LIBERTAS MATHEMATICA, vol XXVIII (2008)

Univalence Sufficient Conditions For Some Integral Operators

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Abstract. In this work some integral operators are studied and the author determines conditions for the univalence of these integral operators.

Keywords: Integral operator, univalence, starlike.
Mathematics Subject Classification (2000): Primary 30C45.

1 Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in C \mid |z| < 1\}$. Let S denote the subclass of A consisting of all univalent functions f in U.

For $f \in A$, the integral operator $G_{\alpha,\gamma,n}$ is defined by

$$G_{\alpha,\gamma,n}f(z) = \left\{\alpha \int_0^z u^{\alpha-1}\left[f'(u^n)\right]^\gamma du\right\}^{\frac{1}{\alpha}} \tag{1.2}$$

for $n \in N^*$ and some complex numbers α, $(\alpha \neq 0)$ and $\gamma\,(\gamma \neq 0)$.

Also, the integral operator M_α is given by

$$M_\alpha f(z) = \left\{\frac{1}{\alpha}\int_0^z u^{-1}\left(f(u)\right)^{\frac{1}{\alpha}} du\right\}^\alpha \tag{1.3}$$

for some complex numbers $\alpha\,(\alpha \neq 0)$. Miller and Mocanu [1] have studied that the integral operator M_α is in the class S for $f \in S^*$, which is the subclass of S consisting of all starlike functions f in U.

2 Preliminary results

We need the following theorems.

Theorem 2.1. [3] *Let α be a complex number, $Re\alpha > 0$ and $f \in A$. If*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.1}$$

for all $z \in U$, then the integral operator F_α define by

$$F_\alpha f(z) = \left\{\alpha \int_0^z u^{\alpha-1} f'(u)du\right\}^{\frac{1}{\alpha}} \tag{2.2}$$

is in the class S.

Schwarz lemma. [2] *If the function f is regular in U, $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in U$, then the following inequalities hold*

$$|f(z)| \leq |z| \tag{2.3}$$

for all $z \in U$ and $|f'(0)| \leq 1$, the equalities (in the inequality (2.3) for $z \neq 0$) hold only in the case $f(z) = Kz$, where $|K| = 1$.

3 Main results

Theorem 3.1. *Let α, γ be complex numbers, $Re\alpha > 0$ and $f \in A$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{n} \tag{3.1}$$

for all $z \in U$ and

$$\frac{|\gamma|}{Re\alpha} \leq \frac{n+2}{2}\left(\frac{n+2}{n}\right)^{\frac{n}{2}}, \text{ for } Re\alpha \in (0,1) \tag{3.2}$$

or

$$|\gamma| \leq \frac{n+2}{2}\left(\frac{n+2}{n}\right)^{\frac{n}{2}}, \text{ for } Re\alpha \in [1,\infty) \tag{3.3}$$

then the function

$$G_{\alpha,\gamma,n}f(z) = \left\{\alpha \int_0^z u^{\alpha-1}\left[f'(u^n)\right]^\gamma du\right\}^{\frac{1}{\alpha}} \tag{3.4}$$

is in the class S. for all $n \in N^*$.

Proof. Let us consider the function

$$g(z) = \int_0^z \left[f'(u^n)\right]^\gamma du. \tag{3.5}$$

The function

$$h(z) = \frac{1}{|\gamma|}\frac{g''(z)}{g'(z)} \tag{3.6}$$

where the constant $|\gamma|$ satisfies the inequality (3.2) or (3.3), is regular in U. From (3.5) and (3.6), we obtain

$$h(z) = \frac{\gamma}{|\gamma|}\left[\frac{nz^{n-1}f''(z^n)}{f'(z^n)}\right]. \tag{3.7}$$

Using (3.1) and (3.7) we have

$$|h(z)| < 1 \tag{3.8}$$

for all $z \in U$. For $z = 0$ we obtain $h(0) = 0$.

From (3.7) and Schwarz Lemma it results that

$$\frac{1}{|\gamma|}\left|\frac{g''(z)}{g'(z)}\right| \leq |z|^{n-1} \leq |z| \tag{3.9}$$

for all $z \in U$, and hence

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq |\gamma|\frac{1-|z|^{2Re\alpha}}{Re\alpha}|z|^n. \tag{3.10}$$

for all $z \in U$.

We have the cases:

Case 1. For $Re\alpha \in (0,1)$ we obtain

$$1-|z|^{2Re\alpha} \leq 1-|z|^2,\ z \in U.$$

From (3.10) we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{|\gamma|}{Re\alpha}\left(1-|z|^2\right)|z|^n, \tag{3.11}$$

for all $z \in U$.

Let us consider $Q : [0,1] \to \Re$, $Q(x) = (1-x)^2x^n$, $x = |z|$. We have

$$Q(x) \leq \frac{2}{n+2}\left(\frac{n}{n+2}\right)^{\frac{n}{2}} \tag{3.12}$$

for all $x \in [0,1]$. From (3.2), (3.11) and (3.12) we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1 \tag{3.13}$$

for all $z \in U$ and $Re\alpha \in (0,1)$.

Case 2. For $Re\alpha \in [1,\infty)$ we have $\frac{1-|z|^{2Re\alpha}}{Re\alpha} \leq 1-|z|^2$, $z \in U$, then from (3.10) we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq |\gamma|\left(1-|z|^2\right)|z|^n,\ z \in U. \tag{3.14}$$

From (3.3), (3.14) and (3.12) we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1 \tag{3.15}$$

for all $z \in U$ and $Re\alpha \in [1,\infty)$.

Then from (3.13), (3.15) and because $g'(z) = [f'(z^n)]^\gamma$, by Theorem 2.1 it follows that the function $G_{\alpha,\gamma,n}f(z)$ is in the class S.

Theorem 3.2. *Let α be a complex number with $Re\frac{1}{\alpha} > 0$, and the function $g \in A$ satisfying*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq 1,\ z \in U. \tag{3.16}$$

If

$$|\alpha| Re\frac{1}{\alpha} \geq \frac{2}{3\sqrt{3}},\ \textit{for}\ Re\frac{1}{\alpha} \in (0,1) \tag{3.17}$$

or

$$|\alpha| \geq \frac{2}{3\sqrt{3}},\ \textit{for}\ Re\frac{1}{\alpha} \in [1,\infty) \tag{3.18}$$

then the integral operator M_α given by

$$M_\alpha g(z) = \left\{\frac{1}{\alpha}\int_0^z u^{-1}\left(g(u)\right)^{\frac{1}{\alpha}} du\right\}^\alpha \tag{3.19}$$

is in the class S.

Proof. We have

$$M_\alpha g(z) = \left\{\frac{1}{\alpha}\int_0^z u^{\frac{1}{\alpha}-1}\left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du\right\}^\alpha. \tag{3.20}$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du. \tag{3.21}$$

Then the function

$$p(z) = |\alpha|\frac{zf''(z)}{f'(z)} \tag{3.22}$$

is regular in U, where $|\alpha|$ satisfies the inequality (3.17) or (3.18).

From (3.21) and (3.22), we have that

$$p(z) = \frac{|\alpha|}{\alpha}\left(\frac{zg'(z)}{g(z)} - 1\right). \tag{3.23}$$

Using (3.23) and (3.16), we obtain

$$|p(z)| \leq 1,\ \ (z \in U). \tag{3.24}$$

Noting that $p(0) = 0$ and applying Schwarz Lemma for $p(z)$, we get

$$|\alpha|\left|\frac{zf''(z)}{f'(z)}\right| \leq |z|, \ (z \in U) \tag{3.25}$$

and hence, we obtain

$$\frac{1-|z|^{2Re\frac{1}{\alpha}}}{Re\frac{1}{\alpha}}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{|\alpha|}\frac{1-|z|^{2Re\frac{1}{\alpha}}}{Re\frac{1}{\alpha}}|z| \tag{3.26}$$

for all $z \in U$.

For $Re\frac{1}{\alpha} \in (0,1)$ we get $1-|z|^{2Re\frac{1}{\alpha}} \leq 1-|z|^2, \ z \in U$.

From (3.26) we obtain

$$\frac{1-|z|^{2Re\frac{1}{\alpha}}}{Re\frac{1}{\alpha}}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{|\alpha| Re\frac{1}{\alpha}}\left(1-|z|^2\right)|z|, \ z \in U. \tag{3.27}$$

Let us consider the function $T : [0,1] \to \Re$, $T(x) = (1-x)^2 x$, $x = |z|$. We have

$$T(x) \leq \frac{2}{3\sqrt{3}}, \ \text{ for all } x \in [0,1]. \tag{3.28}$$

From (3.17) we obtain $\frac{1}{|\alpha| Re\frac{1}{\alpha}} \leq \frac{3\sqrt{3}}{2}$, hence and (3.28), (3.27), we have

$$\frac{1-|z|^{2Re\frac{1}{\alpha}}}{Re\frac{1}{\alpha}}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3.29}$$

for all $z \in U$ and $Re\frac{1}{\alpha} \in (0,1)$.

For $Re\frac{1}{\alpha} \in [1,\infty)$ we have $\frac{1-|z|^{2Re\frac{1}{\alpha}}}{Re\frac{1}{\alpha}} \leq 1-|z|^2, \ z \in U$ and from (3.26) we get

$$\frac{1-|z|^{2Re\frac{1}{\alpha}}}{Re\frac{1}{\alpha}}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{|\alpha|}\left(1-|z|^2\right)|z|, \ z \in U \tag{3.30}$$

for all $z \in U$.

From (3.18) we have $\frac{1}{|\alpha|} \leq \frac{3\sqrt{3}}{2}$, hence and (3.28), (3.30), we obtain

$$\frac{1-|z|^{2Re\frac{1}{\alpha}}}{Re\frac{1}{\alpha}}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3.31}$$

for all $z \in U$, $Re\frac{1}{\alpha} \in [1,\infty)$.

From (3.29), (3.31) and because $f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}$, by Theorem 2.1, we conclude that the integral operator M_α belongs to S.

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