LIBERTAS MATHEMATICA, vol XXVIII (2008)

On a Szüsz's Solution for Gauss' Problem

Ion COLTESCU and Dan LASCU

Abstract. This paper deal with Gauss' problem (on continued fractions) and present another proof of a theorem which Szüsz solves this problem. Note that on obtained for q the value 0.7594..., which will be optimized later by Szüsz in paper "Über einen Kusminschen Satz" in 1961, where is obtained for q the value 0.485. In our proof, we use an important property of Perron–Frobenius operator of τ under γ, where τ is the continued fraction transformation, and γ is the Gauss' measure.

Keywords: continued fractions, Gauss-Kuzmin problem.

1. Introduction

Let $\xi \in [0,1)$, and let $\xi = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \ddots + \cfrac{1}{d_n + \ddots}}} = [0; d_1, d_2, \ldots, d_n, \ldots]$ be the regular continued fraction expansion of ξ . On October 25, 1800, Gauss wrote in his diary that (in modern notation):

$$\lim_{n\to\infty} \lambda\left(\{\xi \in [0,1)\,;\, \tau^n(\xi) \le z\}\right) = \frac{\log(z+1)}{\log 2}, \quad 0 \le z \le 1, \tag{1.1}$$

where λ is the Lebesgue measure and $\tau : [0,1) \to [0,1)$ is the continued fraction transformation defined by

$$\tau(\xi) := \frac{1}{\xi} - \left[\frac{1}{\xi}\right], \quad \xi \ne 0; \quad \tau(0) := 0, \tag{1.2}$$

where $[\cdot]$ denotes the floor (entire) function. Latter, in a letter dated January 30, 1812, Gauss asked Laplace to give an estimate of the error term $r_n(z)$, defined by

$$r_n(z) := \lambda\left(\tau^{-n}([0,z])\right) - \frac{\log(z+1)}{\log 2}, \quad n \ge 1.$$

Gauss' proof has never been found. The first who prove (1.1) and at the same time answer to Gauss' question was Kuzmin. In 1928, Kuzmin [3] showed that $r_n(z) = \mathcal{O}\left(q^{\sqrt{n}}\right)$, with $q \in (0,1)$, uniform in z. Independently, Paul Lévy showed one year later that $r_n(z) = \mathcal{O}(q^n)$, with $q = 0.7\ldots$, uniform in z. From that time on, a great number of such Gauss–Kuzmin theorems followed. To mention a few: F.Schweiger (1968), P. Wirsing [6] (1974 – who determined that the optimal value of q is equal to $0.303663002\ldots$), K.I. Babenko (1978), and more recently by M. Iosifescu (1992).

2. The Gauss–Kuzmin Type Equation

An essential ingredient in any proof of any Gauss–Kuzmin theorem is the following observation. Let $\xi \in [0,1)\setminus\mathbb{Q}$ and put $\tau_k := \tau^k(\xi)$, $k \geq 0$, where $\tau : [0,1) \to [0,1)$ is the continued fraction transformation defined in (1.2). From (1.2) it follows that

$$0 \leq \tau_{n+1} \leq x \Leftrightarrow \tau_n \in \bigcup_{i \in \mathbb{N}_+} \left[\frac{1}{x+i}, \frac{1}{i}\right].$$

Thus, if we put $F_n(x) := \lambda(\{\xi \in [0,1)\,;\, \tau^n(\xi) \leq x\})$, $n \geq 0$, then

$$F_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \left(F_n\left(\frac{1}{i}\right) - F_n\left(\frac{1}{x+i}\right)\right), n \geq 0, \tag{2.1}$$

relation called the Gauss–Kuzmin type equation.

3. Important Result

Let $B(I)$ the Banach space of all bounded measurable functions $f : I \to \mathbb{C}$, $I := [0,1]$.

Proposition. *If $f \in B(I)$ is non-decreasing, then Uf is non-increasing, where U is the Perron–Frobenius operator of τ under γ (γ is the Gauss' measure which is defined on $\mathcal{B}_{[0,1]}$ -σ -algebra of Borel sets in$[0,1]$, by $\gamma(A) = \frac{1}{\log 2}\int\limits_A \frac{dx}{x+1}$, $A \in \mathcal{B}_{[0,1]}$).*

Proof. Let f be a non–decreasing function. Thus, if $x < y$, then $f(x) \leq f(y)$. We evaluate the difference $Uf(y) - Uf(x)$. We have, $Uf(y) = \sum\limits_{i \in \mathbb{N}_+} P_i(y) f\left(\frac{1}{y+i}\right)$ and $Uf(x) = \sum\limits_{i \in \mathbb{N}_+} P_i(x) f\left(\frac{1}{x+i}\right)$, where $P_i(x) = \frac{x+1}{(x+i)(x+i+1)}$. Thus, $Uf(y) - Uf(x) = S_1 + S_2$, where

$$S_1 = \sum_{i \in \mathbb{N}_+} P_i(y)\left(f\left(\frac{1}{y+i}\right) - f\left(\frac{1}{x+i}\right)\right), \quad S_2 = \sum_{i \in \mathbb{N}_+} (P_i(y) - P_i(x)) f\left(\frac{1}{x+i}\right).$$

Since f is non–decreasing, and $\frac{1}{x+i} > \frac{1}{y+1}$, then $f\left(\frac{1}{x+i}\right) \geq f\left(\frac{1}{y+1}\right)$. Thus, $S_1 \leq 0$. We will show that $S_2 \leq 0$ too. We have that $\sum\limits_{i \in \mathbb{N}_+} P_i(u) = 1$, $u \in I$, and therefore we obtain:

$$S_2 = \sum_{i \in \mathbb{N}_+} (P_i(y) - P_i(x)) f\left(\frac{1}{x+i}\right) - \sum_{i \in \mathbb{N}_+} (P_i(y) - P_i(x)) f\left(\frac{1}{x+1}\right) - =$$

$$= -\sum_{i \in \mathbb{N}_+} \left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+i}\right)\right)(P_i(y) - P_i(x)).$$

Now, it is easy to show that the function P_1 is decreasing, while the functions P_i, $i \geq 2$, are all increasing. Also,

$$f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+i}\right) \geq f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right) \geq 0, \quad i \geq 2.$$

Therefore,

$$S_2 = -\sum_{i\geq 2}\left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+i}\right)\right)(P_i(y) - P_i(x)) \leq$$
$$\leq -\left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right)\right)\sum_{i\geq 2}(P_i(y) - P_i(x)) \leq 0$$

Thus, $Uf(y) - Uf(x) \leq 0$.

4. The Gauss–Kuzmin Theorem

We will give a simple proof that $F_n(x) = \frac{\log(x+1)}{\log 2} + \mathcal{O}(q^n)$, where $0 < q < 1$ or, to be exactly, $q = 0.7594...$. In fact, we will proof the following:

Theorem. *Let $f_0(x)$ be any twice differentiable function defined on $[0,1]$ with $f_0(0) = 0$ and $f_0(1) = 1$. Let the sequence of functions $f_1(x), f_2(x), \ldots$ be defined b the recursion formula*

$$f_{n+1}(x) = \sum_{i\in\mathbb{N}_+}\left(f_n\left(\frac{1}{i}\right) - f_n\left(\frac{1}{x+i}\right)\right).$$

Then

$$f_n(x) = \frac{\log(x+1)}{\log 2} + \mathcal{O}(q^n),$$

where $0 < q < 1$ or, to be exactly, $q = 0.7594...$.

It is clear that for $f_0(x) = x = F_0(x)$, this theorem will establish Gauss' claim and provide an answer to his problem.

Proof. Instead of studying $f_n(x)$ directly, we look at the derivative:

$$f'_{n+1}(x) = \sum_{i\in\mathbb{N}_+}\frac{1}{(x+i)^2}f'_n\left(\frac{1}{x+i}\right) \tag{4.1}$$

Let us introduce another sequence of functions $g_0, g_1, \ldots$ defined by $g_n(x) = (x+1)f'_n(x)$. Then the recursion formula (4.1) is transformed into

$$\frac{g_{n+1}(x)}{x+1} = \sum_{i\in\mathbb{N}_+}\frac{1}{(x+i)^2}\frac{g_n\left(\frac{1}{x+i}\right)}{\frac{1}{x+i}+1} = \sum_{i\in\mathbb{N}_+}\frac{1}{(x+i)(x+i+1)}g_n\left(\frac{1}{x+i}\right) \Rightarrow$$
$$g_{n+1}(x) = \sum_{i\in\mathbb{N}_+}\frac{x+1}{(x+i)(x+i+1)}g_n\left(\frac{1}{x+i}\right) = \sum_{i\in\mathbb{N}_+}P_i(x)\,g_n\left(\frac{1}{x+i}\right) = Ug_n,$$

where $P_i(x) = \frac{x+1}{(x+i)(x+i+1)}$, $i \in \mathbb{N}_+$, and U is the Perron–Frobenius operator of τ under γ.

If we can show that $g_n(x) = \frac{1}{\log 2} + \mathcal{O}(q^n)$, then an integration will establish the theorem for $f_n(x)$, because integrating $\frac{1}{x+1}$ will give $\log(x+1)$ term together with a bounded expression on a bounded interval times the $\mathcal{O}(q^n)$ error term, which will remain $\mathcal{O}(q^n)$. To demonstrate that $g_n(x)$ has this desired form, it suffices to establish that $g_n'(x) = \mathcal{O}(q^n)$, as the $\frac{1}{\log 2}$ constant in $g_n(x)$ will follow from the normalization requirement that $f_0(0) = 0$ and $f_0(1) = 1$. We have:

$$P_i(x) = \frac{x+1}{(x+i)(x+i+1)} = \frac{i}{x+i+1} - \frac{i-1}{x+i},$$

thus

$$\begin{aligned}
g_{n+1}(x) &= \sum_{i\in\mathbb{N}_+}\left(\frac{i}{x+i+1} - \frac{i-1}{x+i}\right) g_n\left(\frac{1}{x+i}\right) \Leftrightarrow \\
\Leftrightarrow g_{n+1}'(x) &= -\sum_{i\in\mathbb{N}_+}\left(\frac{i}{(x+i+1)^2} - \frac{i-1}{(x+i)^2}\right) g_n\left(\frac{1}{x+i}\right) + \\
&+ \frac{x+1}{(x+i)(x+i+1)}\frac{1}{(x+i)^2} g_n'\left(\frac{1}{x+i}\right) \\
\Leftrightarrow g_{n+1}'(x) &= -\sum_{i\in\mathbb{N}_+}\left(\frac{i}{(x+i+1)^2}\right)\left(g_n\left(\frac{1}{x+i}\right) - g_n\left(\frac{1}{x+i+1}\right)\right) - \\
&- \sum_{i\in\mathbb{N}_+}\frac{P_i(x)}{(x+i)^2} g_n'\left(\frac{1}{x+i}\right). \qquad (4.2)
\end{aligned}$$

By applying the mean value theorem of calculus to the difference

$$g_n\left(\frac{1}{x+i}\right) - g_n\left(\frac{1}{x+i+1}\right),$$

we obtain

$$g_n\left(\frac{1}{x+i}\right) - g_n\left(\frac{1}{x+i+1}\right) = \left(\frac{1}{x+i} - \frac{1}{x+i+1}\right) g_n'\left(\frac{1}{x+\theta_i}\right),$$

where $1 < \theta_i < i$. Thus, from (4.2), we have:

$$g_{n+1}'(x) = -\sum_{i\in\mathbb{N}_+}\frac{i}{(x+i)(x+i+1)^3} g_n'\left(\frac{1}{x+\theta_i}\right) - \sum_{i\in\mathbb{N}_+}\frac{P_i(x)}{(x+i)^2} g_n'\left(\frac{1}{x+i}\right). \qquad (4.3)$$

Let M_n be the maximum of $|g_n'(x)|$ on $[0,1]$, i.e. $M_n = \max_{x\in[0,1]} |g_n'(x)|$. Then, from (4.3), we have that:

$$M_{n+1} \leq M_n \cdot \max_{x\in[0,1]}\left(\sum_{i\in\mathbb{N}_+}\frac{i}{(x+i)(x+i+1)^3} + \sum_{i\in\mathbb{N}_+}\frac{P_i(x)}{(x+i)^2}\right). \qquad (4.4)$$

We now must calculate the maximum value of the sums in this expression. To this end, define function h by

$$h(x) = \sum_{i \in \mathbb{N}_+} P_i(x) \frac{1}{(x+i)^2}, \quad x \in [0,1].$$

Note that for $\varphi(x) = x^2$, $x \in [0,1]$, we have $h(x) = U\varphi(x)$. Since φ is increasing, and using the proposition from Section 3, we have that h is decreasing. Hence, $h(x) \le h(0)$, and since $h(0) = \sum_{i \in \mathbb{N}_+} \frac{P_i(0)}{i^2} = \sum_{i \in \mathbb{N}_+} \frac{1}{i^3(i+1)}$. Therefore, (4.4) become:

$$\begin{aligned} M_{n+1} &\le M_n \cdot \sum_{i \in \mathbb{N}_+} \left(\frac{i}{i(i+1)^3} + \frac{1}{i^3(i+1)} \right) = \\ &= M_n \cdot \sum_{i \in \mathbb{N}_+} \left(\frac{1}{i^3} - \frac{1}{i^2} + \frac{1}{i} - \frac{1}{i+1} + \frac{1}{(i+1)^3} \right) = \\ &= M_n \cdot (\zeta(3) - \zeta(2) + 1 + \zeta(3) - 1) = M_n \cdot (2\zeta(3) - \zeta(2)), \end{aligned}$$

where $\zeta(n)$ denotes the Riemann zeta function. Hence,

$$2\zeta(3) - \zeta(2) = 0.7594798\ldots .$$

Thus, $M_{n+1} < M_n \cdot q$, where $q = 0.7594798\ldots$, and $g'_{n+1}(x) = \mathcal{O}(q^{n+1})$, which proves the theorem.

References

[1] M. Iosifescu, (1992) *A very simple proof of a generalization of the Gauss-Kuzmin-Lévy theorem on continued fractions, and related questions*, Rev.Roumaine Math. Pures Appl. 37, 901–914.

[2] M. Iosifescu, C. Kraaikamp, (2002) *Metrical theory of continued fractions*, Dordrecht: Kluwer Academic Publisher.

[3] R.O. Kuzmin, (1928) *Sur un probléme de Gauss*, Atti Congr. Bologne.

[4] A.M. Rockett, P. Szüsz, (1992) *Continued fractions*, World Scientific, Singapore.

[5] P. Szüsz, (1961) *Über einen Kusminschen Satz*, Acta Math. Acad. Sci. Hungar.

[6] E. Wirsing, (1974) *On the theorem of Gauss-Kuzmin-Lévy and Frobenius-type theorem for function space*, Acta Arith.