x_1} + \sqrt[4]{x_1^3x_2^3} + x_2\sqrt{x_2}\right)\cdot\right.$$

$$\left.\cdot\left(x_2\sqrt{x_2} + \sqrt[4]{x_2^3x_3^3} + x_3\sqrt{x_3}\right)\left(x_3\sqrt{x_3} + \sqrt[4]{x_3^3x_1^3} + x_1\sqrt{x_1}\right)\right)^{\frac{2}{9}};$$

$$B_{26}(x_1, x_2, \ldots, x_n) = \frac{1}{2\sqrt[3]{2}n} \sum_{cyclic} ((x_1 + x_2)(\sqrt{x_1} + \sqrt{x_2})(x_2 + x_3) \cdot$$
$$\cdot (\sqrt{x_2} + \sqrt{x_3})(x_3 + x_1)(\sqrt{x_3} + \sqrt{x_1}))^{\frac{2}{9}};$$
$$B_{27}(x_1, x_2, \ldots, x_n) = \frac{1}{2\sqrt[3]{2}n} \sum_{cyclic} \left(\left(x_1^{\frac{3}{4}} + x_2^{\frac{3}{4}} \right) \left(x_2^{\frac{3}{4}} + x_3^{\frac{3}{4}} \right) \left(x_3^{\frac{3}{4}} + x_4^{\frac{3}{4}} \right) \right)^{\frac{4}{9}}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{25, 26, 27\}$ for which we have the following:

Theorem 22. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_1 \geq B_{25} \geq B_{26} \geq B_{27} \geq G \geq \overline{B_{27}} \geq \overline{B_{26}} \geq \overline{B_{25}} \geq \overline{B_1} \geq H$*

Proof. We have $A = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{1}{6n} \sum_{cyclic} ((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq B_1 \geq B_{25} \geq B_{26} \geq B_{27} \geq G \geq \overline{B_{27}} \geq \overline{B_{26}} \geq \overline{B_{25}} \geq \overline{B_1} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 23. *If $x, y > 0$, and $\alpha \geq 1$ then $\frac{x+y}{2} \geq \frac{x^{\frac{\alpha+1}{\alpha}} - y^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)\left(x^{\frac{1}{\alpha}} - y^{\frac{1}{\alpha}}\right)} \geq \left(\frac{x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}}}{2}\right)^{\alpha} \geq \sqrt{xy}$.*

Proof. See [2]. If $x = y$, then $\frac{x^{\frac{\alpha+1}{2}} - y^{\frac{\alpha+1}{2}}}{(\alpha+1)\left(x^{\frac{1}{\alpha}} - y^{\frac{1}{\alpha}}\right)} = x$.

We introduce the following new means

$$B_{28}(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{cyclic} \frac{x_1^{\frac{\alpha+1}{\alpha}} - x_2^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)\left(x_1^{\frac{1}{\alpha}} - x_2^{\frac{1}{\alpha}}\right)}; \; B_{29}(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{cyclic} \left(\frac{x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}}}{2}\right)^{\alpha}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{28, 29\}$, for which we have the following:

Theorem 24. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_{28} \geq B_{29} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{29}} \geq \overline{B_{28}} \geq H$.*

Proof. We have $A = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{1}{n} \sum_{cyclic} \frac{x_1 + x_2}{2} \geq B_{28} \geq B_{29} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{29}} \geq \overline{B_{28}} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 25. *If $x, y, z > 0$, and $\alpha \geq 1$ then*

$$(x+y)(y+z)(z+x) \geq 8 \prod_{cyclic} \frac{x^{\frac{\alpha+1}{\alpha}} - y^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)\left(x^{\frac{1}{\alpha}} - y^{\frac{1}{\alpha}}\right)} \geq 8 \prod_{cyclic} \left(\frac{x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}}}{2}\right)^{\alpha} \geq 8xyz.$$

Proof. See the Theorem 23.

These are new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{30}(x_1,x_2,\dots,x_n)=\frac{1}{n}\sum_{cyclic}\left(\left(\frac{x_1^{\frac{\alpha+1}{\alpha}}-x_2^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)\left(x_1^{\frac{1}{\alpha}}-x_2^{\frac{1}{\alpha}}\right)}\right)\left(\frac{x_2^{\frac{\alpha+1}{\alpha}}-x_3^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)\left(x_2^{\frac{1}{\alpha}}-x_3^{\frac{1}{\alpha}}\right)}\right)\cdot\left(\frac{x_3^{\frac{\alpha+1}{\alpha}}-x_1^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)\left(x_3^{\frac{1}{\alpha}}-x_1^{\frac{1}{\alpha}}\right)}\right)\right)^{\frac{1}{3}};$$

$$B_{31}(x_1,x_2,\dots,x_n)=\frac{1}{n}\sum_{cyclic}\left(\left(\frac{x_1^{\frac{1}{\alpha}}+x_2^{\frac{1}{\alpha}}}{2}\right)\left(\frac{x_2^{\frac{1}{\alpha}}+x_3^{\frac{1}{\alpha}}}{2}\right)\left(\frac{x_3^{\frac{1}{\alpha}}+x_1^{\frac{1}{\alpha}}}{2}\right)\right)^{\frac{\alpha}{3}}$$

and $\overline{B_k}(x_1,x_2,\dots,x_n)=\frac{1}{B_k\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}\right)}$, where $k\in\{30,31\}$, for which we have the following:

Theorem 26. *If $x_k>0$ $(k=1,2,\dots,n)$, then $A\geq B_1\geq B_{30}\geq B_{31}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{31}}\geq\overline{B_{30}}\geq\overline{B_1}\geq H$.*

Proof. We have $A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1))\geq B_1\geq B_{30}\geq B_{31}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{31}}\geq\overline{B_{30}}\geq\overline{B_1}\geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 27. *If $x,y,z>0$, then $\frac{x+y}{2}\geq\frac{1}{2}\left(\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}}\right)\geq\frac{1}{2}\left(\sqrt{xy}+\frac{x+y}{2}\right)\geq\sqrt{xy}$.*

Proof. With elementary calculus.

We introduce the following new means

$$B_{32}(x_1,x_2,\dots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(\frac{2x_1x_2}{x_1+x_2}+\sqrt{\frac{x_1^2+x_2^2}{2}}\right);$$

$$B_{33}(x_1,x_2,\dots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(\sqrt{x_1x_2}+\frac{x_1+x_2}{2}\right)$$

and $\overline{B_k}(x_1,x_2,\dots,x_n)=\frac{1}{B_k\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}\right)}$, where $k\in\{32,33\}$, for which we have the following:

Theorem 28. *If $x_k>0$ $(k=1,2,\dots,n)$, then $A\geq B_{32}\geq B_{33}\geq B_9\geq G\geq\overline{B_9}\geq\overline{B_{33}}\geq\overline{B_{32}}\geq H$.*

Proof. We have $A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{n}\sum_{cyclic}\frac{x_1+x_2}{2}\geq B_{32}\geq B_{33}\geq B_9\geq G\geq\overline{B_9}\geq\overline{B_{33}}\geq\overline{B_{32}}\geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 29. *If $x, y, z > 0$, then*

$$(x+y)(y+z)(z+x) \geq \prod_{cyclic}\left(\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}}\right) \geq \prod_{cyclic}\left(\sqrt{xy}+\frac{x+y}{2}\right) \geq 8xyz.$$

Proof. See Theorem 27.

These are new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{34}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\left(\left(\frac{2x_1x_2}{x_1+x_2}+\sqrt{\frac{x_1^2+x_2^2}{2}}\right)\left(\frac{2x_2x_3}{x_2+x_3}+\sqrt{\frac{x_2^2+x_3^2}{2}}\right)\cdot\left(\frac{2x_3x_1}{x_3+x_1}+\sqrt{\frac{x_3^2+x_1^2}{2}}\right)\right)^{\frac{1}{3}};$$

$$B_{35}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\left(\left(\sqrt{x_1x_2}+\frac{x_1+x_2}{2}\right)\left(\sqrt{x_2x_3}+\frac{x_2+x_3}{2}\right)\cdot\left(\sqrt{x_3x_1}+\frac{x_3+x_1}{2}\right)\right)^{\frac{1}{3}}$$

and $\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{34, 35\}$, for which we have the following:

Theorem 30. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_1 \geq B_{34} \geq B_{35} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{35}} \geq \overline{B_{34}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1)) \geq B_1 \geq B_{34} \geq B_{35} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{35}} \geq \overline{B_{34}} \geq \overline{B_1} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 31. *If $x, y, a, b > 0$, then $\frac{x+y}{2} \geq \frac{((2a+b)x+by)(ax+(a+2b)y)}{4(a+b)(ax+by)} \geq \sqrt{xy}$.*

Proof. The inequality $\frac{((2a+b)x+by)(ax+(a+2b)y)}{4(a+b)(ax+by)} \leq \frac{x+y}{2}$ is equivalent with

$$((2a+b)x+by)(ax+(a+2b)y) \leq 2(a+b)(x+y)(ax+by)$$

or $ab(x-y)^2 \geq 0$. Using the AM-GM inequality we have

$$\sqrt{x\left(\frac{ax+by}{a+b}\right)}\sqrt{y\left(\frac{ax+by}{a+b}\right)} \leq \frac{1}{2}\left(x+\frac{ax+by}{a+b}\right)\frac{1}{2}\left(y+\frac{ax+by}{a+b}\right) = \frac{((2a+b)x+by)(ax+(a+2b)y)}{4(a+b)^2},$$

therefore $\sqrt{xy} \leq \frac{((2a+b)x+by)(ax+(a+2b)y)}{4(a+b)(ax+by)}$.

We introduce the following new means

$$B_{36}(x_1,x_2,\dots,x_n)=\frac{1}{n}\left(\frac{((2a_1+b_1)x_1+b_1x_2)(a_1x_1+(a_1+2b_1)x_2)}{4(a_1+b_1)(a_1x_1+b_1x_2)}+\right.$$
$$+\frac{((2a_2+b_2)x_2+b_2x_3)(a_2x_2+(a_2+2b_2)x_3)}{4(a_2+b_2)(a_2x_2+b_2x_3)}+\dots+$$
$$\left.+\frac{((2a_n+b_n)x_n+b_nx_1)(a_nx_n+(a_n+2b_n)x_1)}{4(a_n+b_n)(a_nx_n+b_nx_1)}\right)$$

and $\overline{B_{36}}(x_1,x_2,\dots,x)=\frac{1}{B_{36}\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}\right)}$ where $a_k,b_k>0$ $(k=1,2,\dots,n)$ for which we have the following:

Theorem 32. *If $x_k,a_k,b_k>0$ $(k=1,2,\dots,n)$, then $A\ge B_{36}\ge B_9\ge G\ge\overline{B_9}\ge\overline{B_{36}}\ge H$.*

Proof. We have $A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{n}\sum_{cyclic}\frac{x_1+x_2}{2}\ge B_{36}\ge B_9\ge G\ge\overline{B_9}\ge\overline{B_{36}}\ge H.$

These are new refinements of AM-GM-HM inequalities.

Theorem 33. *If $x,y,z>0$, and $a_k,b_k>0$ $(k=1,2,3)$ then*

$$(x+y)(y+z)(z+x)\ge\frac{1}{8}\frac{((2a_1+b_1)x+b_1y)(a_1x+(a_1+2b_1)y)}{(a_1+b_1)(a_1x+b_1y)}\cdot$$
$$\cdot\frac{((2a_2+b_2)y+b_2z)(a_2y+(a_2+2b_2)z)}{(a_2+b_2)(a_2y+b_2z)}\cdot\frac{((2a_3+b_3)z+b_3x)(a_3z+(a_3+2b_3)x)}{(a_3+b_3)(a_3z+b_3x)}\ge 8xyz.$$

Proof. See Theorem 31.

This is a new refinement of Cesaro's inequality.

We introduce the following new means

$$B_{37}(x_1,x_2,\dots,x_n)=\frac{1}{4n}\prod_{cyclic}\left(\frac{((2a_1+b_1)x_1+b_1x_2)(a_1x_1+(a_1+2b_1)x_2)}{(a_1+b_1)(a_1x_1+b_1x_2)}\right.\cdot$$
$$\left.\cdot\frac{((2a_2+b_2)x_2+b_2x_3)(a_2x_2+(a_2+2b_2)x_3)}{(a_2+b_2)(a_2x_2+b_2x_3)}\cdot\frac{((2a_3+b_3)x_3+b_3x_1)(a_3x_3+(a_3+2b_3)x_1)}{(a_3+b_3)(a_3x_3+b_3x_1)}\right)^{\frac{1}{3}}$$

and $\overline{B_{37}}(x_1,x_2,\dots,x)=\frac{1}{B_{37}\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}\right)}$ for which we have the following:

Theorem 34. *If $x_k,a_k,b_k>0$ $(k=1,2,3)$, then $A\ge B_1\ge B_{37}\ge B_5\ge G\ge\overline{B_5}\ge\overline{B_{37}}\ge\overline{B_1}\ge H$.*

Proof. We have $A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1))\ge B_1\ge B_{37}\ge B_5\ge G\ge\overline{B_5}\ge\overline{B_{37}}\ge\overline{B_1}\ge H.$

These are new refinements of AM-GM-HM inequalities.

Theorem 35. *If $x,y>0$, then $\frac{x+y}{2}\ge\frac{x+\sqrt{xy}+y}{3}\ge\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2\ge\frac{2\left(x+\sqrt{xy}+y\right)^2}{9(x+y)}\ge\sqrt{xy}$.*

Proof. If in inequality $\frac{2\left(x+\sqrt{xy}+y\right)^2}{9(x+y)} \geq \sqrt{xy}$, we take $x = yt^2$, then we obtain $(t-1)^2 \cdot$ $\cdot\left(2t^2 - t + 2\right) \geq 0$. If in inequality $\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \geq \frac{2\left(x+\sqrt{xy}+y\right)^2}{9(x+y)}$, we take $x = yt^2$, then we obtain $(t-1)^2\left(t^2+4t+1\right) \geq 0$. The others inequalities we obtain with elementary calculus.

We introduce the following new means

$$B_{38}(x_1, x_2, \ldots, x_n) = \frac{1}{3n}\sum_{cyclic}\left(x_1 + \sqrt{x_1x_2} + x_2\right);$$

$$B_{39}(x_1, x_2, \ldots, x_n) = \frac{2}{9n}\sum_{cyclic}\frac{\left(x_1 + \sqrt{x_1x_2} + x_2\right)^2}{x_1 + x_2}$$

and $\overline{B_k}(x_1, x_2, \ldots, x) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{38, 39\}$, then we obtain the following:

Theorem 36. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_{38} \geq B_8 \geq B_{39} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{39}} \geq \overline{B_8} \geq \overline{B_{38}} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{cyclic}\frac{x_1+x_2}{2} \geq B_{38} \geq B_8 \geq B_{39} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{39}} \geq$ $\overline{B_8} \geq \overline{B_{38}} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 37. *If $x, y, z > 0$, then*

$$(x+y)(y+z)(z+x) \geq \frac{8}{27}\left(x+\sqrt{xy}+y\right)\left(y+\sqrt{yz}+z\right)\left(z+\sqrt{zx}+x\right) \geq$$
$$\geq \frac{1}{8}\left(\sqrt{x}+\sqrt{y}\right)^2\left(\sqrt{y}+\sqrt{z}\right)^2\left(\sqrt{z}+\sqrt{x}\right)^2 \geq$$
$$\geq \frac{64}{729}\cdot\frac{\left(x+\sqrt{xy}+y\right)^2}{x+y}\cdot\frac{\left(y+\sqrt{yz}+z\right)^2}{y+z}\cdot\frac{\left(z+\sqrt{zx}+x\right)^2}{z+x} \geq 8xyz.$$

Proof. See Theorem 35.

These are new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{40}(x_1, x_2, \ldots, x_n) = \frac{1}{3n}\sum_{cyclic}\left(x_1+\sqrt{x_1x_2}+x_2\right)^{\frac{1}{3}}\left(x_2+\sqrt{x_2x_3}+x_3\right)^{\frac{1}{3}}\left(x_3+\sqrt{x_3x_1}+x_1\right)^{\frac{1}{3}};$$

$$B_{41}(x_1, x_2, \ldots, x_n) = \frac{1}{4n}\sum_{cyclic}\left(\sqrt{x_1}+\sqrt{x_2}\right)^{\frac{2}{3}}\left(\sqrt{x_2}+\sqrt{x_3}\right)^{\frac{2}{3}}\left(\sqrt{x_3}+\sqrt{x_1}\right)^{\frac{2}{3}};$$

$$B_{42}(x_1, x_2, \ldots, x_n) = \frac{2}{9n}\sum_{cyclic}\left(\frac{\left(x_1+\sqrt{x_1x_2}+x_2\right)^2}{x_1+x_2}\right)^{\frac{1}{3}}\left(\frac{\left(x_2+\sqrt{x_2x_3}+x_3\right)^2}{x_2+x_3}\right)^{\frac{1}{3}}\cdot$$
$$\cdot\left(\frac{\left(x_3+\sqrt{x_3x_1}+x_1\right)^2}{x_3+x_1}\right)^{\frac{1}{3}}$$

and $\overline{B_k}(x_1,x_2,\ldots,x) = \frac{1}{B_k\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$, where $k\in\{40,41,42\}$, for which we obtain the following:

Theorem 38. *If $x_k>0$ $(k=1,2,\ldots,n)$, then $A\geq B_1\geq B_{40}\geq B_{41}\geq B_{42}\geq B_5\geq G\geq \overline{B_5}\geq\overline{B_{42}}\geq\overline{B_{41}}\geq\overline{B_{40}}\geq\overline{B_1}\geq H$.*

Proof. We have $A=\frac{1}{n}\sum\limits_{k=1}^{n}x_k=\frac{1}{6n}\sum\limits_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1))\geq B_1\geq B_{40}\geq B_{41}\geq B_{42}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{42}}\geq\overline{B_{41}}\geq\overline{B_{40}}\geq\overline{B_1}\geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 39. *If $x,y,z>0$, and $\alpha\geq 1$ then*

$$\frac{x+y}{2}\geq\frac{\left(x^{\frac{1}{\alpha}}-(xy)^{\frac{1}{2\alpha}}+y^{\frac{1}{\alpha}}\right)^{\alpha}+(\sqrt{xy})^{\alpha}}{2}\geq\left(\frac{x^{\frac{1}{\alpha}}+y^{\frac{1}{\alpha}}}{2}\right)^{\alpha}\geq\sqrt{xy}.$$

Proof. For $t\geq 0$ we prove with elementary calculus that $t^{2\alpha}-t^{\alpha}+1\geq\left(t^2-t+1\right)^{\alpha}$ or $t^{2\alpha}+1\geq\left(t^2-t+1\right)^{\alpha}+t^{\alpha}$, therefore $\frac{t^{2\alpha}+1}{2}\geq\frac{\left(t^2-t+1\right)^{\alpha}+t^{\alpha}}{2}\geq\frac{\left(t^2-t+1+t\right)^{\alpha}}{2}=\left(\frac{t^2+1}{2}\right)^{\alpha}$ in these we take $t=\sqrt{\frac{x}{y}}$.

We introduce the following new means

$$B_{43}(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(\left(\left(x_1^{\frac{1}{\alpha}}-(x_1x_2)\right)^{\frac{1}{2\alpha}}+x_2^{\frac{1}{\alpha}}\right)^{\alpha}+(\sqrt{x_1x_2})^{\alpha}\right)$$

and $\overline{B_{43}}(x_1,x_2,\ldots,x)=\frac{1}{B_{43}\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 40. *If $x_k>0$ $(k=1,2,\ldots,n)$, then $A\geq B_{43}\geq B_{29}\geq B_9\geq G\geq\overline{B_9}\geq\overline{B_{29}}\geq\overline{B_{43}}\geq H$.*

Proof. We have $A=\frac{1}{n}\sum\limits_{k=1}^{n}x_k=\frac{1}{n}\sum\limits_{cyclic}\frac{x_1+x_2}{2}\geq B_{43}\geq B_{29}\geq B_9\geq G\geq\overline{B_9}\geq\overline{B_{29}}\geq\overline{B_{43}}\geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 41. *If $x,y,z>0$, and $\alpha\geq 1$ then*

$$(x+y)(y+z)(z+x)\geq\prod\left(\left(x^{\frac{1}{\alpha}}-(xy)^{\frac{1}{2\alpha}}+y^{\frac{1}{\alpha}}\right)^{\alpha}+(\sqrt{xy})^{\alpha}\right)\geq 8\prod\left(\frac{x^{\frac{1}{\alpha}}+y^{\frac{1}{\alpha}}}{2}\right)^{\alpha}\geq 8xyz.$$

Proof. See Theorem 39.

These are new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{44}(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\prod_{cyclic}\left(\left(x_1^{\frac{1}{\alpha}}-(x_1x_2)^{\frac{1}{2\alpha}}+x_2^{\frac{1}{\alpha}}\right)^{\alpha}+(\sqrt{x_1x_2})^{\alpha}\right)^{\frac{1}{3}}\cdot$$

$$\cdot\left(\left(x_2^{\frac{1}{\alpha}}-(x_2x_3)^{\frac{1}{2\alpha}}+x_3^{\frac{1}{\alpha}}\right)^{\alpha}+(\sqrt{x_2x_3})^{\alpha}\right)^{\frac{1}{3}}\cdot\left(\left(x_3^{\frac{1}{\alpha}}-(x_3x_1)^{\frac{1}{2\alpha}}+x_1^{\frac{1}{\alpha}}\right)^{\alpha}+(\sqrt{x_3x_1})^{\alpha}\right)^{\frac{1}{3}}$$

and $\overline{B_{44}}(x_1, x_2, \ldots, x) = \frac{1}{B_{44}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 42. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_1 \geq B_{44} \geq B_{31} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{31}} \geq \overline{B_{44}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq B_1 \geq B_{44} \geq B_{31} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{31}} \geq \overline{B_{44}} \geq \overline{B_1} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 43. *If $x, y, z > 0$, then $(x+y)(y+z)(z+x) \geq \frac{8(xy+yz+zx)^2}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2} \geq 8xyz$.*

Proof. First we show that if $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 1$, then $(u^2+v^2)(v^2+w^2)(w^2+u^2) \geq \left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)^2 \geq 8(u^2+v^2+w^2)^2$, for any $u, v, w > 0$.

Write $u^2+v^2 = 2c$, $v^2+w^2 = 2a$, $w^2+u^2 = 2b$. Then the inequality becomes $\sum\sqrt{\frac{abc}{b+c-a}} \geq \sum a$. Recall Schur's inequality

$$\sum a^4 + abc\sum a \geq \sum a^3(b+c) \Leftrightarrow abc\left(\sum a\right) \geq \sum a^3(b+c-a).$$

Now, using Hölder's inequality, we find that

$$\sum a^3(b+c-a) = \sum \frac{a^3}{\left(\frac{1}{\sqrt{b+c-a}}\right)^2} \geq \frac{\left(\sum a\right)^3}{\left(\sum \frac{1}{\sqrt{b+c-a}}\right)^2}.$$

Combining the two inequalities, we find that $\sum\sqrt{\frac{abc}{b+c-a}} \geq \sum a$ and so the inequality is proved.

If $u_1 = \frac{1}{u}, v_1 = \frac{1}{v}, w_1 = \frac{1}{w}$, then for $u_1 + v_1 + w_1 = 1$ and holds

$$\left(u_1^2+v_1^2\right)\left(v_1^2+w_1^2\right)\left(w_1^2+u_1^2\right) \geq 8\left(u_1^2v_1^2 + v_1^2w_1^2 + w_1^2u_1^2\right)^2.$$

If $u_1 = \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}, v_1 = \frac{\sqrt{y}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}, w_1 = \frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$, then we obtain

$$(x+y)(y+z)(z+x) \geq \frac{8(xy+yz+zx)^2}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}$$

but $\left(\sum xy\right)^2 \geq 3xyz\sum x \geq xyz\left(\sum\sqrt{x}\right)^2$, therefore $\frac{8(xy+yz+zx)^2}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2} \geq 8xyz$. Equality holds if and only if $x = y = z$. This is a new refinement of Cesaro's inequality.

We introduce the following new means

$$B_{45}(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{cyclic}\left(\frac{x_1x_2 + x_2x_3 + x_3x_1}{\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}}\right)^{\frac{2}{3}}$$

and $\overline{B_{45}}(x_1, x_2, \ldots, x) = \frac{1}{B_{45}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 44. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$, *then* $A \geq B_1 \geq B_{45} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{45}} \geq \overline{B_1} \geq H$.

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}\left((x_1+x_2)+(x_2+x_3)+(x_3+x_1)\right) \geq B_1 \geq B_{45} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{45}} \geq \overline{B_1} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 45. *If* $x, y, z > 0$, *then*

$$(x+y)(y+z)(z+x) \geq \left\{\sqrt[3]{xyz}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)\left(y^{\frac{2}{3}}+z^{\frac{2}{3}}\right)\left(z^{\frac{2}{3}}+x^{\frac{2}{3}}\right); \right.$$
$$\left. 2^{\frac{9}{4}}(xyz)^{\frac{2}{3}}\left(x^{\frac{4}{3}}+y^{\frac{4}{3}}\right)^{\frac{1}{4}}\left(y^{\frac{4}{3}}+z^{\frac{4}{3}}\right)^{\frac{1}{4}}\left(z^{\frac{4}{3}}+x^{\frac{4}{3}}\right)^{\frac{1}{4}}\right\} \geq 8xyz.$$

Proof. If $a > 0$, then $2\left(a^3+1\right)^4 \geq \left(a^4+1\right)\left(a^2+1\right)^4$. But $\left(a^2+1\right)^4 \leq (a+1)^2\left(a^3+1\right)^2$ and we are left with the inequality

$$2\left(a^2+1\right)^2 \geq (a+1)^2\left(a^4+1\right) \Leftrightarrow 2\left(a^2-a+1\right)^2 \geq a^4+1 \Leftrightarrow (a-1)^4 \geq 0,$$

which follows. Using this inequality we have for all $a, b, c > 0$ the following inequality $8\prod\left(a^3+1\right)^4 \geq \prod\left(a^4+1\right)\left(a^2+1\right)$. Equality holds if and only if $a = b = c = 1$. We take $a = \frac{u}{v}$, $b = \frac{v}{w}$, $c = \frac{w}{u}$ and for this we have

$$8\prod\left(u^3+v^3\right)^4 \geq \prod\left(u^4+v^4\right)\left(u^2+v^2\right)^4 \geq$$
$$\geq \left\{8u^4v^4w^4\prod\left(u^2+v^2\right)^4; 2^{12}u^8v^8w^8\prod\left(u^4+v^4\right)\right\} \geq 8\cdot 2^{12}u^{12}v^{12}w^{12}$$

or $\prod\left(u^3+v^3\right) \geq \left\{uvw\prod\left(u^2+v^2\right); 2^{\frac{9}{4}}u^2v^2w^2\prod\left(u^4+v^4\right)^{\frac{1}{4}}\right\} \geq 8u^3v^3w^3$ and in these we take $u = \sqrt[3]{x}$, $v = \sqrt[3]{y}$, $w = \sqrt[3]{z}$, for which holds the desired inequalities. Equality holds if and only if $x = y = z$.

These are new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{46}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\left\{\sum_{cyclic}(x_1x_2x_3)^{\frac{1}{9}}\left(x_1^{\frac{2}{3}}+x_2^{\frac{2}{3}}\right)^{\frac{1}{3}}\left(x_2^{\frac{2}{3}}+x_3^{\frac{2}{3}}\right)^{\frac{1}{3}}\left(x_3^{\frac{2}{3}}+x_1^{\frac{2}{3}}\right)^{\frac{1}{3}}; \right.$$
$$\left. 2^{\frac{3}{4}}\sum_{cyclic}(x_1x_2x_3)^{\frac{2}{9}}\left(x_1^{\frac{4}{3}}+x_2^{\frac{4}{3}}\right)^{\frac{1}{12}}\left(x_2^{\frac{4}{3}}+x_3^{\frac{4}{3}}\right)^{\frac{1}{12}}\left(x_3^{\frac{4}{3}}+x_1^{\frac{4}{3}}\right)^{\frac{1}{12}}\right\}$$

and $\overline{B_{46}}(x_1, x_2, \ldots, x) = \frac{1}{B_{46}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 46. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$, *then* $A \geq B_1 \geq B_{46} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{46}} \geq \overline{B_1} \geq H$.

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1)) \geq B_1 \geq B_{46} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{46}} \geq \overline{B_1} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Remark. If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $2^n\left(\frac{x_1^3+x_2^3}{x_1^2+x_2^2}\right)^4 \ldots \left(\frac{x_n^3+x_1^3}{x_n^2+x_1^2}\right)^4 \geq (x_1^4+x_2^4)\ldots(x_n^4+x_1^4)$.

Proof. Using the proof of the Theorem 45 we have $\prod_{k=1}^{n} 2\left(\frac{a_k^3+1}{a_k^2+1}\right)^4 \geq \prod_{k=1}^{n}(a_k^4+1)$ for all $a_k > 0$ $(k = 1, 2, \ldots, n)$. If $a_1 = \frac{x_1}{x_2}$, $a_2 = \frac{x_2}{x_3}, \ldots, a_n = \frac{x_n}{x_1}$, then we obtain the desired inequality.

Theorem 47. *If $x, y, z > 0$, then*

$$(x+y)(y+z)(z+x) \geq \sqrt[3]{xyz}\left(\sqrt[3]{x^2}+\sqrt[3]{yz}\right)\left(\sqrt[3]{y^2}+\sqrt[3]{zx}\right)\left(\sqrt[3]{z^2}+\sqrt[3]{xy}\right) \geq 8xyz.$$

Proof. With elementary calculus we prove that

$$(u^3+1)(v^3+1)(w^3+1) \geq (u^2v+1)(v^2w+1)(w^2u+1)$$

in which we take $u = \sqrt[3]{\frac{x}{y}}$, $v = \sqrt[3]{\frac{y}{z}}$, $w = \sqrt[3]{\frac{z}{x}}$.

This is a new refinement of Cesaro's inequality.

We introduce the following new means

$$B_{47}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\sqrt[9]{x_1x_2x_3}\left(\sqrt[3]{x_1^2}+\sqrt[3]{x_2x_3}\right)^{\frac{1}{3}}\left(\sqrt[3]{x_2^2}+\sqrt[3]{x_3x_1}\right)^{\frac{1}{3}}\left(\sqrt[3]{x_3^2}+\sqrt[3]{x_1x_2}\right)^{\frac{1}{3}}$$

and $\overline{B_{47}}(x_1, x_2, \ldots, x) = \frac{1}{B_{47}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 48. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_1 \geq B_{47} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{47}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1)) \geq B_1 \geq B_{47} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{47}} \geq \overline{B_1} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Remark. If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then

$$(x_1+x_2)(x_2+x_3)\ldots(x_n+x_1) \geq \sqrt[3]{\prod_{k=1}^{n} x_k \prod_{cyclic}\left(\sqrt[3]{x_1^2}+\sqrt[3]{x_2x_3}\right)} \geq 2^n\prod_{k=1}^{n} x_k.$$

Proof. First we show that $\prod_{k=1}^{n}(u_k^3+1) \geq \prod_{cyclic}(u_1^2u_2+1)$ after then we take $u_k = \sqrt[3]{\frac{x_k}{x_{k+1}}}$ $(x_{n+1} = 1)$ $(k = 1, 2, \ldots, n)$.

This is a new generalization and a new refinement of Cesaro's inequality.

Theorem 49. *If $x, y > 0$ and $m \in N^*$, then*

$$\frac{x+y}{2} \geq \left(2^{-m}\left(\left(\sqrt{x}\right)^m - \left(\sqrt{y}\right)^m\right)^2 + \left(\sqrt{xy}\right)^m\right)^{\frac{1}{m}} \geq \sqrt{xy}.$$

Proof. Let be $x = a^2$ and $b = y^2$, then the left inequality is equivalent with

$$\left(a^2+b^2\right)^m - 2^m a^m b^m \geq a^{2m} + b^{2m} - 2a^m b^m \quad \text{or}$$

$$a^{2m} + b^{2m} + \sum_{k=1}^{m-1}\binom{m}{k} a^{2m-2k} b^{2k} - \left(2^m - 2\right) a^m b^m \geq a^{2m} + b^{2m} \quad \text{or}$$

$$\sum_{k=1}^{m-1}\binom{m}{k} a^{2m-2k} b^{2k} - \sum_{k=1}^{m-1}\binom{m}{k} a^m b^m \geq 0 \quad \text{or}$$

$$\sum_{k=1}^{m-1}\binom{m}{k} a^{2m-2k} b^{2k} + \sum_{k=1}^{m-1}\binom{m}{k} a^{2k} b^{2m-2k} - 2\sum_{k=1}^{m-1}\binom{m}{k} a^m b^m \geq 0.$$

With elementary transformation we have:

$$\sum_{k=1}^{m-1}\binom{m}{k} a^m b^{2k}\left(a^{m-2k} - b^{m-2k}\right) + \sum_{k=1}^{m-1}\binom{m}{k} a^{2k} b^m \left(b^{m-2k} - a^{m-2k}\right) \geq 0 \quad \text{or}$$

$$\sum_{k=1}^{m-1}\binom{m}{k}\left(a^{m-2k} - b^{m-2k}\right)^2 a^{2k} b^{2k} \geq 0$$

and this is true for all $n \geq 2$. For $m = 1$ we obtain the identity $\frac{x+y}{2} = \frac{\left(\sqrt{x}-\sqrt{y}\right)^2}{2} + \sqrt{xy}$.

We introduce the following new means

$$B_{48}\left(x_1, x_2, \ldots, x_n\right) = \frac{1}{n}\sum_{cyclic}\left(\sum (2^{-m}((\sqrt{x_1})^m - (\sqrt{x_2})^m)^2 + (\sqrt{x_1 x_2})^m\right)^{\frac{1}{m}}$$

and $\overline{B_{48}}\left(x_1, x_2, \ldots, x\right) = \frac{1}{B_{48}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 50. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $A \geq B_{48} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{48}} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{cyclic}\frac{x_1+x_2}{2} \geq B_{48} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{48}} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 51. *If $x, y, z > 0$ and $m \in N^*$ then*

$$(x+y)(y+z)(z+x) \geq \prod_{cyclic}\left(x^m + y^m + \left(2^m - 2\right)\left(\sqrt{xy}\right)^m\right)^{\frac{1}{m}} \geq 8xyz.$$

Proof. See the proof of Theorem 49.

This is a new refinement of Cesaro's inequality.

We introduce the following new means

$$B_{49}(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\sum_{cyclic}\left(x_1^m+x_2^m+(2^m-2)\left(\sqrt{x_1x_2}\right)^m\right)^{\frac{1}{3m}}\cdot$$

$$\cdot\left(x_2^m+x_3^m+(2^m-2)\left(\sqrt{x_2x_3}\right)^m\right)^{\frac{1}{3m}}\left(x_3^m+x_1^m+(2^m-2)\left(\sqrt{x_3x_1}\right)^m\right)^{\frac{1}{3m}}$$

and $\overline{B_{49}}(x_1,x_2,\ldots,x)=\frac{1}{B_{49}\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 52. *If $x_k>0$ $(k=1,2,\ldots,n)$, then $A\ge B_1\ge B_{49}\ge B_5\ge G\ge\overline{B_5}\ge\overline{B_{49}}\ge\overline{B_1}\ge H$.*

Proof. We have $A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{6n}\sum_{cyclic}((x_1+x_2)(x_2+x_3)(x_3+x_1))\ge B_1\ge B_{49}\ge B_5\ge G\ge\overline{B_5}\ge\overline{B_{49}}\ge\overline{B_1}\ge H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 53. *If $x_k>0$ $(k=1,2,\ldots,n)$ and*

$$F_m(x_1,x_2,\ldots,x_n)=\left(\prod_{k=1}^{n}x_k\right)^{1-\frac{1}{3^m}}\left(x_1^{\left(\frac{1}{3}\right)^m}+x_2^{\left(\frac{1}{3}\right)^m}\right)\left(x_2^{\left(\frac{1}{3}\right)^m}+x_3^{\left(\frac{1}{3}\right)^m}\right)\cdots\left(x_n^{\left(\frac{1}{3}\right)^m}+x_1^{\left(\frac{1}{3}\right)^m}\right),$$

then

$$(x_1+x_2)(x_2+x_3)\ldots(x_n+x_1)=F_0(x_1,x_2,\ldots,x_n)\ge F_1(x_1,x_2,\ldots,x_n)\ge\cdots\ge$$

$$\ge F_m(x_1,x_2,\ldots,x_n)\ge F_{m+1}(x_1,x_2,\ldots,x_n)\ge\cdots\ge\lim_{m\to\infty}F_m(x_1,x_2,\ldots,x_n)=2^n\prod_{k=1}^{n}x_k.$$

Proof. For $m=0$ it is true. We have

$$x_1+x_2=\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}\right)\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}-\sqrt[3]{x_1x_2}\right)\ge\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}\right)\sqrt[3]{x_1x_2},$$

therefore

$$\prod_{cyclic}(x_1+x_2)\ge\prod_{cyclic}\sqrt[3]{x_1x_2}\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}\right)=$$

$$=\sqrt[3]{\prod_{k=1}^{n}x_k^2}\prod_{cyclic}\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}\right)\ge\sqrt[3]{\prod_{k=1}^{n}x_k^2}\prod_{cyclic}\left(2\sqrt[6]{x_1x_3}\right)=2^n\prod_{k=1}^{n}x_k,$$

and the statement it is true for $m=1$.

We suppose true for m namely $F_m(x_1,x_2,\ldots,x_n)\ge F_{m+1}(x_1,x_2,\ldots,x_n)$ and we prove for $m+1$, but this follows from

$$F_{m+1}(x_1,x_2,\ldots,x_n)=\sqrt[3]{\prod_{k=1}^{n}x_kF_m\left(\sqrt[3]{x_1},\sqrt[3]{x_2},\ldots,\sqrt[3]{x_n}\right)}\ge$$

$$\geq \sqrt[3]{\prod_{k=1}^{n} x_k F_{m+1}\left(\sqrt[3]{x_1}, \sqrt[3]{x_2}, \ldots, \sqrt[3]{x_n}\right)} = F_{m+2}\left(x_1, x_2, \ldots, x_n\right),$$

which end the proof. These offer a lot of refinements for generalized Cesaro's inequality.

We introduce the following new means $B_{50}(x_1, x_2, \ldots, x_n) = \frac{1}{2n} \sum_{cyclic} \sqrt[3]{F_m(x_1x_2x_3)}$ and $\overline{B_{50}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{50}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ for which we have the following:

Theorem 54. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$ then $A \geq B_1 \geq B_{50} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{50}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq B_1 \geq B_{50} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{50}} \geq \overline{B_1} \geq H$.

This are new refinements of AM-GM-HM inequalities.

Theorem 55. *If $x, y > 0$ and $m \in N^*$ $(m \geq 2)$ then $x + y \geq \sqrt[m]{xy^{m-1}} + \sqrt[m]{yx^{m-1}} \geq 2\sqrt{xy}$.*

Proof. Using the pondered AM-GM inequality, we have: $\sqrt[m]{xy^{m-1}} + \sqrt[m]{yx^{m-1}} \leq \frac{x+(m-1)y}{m} + \frac{y+(m-1)x}{m} = x + y$ and $\sqrt[m]{xy^{m-1}} + \sqrt[m]{yx^{m-1}} \geq 2\sqrt{\sqrt[m]{x^m y^m}} = 2\sqrt{xy}$.

We introduce the following new means

$$B_{51}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\left(\sqrt[m]{x_1x_2^{m-1}} + \sqrt[m]{x_2x_1^{m-1}}\right)$$

and $\overline{B_{51}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{51}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ for which we have the following:

Theorem 56. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$ then $A \geq B_{51} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{51}} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{cyclic}\frac{x_1+x_2}{2} \geq B_{51} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{51}} \geq H$.

This are new refinements of AM-GM-HM inequalities.

Theorem 57. *If $x, y, z > 0$ and $m \in N^*$ $(m \geq 2)$ then*

$$(x+y)(y+z)(z+x) \geq \prod_{cyclic}\left(\sqrt[m]{xy^{m-1}} + \sqrt[m]{yx^{m-1}}\right) \geq 8xyz.$$

Proof. See the proof of Theorem 55.

This is a new refinement of Cesaro's inequality.

We introduce the following new means

$$B_{52}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\left(\sqrt[m]{x_1x_2^{m-1}} + \sqrt[m]{x_2x_1^{m-1}}\right)^{\frac{1}{3}}\left(\sqrt[m]{x_2x_3^{m-1}} + \sqrt[m]{x_3x_2^{m-1}}\right)^{\frac{1}{3}} \cdot$$

$$\cdot\left(\sqrt[m]{x_3x_1^{m-1}} + \sqrt[m]{x_1x_3^{m-1}}\right)^{\frac{1}{3}}$$

and $\overline{B_{52}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{52}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ for which we have the following:

Theorem 58. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$ *and* $m \in N^*$ $(m \geq 2)$ *then* $A \geq B_1 \geq B_{52} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{52}} \geq \overline{B_1} \geq H$.

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq B_1 \geq B_{52} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{52}} \geq \overline{B_1} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Remark. If $x_k > 0$ $(k = 1, 2, \ldots, n)$ and $m \in N^*$ $(m \geq 2)$ then

$$\sum_{k=1}^{n} x_k \geq \sqrt[m]{x_1 x_2^{m-1}} + \sqrt[m]{x_2 x_3^{m-1}} + \cdots + \sqrt[m]{x_n x_1^{m-1}} \geq n\sqrt[n]{\prod_{k=1}^{n} x_k}.$$

Proof. We have $\sum_{cyclic}\sqrt[m]{x_1 x_2^{m-1}} \leq \sum_{cyclic}\frac{x_1 + (m-1)x_2}{m} = \sum_{k=1}^{n} x_k$ etc. This is a new refinement for AM-GM inequality. If we take $x_k \to \frac{1}{x_k}$, then we obtain a new refinement for GM-HM inequality.

Theorem 59. *If* $x, y > 0$ *then*

$$x + y \geq \sqrt[3]{xy}\sqrt{2\left(\sqrt[3]{x^2} + \sqrt[3]{y^2}\right)} \geq \sqrt[3]{xy}\left(\sqrt[3]{x^2} + \sqrt[3]{y^2}\right) \geq 2\sqrt{xy}.$$

Proof. First we show the following inequality: $1 + t^3 \geq t\sqrt{2(1 + t^2)}$ for all $t \geq 0$ or $t^6 + 2t^3 + 1 \geq 2t^2 + 2t^4$ or $t^6 - 2t^4 + 2t^3 - 2t^2 + 1 \geq 0$ which is equivalent with $(u - 2)(u^2 + 2u - 1) \geq 0$, where $u = t + \frac{1}{t} \geq 2$.

If $t = \sqrt[3]{\frac{x}{y}}$, then we obtain the inequality $x + y \geq \sqrt[3]{xy}\sqrt{2\left(\sqrt[3]{x^2} + \sqrt[3]{y^2}\right)}$, but

$$\sqrt[3]{xy}\sqrt{2\left(\sqrt[3]{x^2} + \sqrt[3]{y^2}\right)} \geq \sqrt[3]{xy}\sqrt{\left(\sqrt[3]{x} + \sqrt[3]{y}\right)^2} = \sqrt[3]{xy}\left(\sqrt[3]{x^2} + \sqrt[3]{y^2}\right) \geq 2\sqrt[3]{xy}\sqrt{\sqrt[3]{xy}} = 2\sqrt{xy}.$$

We introduce the following new means

$$B_{53}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\sqrt[3]{x_1 x_2}\sqrt{2\left(\sqrt[3]{x_1^2} + \sqrt[3]{x_2^2}\right)},$$

$$B_{54}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\sqrt[3]{x_1 x_2}\left(\sqrt[3]{x_1^2} + \sqrt[3]{x_2^2}\right),$$

$\overline{B_k}(x_1, x_2, \ldots, x_n) = \frac{1}{B_k\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, where $k \in \{53, 54\}$ for which we have the following:

Theorem 60. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$ *then* $A \geq B_{53} \geq B_{54} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{54}} \geq \overline{B_{53}} \geq H$.

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{cyclic} \frac{x_1+x_2}{2} \geq B_{53} \geq B_{54} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{54}} \geq \overline{B_{53}} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 61. *If $x, y, z > 0$ then*

$$(x+y)(y+z)(z+x) \geq$$
$$\geq 2\sqrt[3]{x^2y^2z^2}\sqrt{2\left(\sqrt[3]{x^2}+\sqrt[3]{y^2}\right)\left(\sqrt[3]{y^2}+\sqrt[3]{z^2}\right)\left(\sqrt[3]{z^2}+\sqrt[3]{x^2}\right)} \geq$$
$$\geq \sqrt[3]{x^2y^2z^2}\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\left(\sqrt[3]{y}+\sqrt[3]{z}\right)\left(\sqrt[3]{z}+\sqrt[3]{x}\right) \geq 8xyz.$$

Proof. Using the Theorem 59, we have

$$\prod_{cyclic}(x+y) \geq \prod_{cyclic}\sqrt[3]{xy}\sqrt{2\left(\sqrt[3]{x^2}+\sqrt[3]{y^2}\right)} \geq \prod_{cyclic}\sqrt[3]{xy}\left(\sqrt[3]{x}+\sqrt[3]{y}\right) \geq \prod_{cyclic} 2\sqrt{xy} = 8xyz.$$

These are new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{55}(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\sqrt[3]{2\sqrt[3]{x_1^2x_2^2x_3^2}}\left(2\left(\sqrt[3]{x_1^2}+\sqrt[3]{x_2^2}\right)\right)\left(\sqrt[3]{x_2^2}+\sqrt[3]{x_3^2}\right)\left(\sqrt[3]{x_3^2}+\sqrt[3]{x_1^2}\right)^{\frac{1}{6}}$$

and $\overline{B_{55}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{53}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following:

Theorem 62. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$ then $A \geq B_1 \geq B_{50} \geq B_{55} \geq G \geq \overline{B_5} \geq \overline{B_{55}} \geq \overline{B_{50}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1)) \geq B_1 \geq B_{50} \geq B_{55} \geq G \geq \overline{B_5} \geq \overline{B_{55}} \geq \overline{B_{50}} \geq \overline{B_1} \geq H$.

These are new refinements of AM-GM-HM inequalities.

Theorem 63. *If $x, y, z > 0$ and $f_{m+1}(x) = f_m(y) + f_m(z)$; $f_{m+1}(y) = f_m(z) + f_m(x)$; $f_{m+1}(z) = f_m(x) + f_m(y)$, where $f_0(x) = x$ and*

$$F_m(x, y, z) = \frac{(f_m(x)+f_m(y))(f_m(y)+f_m(z))(f_m(z)+f_m(x))}{f_m(x)f_m(y)f_m(z)},$$

then

$$F_0(x,y,z) \geq F_1(x,y,z) \geq \cdots \geq F_m(x,y,z) \geq F_{m+1}(x,y,z) \geq \cdots \geq \lim_{m\to\infty} F_m(x,y,z) = 8.$$

Proof. We have $\frac{x+y}{\sqrt{xy}} \geq \frac{f_1(x)+f_1(y)}{\sqrt{f_1(x)f_1(y)}}$ or $\frac{x}{y} + \frac{y}{x} \geq \frac{f_1(x)}{f_1(y)} + \frac{f_1(y)}{f_1(x)}$ or $\frac{x}{y} + \frac{y}{x} \geq \frac{y+z}{x+y} + \frac{x+z}{y+z}$ which is equivalent with $(x-y)^2(x+y+z) \geq 0$, therefore

$$\prod_{cyclic}\frac{x+y}{x} = \prod_{cyclic}\frac{x+y}{\sqrt{xy}} \geq \prod_{cyclic}\frac{f_1(x)+f_1(y)}{\sqrt{f_1(x)f_1(y)}} = \prod\frac{f_1(x)+f_1(y)}{f_1(x)}.$$

From principie of induction holds that the sequence $(F_m(x,y,z))_{m\geq 0}$ is decreasing. This is a very interesting refinement of Cesaro's inequality.

We introduce the following new means

$$B_{56}(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\sum_{cyclic}\sqrt[3]{x_1x_2x_3F_m(x_1,x_2,x_3)}$$

and $\overline{B_{56}}(x_1,x_2,\ldots,x_n)=\frac{1}{B_{56}\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ for which we obtain the following:

Theorem 64. *If* $x_k>0$ $(k=1,2,\ldots,n)$ *then* $A\geq B_1\geq B_{56}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{56}}\geq\overline{B_1}\geq H$.

Proof. We have $A=\frac{1}{n}\sum\limits_{k=1}^{n}x_k=\frac{1}{6n}\sum\limits_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1))\geq B_1\geq B_{56}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{56}}\geq\overline{B_1}\geq H$. These are new refinements of AM-GM-HM inequalities.

Theorem 65. *Let be* $F(x,y)=\sum\limits_{k=1}^{m}\left(\frac{x+b_k}{y+b_k}+\frac{y+b_k}{x+b_k}\right)$, *where* $b_k>0$ $(k=1,2,\ldots,m)$. *If* $x_i>0$ $(i=1,2,\ldots,n)$, *then*

$$(x_1+x_2)(x_2+x_3)\ldots(x_n+x_1)\geq$$
$$\geq\frac{x_1x_2\ldots x_n}{m^n}F(\sqrt{x_1},\sqrt{x_2})F(\sqrt{x_2},\sqrt{x_3})\ldots F(\sqrt{x_n},\sqrt{x_1})\geq 2^nx_1x_2\ldots x_n.$$

Proof. We have $\frac{x}{y}+\frac{y}{x}\geq\frac{x+b_k}{y+b_k}+\frac{y+b_k}{x+b_k}$, which is equivalent with $(x-y)^2(x+y+b_k)\geq 0$ for all $k\in\{1,2,\ldots,m\}$ and $\frac{x+b_k}{y+b_k}+\frac{y+b_k}{x+b_k}\geq 2$, therefore

$$m\left(\frac{x}{y}+\frac{y}{x}\right)=\sum_{k=1}^{m}\left(\frac{x}{y}+\frac{y}{x}\right)\geq\sum_{k=1}^{m}\left(\frac{x+b_k}{y+b_k}+\frac{y+b_k}{x+b_k}\right)=F(x,y)\geq\sum_{i=1}^{m}2=2m$$

or $\frac{x}{y}+\frac{y}{x}\geq\frac{1}{m}F(x,y)\geq 2$, therefore

$$\prod_{cyclic}\left(\frac{\sqrt{x_1}}{\sqrt{x_2}}+\frac{\sqrt{x_2}}{\sqrt{x_1}}\right)\geq\prod_{cyclic}\frac{1}{m}F(\sqrt{x_1},\sqrt{x_2})=\frac{1}{m^n}\prod_{cyclic}F(\sqrt{x_1},\sqrt{x_2})\geq 2^n\quad\text{or}$$

$$\prod_{cyclic}(x_1+x_2)\geq\frac{x_1\ldots x_n}{m^n}\prod_{cyclic}F(\sqrt{x_1},\sqrt{x_2})\geq 2^nx_1x_2\ldots x_n.$$

This offer a lot of new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{57}(x_1,x_2,\ldots,x_n)=\frac{1}{2nm}\sum_{cyclic}\sqrt[3]{x_1x_2x_3F(\sqrt{x_1},\sqrt{x_2})F(\sqrt{x_2},\sqrt{x_3})F(\sqrt{x_3},\sqrt{x_1})}$$

and $\overline{B_{57}}(x_1,x_2,\ldots,x_n)=\frac{1}{B_{57}\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ for which we obtain the following:

Theorem 66. *If $x_k > 0$ $(k = 1, 2, \dots, n)$ then $A \geq B_1 \geq B_{57} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{57}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1)) \geq B_1 \geq B_{57} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{57}} \geq \overline{B_1} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 67. *Let be $G(x,y) = \sqrt[m]{\prod_{k=1}^{m}\left(\frac{x+b_k}{y+b_k}+\frac{y+b_k}{x+b_k}\right)}$, where $b_k > 0$. If $x_i > 0$ $(i = 1, 2, \dots, n)$, then*

$$(x_1+x_2)(x_2+x_3)\dots(x_n+x_1) \geq$$
$$\geq x_1x_2\dots x_n G(\sqrt{x_1},\sqrt{x_2})\,G(\sqrt{x_2},\sqrt{x_3})\dots G(\sqrt{x_n},\sqrt{x_1}) \geq 2^n x_1x_2\dots x_n.$$

Proof. Using the proof of Theorem 65, we can write:

$$\left(\frac{x}{y}+\frac{y}{x}\right)^m = \prod_{k=1}^{m}\left(\frac{x}{y}+\frac{y}{x}\right) \geq \prod_{k=1}^{m}\left(\frac{x+b_k}{y+b_k}+\frac{y+b_k}{x+b_k}\right) = G^m(x,y) \geq 2^m$$

or $\frac{x}{y}+\frac{y}{x} \geq G(x,y) \geq 2$. Using this, we obtain:

$$\prod_{cyclic}\left(\frac{\sqrt{x_1}}{\sqrt{x_2}}+\frac{\sqrt{x_2}}{\sqrt{x_1}}\right) \geq \prod_{cyclic} G(\sqrt{x_1},\sqrt{x_2}) \geq 2^n \quad \text{or}$$

$$\prod_{cyclic}(x_1+x_2) \geq x_1x_2\dots x_n G(\sqrt{x_1},\sqrt{x_2})\,G(\sqrt{x_2},\sqrt{x_3})\dots G(\sqrt{x_n},\sqrt{x_1}) \geq 2^n x_1x_2\dots x_n.$$

This offer a lot of new refinements of Cesaro's inequality.

We introduce the following new means

$$B_{58}(x_1,x_2,\dots,x_n) = \frac{1}{2n}\sum_{cyclic}\sqrt[3]{x_1x_2x_3G(\sqrt{x_1},\sqrt{x_2})\,G(\sqrt{x_2},\sqrt{x_3})\,G(\sqrt{x_3},\sqrt{x_1})}$$

and $\overline{B_{58}}(x_1,x_2,\dots,x_n) = \frac{1}{B_{58}\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}\right)}$ for which we obtain the following:

Theorem 68. *If $x_k > 0$ $(k = 1, 2, \dots, n)$ then $A \geq B_1 \geq B_{58} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{58}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1)) \geq B_1 \geq B_{58} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{58}} \geq \overline{B_1} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 69. *Let be*

$$H_m(a,b,c) = (abc)^{1-\left(\frac{1}{2}\right)^m}\left(a^{\left(\frac{1}{2}\right)^m}+b^{\left(\frac{1}{2}\right)^m}\right)\left(b^{\left(\frac{1}{2}\right)^m}+c^{\left(\frac{1}{2}\right)^m}\right)\left(c^{\left(\frac{1}{2}\right)^m}+a^{\left(\frac{1}{2}\right)^m}\right),$$

where $a,b,c>0$, then we have the following chain of inequalities:

$$(a+b)(b+c)(c+a)=H_0(a,b,c)\geq$$
$$\geq H_1(a,b,c)\geq\cdots\geq H_m(a,b,c)\geq H_{m+1}(a,b,c)\geq\cdots\geq\lim_{m\to\infty}H_m(a,b,c)=8abc.$$

Proof. With mathematical induction. The inequality $(a+b)(a+c)\geq a\left(\sqrt{b}+\sqrt{c}\right)^2$ is equivalent with $\left(a-\sqrt{bc}\right)^2\geq 0$, therefore

$$\prod_{cyclic}(a+b)^2=\prod_{cyclic}(a+b)(a+c)\geq\prod_{cyclic}a\left(\sqrt{b}+\sqrt{c}\right)^2=abc\prod_{cyclic}\left(\sqrt{a}+\sqrt{b}\right)^2\quad\text{or}$$

$$\prod_{cyclic}(a+b)\geq\sqrt{abc}\prod_{cyclic}\left(\sqrt{a}+\sqrt{b}\right)\geq 8abc,$$

because $\prod_{cyclic}\left(\sqrt{a}+\sqrt{b}\right)\geq\prod_{cyclic}\left(2\sqrt[4]{ab}\right)=8\sqrt{abc}.$

Therefore the statement is true for $m=0$ and $m=1$. We suppose true for m, namely $H_m(a,b,c)\geq H_{m+1}(a,b,c)$ and we proof for $m+1$, namely $H_{m+1}(a,b,c)\geq H_{m+2}(a,b,c)$. In inequality $H_m(a,b,c)\geq H_{m+1}(a,b,c)$ we take $a\to\sqrt{a}$, $b\to\sqrt{b}$, $c\to\sqrt{c}$ and we obtain $H_m\left(\sqrt{a},\sqrt{b},\sqrt{c}\right)\geq H_{m+1}\left(\sqrt{a},\sqrt{b},\sqrt{c}\right)$. After multiplication with $\sqrt{abc}$ holds $H_{m+1}(a,b,c)=\sqrt{abc}H_m\left(\sqrt{a},\sqrt{b},\sqrt{c}\right)\geq\sqrt{abc}H_{m+1}\left(\sqrt{a},\sqrt{b},\sqrt{c}\right)=H_{m+2}(a,b,c)$ and this finish the proof.

This offer a lot of new refinements of Ceasro's inequality.

We introduce the following new means $B_{59}(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\sum_{cyclic}\sqrt[3]{H_m(a,b,c)}$ and $\overline{B_{59}}(x_1,x_2,\ldots,x_n)=\frac{1}{B_{59}\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ for which we obtain the following:

Theorem 70. *If $x_k>0$ $(k=1,2,\ldots,n)$ then $A\geq B_1\geq B_{59}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{59}}\geq\overline{B_1}\geq H$.*

Proof. We have $A=\frac{1}{n}\sum_{k=1}^{n}x_k=\frac{1}{6n}\sum_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1))\geq B_1\geq B_{59}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{59}}\geq\overline{B_1}\geq H.$

These are new refinements for AM-GM-HM inequalities.

Theorem 71. *If $x_k>0$ $(k=1,2,\ldots,n)$, $\alpha\geq 1$, and*

$$M_m(x_1,x_2,\ldots,x_n)=\left(\prod_{k=1}^{n}x_k\right)^{1-\left(\frac{1}{\alpha}\right)^m}\left(x_1^{\left(\frac{1}{\alpha}\right)^m}+x_2^{\left(\frac{1}{\alpha}\right)^m}\right)\left(x_2^{\left(\frac{1}{\alpha}\right)^m}+x_3^{\left(\frac{1}{\alpha}\right)^m}\right)\cdots\left(x_n^{\left(\frac{1}{\alpha}\right)^m}+x_1^{\left(\frac{1}{\alpha}\right)^m}\right),$$

then

$$(x_1+x_2)(x_2+x_3)\ldots(x_n+x_1)=M_0(x_1,x_2,\ldots,x_n)\geq M_1(x_1,x_2,\ldots,x_n)\geq$$
$$\geq M_2(x_1,x_2,\ldots,x_n)\geq\cdots\geq M_m(x_1,x_2,\ldots,x_n)\geq M_{m+1}(x_1,x_2,\ldots,x_n)\geq\cdots\geq$$
$$\geq\lim_{m\to\infty}M_m(x_1,x_2,\ldots,x_n)=2^n\prod_{k=1}^{n}x_k.$$

Proof. The inequality $(x^\alpha + y^\alpha)^2 \geq (x+y)^2 (xy)^{\alpha-1}$ is equivalent with the inequality $(x^{\alpha+1} - y^{\alpha+1})(x^{\alpha-1} - y^{\alpha-1}) \geq 0$, which is true.

If we take $x = x_1^{\frac{1}{\alpha}}$ and $y = x_2^{\frac{1}{\alpha}}$, then we obtain: $(x_1+x_2)^2 \geq \left(x_1^{\frac{1}{\alpha}} + x_2^{\frac{1}{\alpha}}\right)^2 (x_1x_2)^{\frac{\alpha-1}{\alpha}}$, therefore $\prod\limits_{cyclic} (x_1+x_2)^2 \geq \prod\limits_{cyclic} \left(x_1^{\frac{1}{\alpha}} + x_2^{\frac{1}{\alpha}}\right)^2 (x_1x_2)^{\frac{\alpha-1}{\alpha}} = \prod\limits_{k=1}^{n} x_k^{\frac{2(\alpha-1)}{\alpha}} \prod\limits_{cyclic} \left(x_1^{\frac{1}{\alpha}} + x_2^{\frac{1}{\alpha}}\right)^2$ or $\prod\limits_{cyclic} (x_1+x_2) \geq \prod\limits_{k=1}^{n} x_k^{\frac{\alpha-1}{\alpha}} \prod\limits_{cyclic} \left(x_1^{\frac{1}{\alpha}} + x_2^{\frac{1}{\alpha}}\right) \geq 2^n \prod\limits_{k=1}^{n} x_k.$.

Therefore the statement it's true for $m=0$ and $m=1$.

The induction step holds from

$$M_{m+1}(x_1, x_2, \ldots, x_n) = \left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{\alpha}} M_m\left(x_1^{\frac{1}{\alpha}}, x_2^{\frac{1}{\alpha}}, \ldots, x_n^{\frac{1}{\alpha}}\right) \geq$$

$$\geq \left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{\alpha}} M_{m+1}\left(x_1^{\frac{1}{\alpha}}, x_2^{\frac{1}{\alpha}}, \ldots, x_n^{\frac{1}{\alpha}}\right) = M_{m+2}(x_1, x_2, \ldots, x_n)$$

and this finish the proof.

These inequalities offer a lot of new refinements for Cesaro's inequality.

We introduce the following new means $B_{60}(x_1, x_2, \ldots, x_n) = \frac{1}{2n} \sum\limits_{cyclic} \sqrt[3]{M_m(x_1, x_2, x_3)}$ and $\overline{B_{60}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{60}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ for which we obtain the following:

Theorem 72. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$ then $A \geq B_1 \geq B_{60} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{60}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n} \sum\limits_{k=1}^{n} x_k = \frac{1}{6n} \sum\limits_{cyclic} ((x_1+x_2) + (x_2+x_3) + (x_3+x_1)) \geq B_1 \geq B_{60} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{60}} \geq \overline{B_1} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 73. *If $x, y > 0$, then $\frac{x+y}{2} \geq I(x,y) \geq P(x,y) \geq \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \geq L(x,y) \geq \sqrt{xy}$, where $I(x,y) = \begin{cases} \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}} & if\ x \neq y \\ x & if\ x = y \end{cases}$ is the identric mean, $P(x,y) = \begin{cases} \frac{x-y}{2sh\left(\frac{x-y}{x+y}\right)} & if\ x \neq y \\ x & if\ x = y \end{cases}$, is the Seiffert mean, and $L(x,y) = \begin{cases} \frac{x-y}{\ln x - \ln y} & if\ x \neq y \\ x & if\ x = y \end{cases}$ is the logarithmic mean.*

Proof. See [2] and [5].

We introduce the following new means

$$B_{61}(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{cyclic} I(x_1, x_2), \quad B_{62}(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{cyclic} P(x_1, x_2),$$

$$B_{63}(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{cyclic} L(x_1, x_2)$$

and $\overline{B_k}(x_1,x_2,\ldots,x_n)=\frac{1}{B_k\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ where $k\in\{61,62,63\}$ for which we obtain the following:

Theorem 74. *If $x_k>0$ $(k=1,2,\ldots,n)$ then $A\geq B_{61}\geq B_{62}\geq B_8\geq B_{63}\geq B_9\geq G\geq \overline{B_9}\geq\overline{B_{63}}\geq\overline{B_8}\geq\overline{B_{62}}\geq\overline{B_{61}}\geq H$.*

Proof. We have $A=\frac{1}{n}\sum\limits_{k=1}^{n}x_k=\frac{1}{n}\sum\limits_{cyclic}\frac{x_1+x_2}{2}\geq B_{61}\geq B_{62}\geq B_8\geq B_{63}\geq B_9\geq G\geq\overline{B_9}\geq \overline{B_{63}}\geq\overline{B_8}\geq\overline{B_{62}}\geq\overline{B_{61}}\geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 75. *If $x,y,z>0$, then*

$$(x+y)(y+z)(z+x)\geq 8I(x,y)I(y,z)I(z,x)\geq 8P(x,y)P(y,z)P(z,x)\geq$$
$$\geq\frac{1}{8}\left(\sqrt{x}+\sqrt{y}\right)^2\left(\sqrt{y}+\sqrt{z}\right)^2\left(\sqrt{z}+\sqrt{x}\right)^2\geq 8L(x,y)L(y,z)L(z,x)\geq 8xyz.$$

Proof. See the Theorem 74.

These are new refinements for Cesaro's inequality.

We introduce the following new means

$$B_{64}(x_1,x_2,\ldots,x_n)=\frac{1}{n}\sum_{cyclic}\sqrt[3]{I(x_1,x_2)I(x_2,x_3)I(x_3,x_1)},$$

$$B_{65}(x_1,x_2,\ldots,x_n)=\frac{1}{n}\sum_{cyclic}\sqrt[3]{P(x_1,x_2)P(x_2,x_3)P(x_3,x_1)},$$

$$B_{66}(x_1,x_2,\ldots,x_n)=\frac{1}{n}\sum_{cyclic}\sqrt[3]{L(x_1,x_2)L(x_2,x_3)L(x_3,x_1)}$$

and $\overline{B_k}(x_1,x_2,\ldots,x_n)=\frac{1}{B_k\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ where $k\in\{64,65,66\}$ for which we obtain the following:

Theorem 76. *If $x_k>0$ $(k=1,2,\ldots,n)$ then $A\geq B_1\geq B_{64}\geq B_{65}\geq B_{11}\geq B_{66}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{66}}\geq\overline{B_{11}}\geq\overline{B_{65}}\geq\overline{B_{64}}\geq\overline{B_1}\geq H$.*

Proof. We have $A=\frac{1}{n}\sum\limits_{k=1}^{n}x_k=\frac{1}{6n}\sum\limits_{cyclic}((x_1+x_2)+(x_2+x_3)+(x_3+x_1))\geq B_1\geq B_{64}\geq B_{65}\geq B_{11}\geq B_{66}\geq B_5\geq G\geq\overline{B_5}\geq\overline{B_{66}}\geq\overline{B_{11}}\geq\overline{B_{65}}\geq\overline{B_{64}}\geq\overline{B_1}\geq H$.

These are new refinements for AM-GM-HM inequalities.

Theorem 77. *If $G(x,y)=\sqrt{xy}$, $A(x,y)=\frac{x+y}{2}$, $x,y>0$ and $f:(0,+\infty)\to(0,+\infty)$, where $f(t)=(\alpha G^t(x,y)+\beta A^t(x,y))^{\frac{1}{t}}$, where $\alpha,\beta>0$, $\alpha+\beta=1$, then $G(x,y)\leq f(t)\leq A(x,y)$ for all $t>0$.*

Proof. We have $f'(t)=\frac{t}{(\alpha G^t(x,y)+\beta A^t(x,y))^2}((\alpha G^t(x,y)+\beta A^t(x,y))(\alpha G^t(x,y)\ln^2G(x,y)+\beta A^t(x,y)\ln^2A(x,y))-(\alpha G^t(x,y)lnG(x,y)+\beta A^t(x,y)\ln A(x,y))^2)\geq 0$, therefore the function f is increasing and $G(x,y)=\lim\limits_{t\to 0}f(t)$ and $A(x,y)=\lim\limits_{t\to\infty}f(t)$. Because f is a continuous function, therefore f is a continuous mean which give infinitely many new refinements for the AM-GM inequality.

In following we use the notation $F_{t_k}(x_1, x_2; \alpha_k, \beta_k) = (\alpha_k G^{t_k}(x_1, x_2) + \beta_k A^{t_k}(x_1, x_2))^{\frac{1}{t_k}}$, where $t_k, \alpha_k, \beta_k > 0$ and $\alpha_k + \beta_k = 1$, $(k = 1, 2, \dots, n)$.

We introduce the following new means: $B_{67}(x_1, x_2, \dots, x_n) = \frac{1}{n} \sum\limits_{cyclic} F_{t_1}(x_1, x_2; \alpha_1, \beta_1)$ and $\overline{B_{67}}(x_1, x_2, \dots, x_n) = \frac{1}{B_{67}\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right)}$ for which we have the following:

Theorem 78. *If $x_k > 0$, $(k = 1, 2, \dots, n)$, then $A \geq B_{67} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{67}} \geq H$.*

Proof. We have $A = \frac{1}{n} \sum\limits_{k=1}^{n} x_k = \frac{1}{n} \sum\limits_{cyclic} \frac{x_1+x_2}{2} \geq B_{67} \geq B_9 \geq G \geq \overline{B_9} \geq \overline{B_{67}} \geq H$. These are new refinements for AM-GM-HM inequalities.

Theorem 79. *If $x, y, z > 0$, then $(x+y)(y+z)(z+x) \geq 8F_{t_1}(x, y; \alpha_1, \beta_1) \cdot F_{t_2}(y, z; \alpha_2, \beta_2) \cdot F_{t_3}(z, x; \alpha_3, \beta_3) \geq 8xyz$.*

Proof. See the proof of the Theorem 77. These are new refinements of Cesaro's inequality.

We introduce the following new means:

$$B_{68}(x_1, x_2, \dots, x_n) = \frac{1}{n} \sum_{cyclic} \sqrt[3]{F_{t_1}(x_1, x_2; \alpha_1, \beta_1) \cdot F_{t_2}(x_2, x_3; \alpha_2, \beta_2) \cdot F_{t_3}(x_3, x_1; \alpha_3, \beta_3)}$$

and $\overline{B_{68}}(x_1, x_2, \dots, x_n) = \frac{1}{B_{68}\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right)}$ for which we have the following:

Theorem 80. *If $x_k > 0$, $(k = 1, 2, \dots, n)$, then $A \geq B_1 \geq B_{68} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{68}} \geq \overline{B_1} \geq H$.*

Proof. We have $A = \frac{1}{n} \sum\limits_{k=1}^{n} x_k = \frac{1}{6n} \sum\limits_{cyclic} ((x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1)) \geq B_1 \geq B_{68} \geq B_5 \geq G \geq \overline{B_5} \geq \overline{B_{68}} \geq \overline{B_1} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Remark 1. Using the method of Application 2.1 and Application 2.2 for the new means $B_1 - B_{66}$ we obtain a lot of new refinements for the classical triangle (Euler, Gerretsen etc.) and tetrahedron inequalities.

Remark 2. It is an open question to ordered all means $B_1 - B_{68}$.

2. Generalization

Denote $M(x_1, x_2, \dots, x_k)$ a mean such that $\frac{x_1+x_2+\dots+x_k}{k} \geq M(x_1, x_2, \dots, x_k) \geq \sqrt[k]{x_1 x_2 \dots x_k}$ for all $x_i > 0$ $(i = 1, 2, \dots, k)$, $k \in \{1, 2, \dots, n\}$.

We introduce the following new means $M_1(x_1, x_2, \dots, x_n) = \frac{1}{n} \sum\limits_{cyclic} M(x_1, x_2, \dots, x_k)$, $M_2(x_1, x_2, \dots, x_n) = \frac{1}{n} \sum\limits_{cyclic} \sqrt[k]{x_1 x_2 \dots x_k}$ and $\overline{M_r}(x_1, x_2, \dots, x_n) = \frac{1}{M_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right)}$, where $r \in \{1, 2\}$ for which we obtain the following:

Theorem G_1. *If $x_j > 0$ $(j = 1, 2, \ldots, n)$ and $k \in \{1, 2, \ldots, n\}$, then $A \geq M_1 \geq M_2 \geq G \geq \overline{M_2} \geq \overline{M_1} \geq H$.*

Proof. We have $A = \frac{1}{n}\sum\limits_{j=1}^{n} x_j = \frac{1}{n}\sum\limits_{cyclic} \frac{x_1+x_2+\cdots+x_k}{k} \geq M_1 \geq M_2 \geq G \geq \overline{M_2} \geq \overline{M_1} \geq H$.

These givs new refinements for AM-GM-HM inequalities.

In following we give one example for this general algorithm.

Lemma. *If $x_i > 0 (i = 1, 2, \ldots, k)$ and $m \in N$, then*

$$\left(\sum_{i=1}^{k} x_i\right)^m - \sum_{i=1}^{k} x_i^m \geq (k^m - k)\left(\prod_{i=1}^{k} x_i\right)^{\frac{m}{k}}.$$

Proof. We have $\left(\sum\limits_{i=1}^{k} x_i\right)^m = \sum\limits_{i_1=1}^{k}\sum\limits_{i_2=1}^{k}\cdots\sum\limits_{i_m=1}^{k} x_{i_1}x_{i_2}\ldots x_{i_m}$. Let a denote the total number of times that the variable x_1 occurs as a factor in these terms. Letting $S = \sum\limits_{i=2}^{k} x_i$, we have $\left(\sum\limits_{i=1}^{k} x_i\right)^m = (x_1 + s)^m = \sum\limits_{p=0}^{m}\binom{m}{p} x_1^{m-p}s^p$.

The expansion s^p contains $(k-1)^m$ elementary terms, so $a = \sum\limits_{p=0}^{m-1}\binom{m}{p}(m-p)(k-1)^p = mk^{m-1}$. The number of terms in expression $\left(\sum\limits_{i=1}^{k} x_i\right)^m - \sum\limits_{i=1}^{k} x_i^m$ is $k^m - k$, therefore $\frac{1}{k^m-k}\left(\left(\sum\limits_{i=1}^{k} x_i\right)^m - \sum\limits_{i=1}^{k} x_i^m\right) \geq \left(\prod\limits_{i=1}^{k} x_i^{mk^{k-1}-m}\right)^{\frac{1}{k^m-k}} = \left(\prod\limits_{i=1}^{k} x_i\right)^{\frac{m}{k}}$ which gives the desired result.

Theorem G_2. *If $x_i > 0$ $(i = 1, 2, \ldots, k)$, $m \in N$, $t \geq 1$, then*

$$\left(\prod_{i=1}^{k} x_i\right)^{\frac{1}{k}} \leq \left(\frac{t\left(\sum_{i=1}^{k} x_i\right)^m + (1-t)\sum_{i=1}^{k} x_i^m}{tk^m + (1-t)k}\right)^{\frac{1}{m}} \leq \frac{1}{k}\sum_{i=1}^{n} x_i.$$

Proof. If $x = \sum\limits_{i=1}^{k} x_i^m$, $y = \left(\sum\limits_{i=1}^{k} x_i\right)^m$ and $f(t) = \frac{x+t(y-x)}{k+t(k^m-k)}$, then $f'(t) = \frac{k(y-xk^{m-1})}{(k+t(k^m-k))^2} < 0$ because $y \leq k^{m-1}x$, therefore $f(t) \geq \lim\limits_{t\to\infty} f(t) = \frac{y-x}{k^m-k}$.

Using the inequality from Lemma, rearranging we obtain: $\frac{y-x}{k^m-k} \geq \left(\prod\limits_{i=1}^{k} x_i\right)^{\frac{m}{k}}$ or $f(1) \geq f(t) \geq \left(\prod\limits_{i=1}^{k} x_i\right)^{\frac{m}{k}}$ which is the equivalent form of the desired inequality. This inequality is a new refinement of AM-GM inequality, and is a new mean for which we use the notation $B(t, m, x_1, x_2, \ldots, x_k) = \left(\frac{t(\sum_{i=1}^{k} x_i)^m + (1-t)\sum_{i=1}^{k} x_i^m}{tk^m + (1-t)k}\right)^{\frac{1}{m}}$ and for which we have $A \geq B(t, m, x_1, \ldots, x_n) \geq G$ for all $m \in N$ and $t \geq 1$.

We introduce the following new means:

$$B_{69}(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{cyclic} B(t, m, x_1, x_2, \ldots, x_k), \quad B_{70}(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{cyclic} \left(\prod_{i=1}^{k} x_i \right)^{\frac{1}{k}}$$

and $\overline{B_{69}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{69}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, $\overline{B_{70}}(x_1, x_2, \ldots, x_n) = \frac{1}{B_{70}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we have the following:

Theorem G$_3$. *If $x_i > 0 (i = 1, 2, \ldots, n)$, $k \in \{1, 2, \ldots, n\}$, $m \in N$, $t \geq 1$, then $A \geq B_{69} \geq B_{70} \geq G \geq \overline{B_{70}} \geq \overline{B_{69}} \geq H$.*

Proof. We have $A = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{n} \sum_{cyclic} \left(\frac{1}{k} \sum_{i=1}^{k} x_i \right) \geq B_{69} \geq B_{70} \geq G \geq \overline{B_{70}} \geq \overline{B_{69}} \geq H$.

These are new refinements for AM-GM-HM inequalities.

Using the idea of Theorem 77, we define the function $F : R \times [0, 1] \to (0, +\infty)$, where $F(u, v, N, M) = ((1-u)N^v + uM^v)^{\frac{1}{v}}$, where $N \leq M$ are two means $M = M(x_1, x_2, \ldots, x_n)$, $N = N(x_1, x_2, \ldots, x_n)$, $x_k > 0$ $(k = 1, 2, \ldots, n)$. The function F is continuous and increasing and $M \geq F(u, v, N, M) \geq N$.

Now we construct the following sequences: $M_1 = F(u, v, N, M)$, $M_1 \leq F(u, v, N, M_1) \leq M$, $M_2 = F(u, v, N, M_1)$, $M_2 \leq F(u, v, N, M_2) \leq M, \ldots$, $M_{m+1} = F(u, v, N, M_m)$ for which holds $M_1 \leq M_2 \leq \cdots \leq M_m \leq M_{m+1} \leq \cdots \leq M$ and $N \leq F(u, v, N, M) \leq M$, $N_1 = F(u, v, N, M)$; $N \leq F(u, v, N_1, M) \leq N_1$, $N_2 = F(u, v, N_1, M)$; $N \leq F(u, v, N_2, M) \leq N_2, \ldots$, $N_{m+1} = F(u, v, N_m, M)$ for which holds $N \leq \cdots \leq N_{m+1} \leq N_n \leq \cdots \leq N_1$.

Finally we obtain the following general theorem which gives infinitely many continuous refinements.

Theorem G$_4$. *We have the following inequalities: $M \geq \cdots \geq M_{m+1} \geq M_m \geq \cdots \geq M_2 \geq M_1 \geq N_1 \geq N_2 \geq \cdots \geq N_m \geq N_{m+1} \geq \cdots \geq N$.*

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