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A NOTE ON RELLICH TYPE INEQUALITY

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Abstract. In the present note an inequality of Rellich type involving functions of several variables and their first and second order partial derivatives is established.

1. INTRODUCTION

In [9] F. Rellich proved the following inequality

$$\int_{R^n} |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{R^n} |x|^{-4} |u|^2 dx, \tag{1}$$

where u is a function in $C_0^\infty(R^n - \{0\})$ which is not identically zero and $n \neq 2$.

The inequalities of this type have significant applications in the theory of partial differential equations. In [10] Schmincke established an extension of (1) in exploring self-adjointness criteria for Schrödinger operator. Another extension of (1) was proved by Allegretto [2] in dealing with elliptic equations of order 2n. For other interesting extensions of (1), see the recent papers by Lewis [6] and Bennett [3]. The aim of the present note is to establish an inequality of Rellich type which will allow for a broader range of application. The analysis used in the proof is elementary and based on the idea used by Schmincke [10] to obtain an extension of Rellich's inequality.

2. BASIC INEQUALITY

Throughout we assume that H is an open, connected subset of R^n that is not necessarily bounded, and that the boundary of H, ∂H, is sufficiently smooth in order that the Green formulas applies. A point in R^n is denoted by $x=(x_1,\cdots,x_n)$ and its norm is given by $|x|=(\sum_{i=1}^{n}|x_i|^2)^{\frac{1}{2}}$. We denote by $C^m(H)$ the vector space consisting of all functions ϕ which, together with all their partial derivative $D^\alpha\phi$ of orders $|\alpha|\leq m$ are continuous on H and denote by $C_0^\infty(H)$ the vector space of infinitely differentiable functions with compact support (see, [1, p.9]).

In this section, we will prove an inequality that will be crucial in proving our main result in the next section. This inequality is given in the following theorem.

THEOREM 1. Let $p\geq 2$ be a constant, $g\in C^2(H)$, $\Delta g\neq 0$ in H and $u_r\in C_0^\infty(H)$, $r=1,\cdots,N$ be real valued functions. Then

(2)
$$\int_H |\Delta g|\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{p}{p-1}}dx$$
$$\leq\left(\frac{2p}{p-1}\right)^{\left(\frac{2p}{p-1}\right)}\int_H |\Delta g|^{-\left(\frac{p+1}{p-1}\right)}|\nabla g|^{\left(\frac{2p}{p-1}\right)}\left\{\sum_{r=1}^{N}|\nabla u_r|^2\right\}^{\frac{p}{p-1}}dx,$$

where $\nabla=\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right)$ and $\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$.

Proof. By applying Green's first formula to $\int_H |\Delta g|\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{p}{p-1}}dx$ and using the definition $\operatorname{sgn}(\Delta g)=\frac{\Delta g}{|\Delta g|}$ we observe that

(3)
$$\int_H |\Delta g|\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{p}{p-1}}dx=-\operatorname{sgn}(\Delta g)\int_H(\nabla g)\nabla\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{p}{p-1}}dx$$
$$\leq\int_H|\nabla g||\nabla\left\{\sum_{r=1}^{N}|\nabla u_r|^2\right\}^{\frac{p}{p-1}}|\,dx.$$

By simple calculation it is easy to see that

$$(4) \qquad \left| \nabla \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} \right| \leq \left(\frac{2p}{p-1} \right) \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p+1}{2(p-1)}} \left\{ \sum_{r=1}^{N} |\nabla u_r|^2 \right\}^{\frac{1}{2}} .$$

Using (4) in (3) and applying the Hölder's inequality with indices $\frac{2p}{p+1}$, $\frac{2p}{p-1}$ we have

$$(5) \qquad \int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx \leq \left(\frac{2p}{p-1} \right) \int_H \left[|\Delta g|^{\frac{p+1}{2p}} \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p+1}{2(p-1)}} \right]$$

$$\cdot \left[|\Delta g|^{-\left(\frac{p+1}{2p}\right)} |\nabla g| \left\{ \sum_{r=1}^{N} |\nabla u_r|^2 \right\}^{\frac{1}{2}} \right] dx$$

$$\leq \left(\frac{2p}{p-1} \right) \left\{ \int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx \right\}^{\frac{p+1}{2p}}$$

$$\cdot \left\{ \int_H |\Delta g|^{-\left(\frac{p+1}{p-1}\right)} |\nabla g|^{\frac{2p}{p-1}} \left\{ \sum_{r=1}^{N} |\nabla u_r|^2 \right\}^{\frac{p}{p-1}} dx \right\}^{\frac{p+1}{2p}} .$$

If $\int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx = 0$, then (2) is trivially true, otherwise we divide both sides of (5) by $\left\{ \int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx \right\}^{\frac{p+1}{2p}}$ and raise both sides to the power $\frac{2p}{p-1}$ to get the inequality (2). The proof is complete.

Remark 1. We note the inequality obtained in (2) is a variant of the Friedrichs inequality given in [3, p.989]. By rewriting (5) as

$$(6)\qquad \int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx \le \left(\frac{2p}{p-1}\right) \int_H \left[|\Delta g|^{\frac{1}{p}} \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{1}{p-1}} \right]$$

$$\cdot \left[|\Delta g|^{-\frac{1}{p}} |\nabla g| \left\{ \left(\sum_{r=1}^{N} |u_r|^2 \right) \left(\sum_{r=1}^{N} |\nabla u_r|^2 \right) \right\}^{\frac{1}{2}} \right] dx,$$

and applying the Hölder's inequality with indices p, $\frac{p}{p-1}$ on the right side of (6) and following the last arguments in the proof of Theorem 1 with suitable modifications, we get the following Dubinskii type inequality (see, [4,p.168]):

$$(7)\qquad \int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx$$

$$\le \left(\frac{2p}{p-1}\right)^{\frac{p}{p-1}} \cdot \int_H |\Delta g|^{-\frac{1}{p-1}} |\nabla g|^{\frac{p}{p-1}} \left\{ \left(\sum_{r=1}^{N} |u_r|^2 \right) \left(\sum_{r=1}^{N} |\nabla u_r|^2 \right) \right\}^{\frac{p}{2(p-1)}} dx .$$

Furthermore, by raising both sides of (2) and (7) to the power $\frac{2m(p-1)}{p}$ and applying the Hölder's inequality with indices $\frac{2m(p-1)}{p}$, $\frac{2m(p-1)}{2m(p-1)-2}$ suitably on the right sides of the resulting inequalities we get respectively the following Sobolev-Lieb-Thirring type inequalities (see, [5,7,8]):

$$(8)\qquad \left[\int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx \right]^{\frac{2m(p-1)}{p}}$$

$$\leq \left(\frac{2p}{p-1}\right)^{4m} \{D(H)\}^{\frac{2m(p-1)}{p}}$$

$$\cdot \int_H |\Delta g|^{-\frac{2m(p+1)}{p}} |\nabla g|^{4m} \left\{ \sum_{r=1}^{N} |\nabla u_r|^2 \right\}^{2m} dx \ ,$$

and

$$(9) \qquad \left[\int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx \right]^{\frac{2m(p-1)}{p}}$$

$$\leq \left(\frac{2p}{p-1}\right)^{2m} \{D(H)\}^{\frac{2m(p-1)-p}{p}}$$

$$\cdot \int_H |\Delta g|^{-\frac{2m}{p}} |\nabla g|^{2m} \left\{ \left(\sum_{r=1}^{N} |u_r|^2 \right) \left(\sum_{r=1}^{N} |\nabla u_r|^2 \right) \right\}^m dx,$$

where $p \geq 2$, $m \geq 1$ are constants and $D(H)$ is the n-dimensional measure of H.

3. MAIN RESULT.

Our main result is given in the following theorem.

THEOREM 2. Let p, g, u_r be as defined in Theorem 1. Then for any constants $\delta \geq 0$, $\epsilon > 0$,

$$(10) \qquad \int_H |\Delta g|^{-\left(\frac{p+1}{p-1}\right)} |g|^{\frac{2p}{p-1}} \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx$$

$$\leq -\epsilon \left[2 + \frac{4}{p-1} \epsilon^{-\frac{1}{p-1}} \right] \cdot \int_H |\Delta g|^{-\frac{1}{p-1}} |g|^{\frac{p}{p-1}} \left(\sum_{r=1}^{N} |u_r|^2 \right)^{\frac{p}{p-1}} dx$$

$$-\delta\epsilon\int_H|\Delta g|^{-(\frac{p+1}{p-1})}|\nabla g|^{\frac{2p}{p-1}}\left\{\sum_{r=1}^N|\nabla u_r|^2\right\}^{\frac{p}{p-1}}dx$$

$$+\epsilon\left[1-\epsilon-\frac{4}{p-1}+\delta\left\{\frac{2p}{p-1}\right\}^{\frac{2p}{p-1}}\right]\int_H|\Delta g|\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{p}{p-1}}dx,$$

where ∇ and Δ are as defined in Theorem 1.

Proof. Let A,B,C,D denote the integrals (without the exterior constants) in (10) successively. Applying Green's second formula to $\int_H|\Delta g|\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{p}{p-1}}dx$ and using the definition $\operatorname{sgn}(\Delta g)=\frac{\Delta g}{|\Delta g|}$ we observe that

$$D=\operatorname{sign}(\Delta g)\int_H|\Delta g|\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{p}{p-1}}dx\,. \tag{11}$$

By the simple partial differentiation we have the following identity

$$\begin{aligned}\Delta\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{p}{p-1}}&=\left(\frac{2p}{p-1}\right)\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{1}{p-1}}\sum_{r=1}^N|u_r|\Delta u_r\operatorname{sgn}u_r\\&+\left(\frac{2p}{p-1}\right)\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{1}{p-1}}\sum_{r=1}^N|\nabla u_r|^2\operatorname{sgn}u_r\\&+\frac{4p}{(p-1)^2}\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{-p+2}{p-1}}\sum_{i=1}^n\left\{\sum_{r=1}^N|u_r|\frac{\partial u_r}{\partial x_i}\operatorname{sgn}u_r\right\}^2.\end{aligned} \tag{12}$$

Using (12) in (11) and applying Schwarz inequality for sum we see that

$$D\le\left(\frac{2p}{p-1}\right)\int_H|g|\left\{\sum_{r=1}^N|u_r|^2\right\}^{\frac{1}{p-1}}\sum_{r=1}^N|u_r|\,|\Delta u_r|dx \tag{13}$$

$$+\left(\frac{2p}{p-1}\right)\int_H |g|\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{1}{p-1}}\sum_{r=1}^{N}|\nabla u_r|^2\,dx$$

$$+\frac{4p}{(p-1)^2}\int_H |g|\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{-p+2}{p-1}}\sum_{i=1}^{n}\left(\sum_{r=1}^{N}|u_r|^2\right)\left(\sum_{r=1}^{N}\left|\frac{\partial u_r}{\partial x_i}\right|^2\right)dx.$$

Let I_1, I_2, I_3 denote the integrals (without the exterior constants) on the right in (13) successively. From the definition of I_1 and applying Young's inequality with indices p, $\frac{p}{p-1}$, Schwarz inequality first for sum and then Schwarz inequality for integrals we observe that

$$(14)\qquad I_1=\int_H\left[|\Delta g|^{\frac{1}{p}}\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{1}{p-1}}\right]\left[|\Delta g|^{-\frac{1}{p}}|g|\sum_{r=1}^{N}|u_r||\Delta u_r|\right]dx$$

$$\leq\int_H\left(\frac{1}{p}|\Delta g|\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{p}{p-1}}\right.$$

$$\left.+\left(\frac{p-1}{p}\right)|\Delta g|^{-\frac{1}{p-1}}|g|^{\frac{p}{p-1}}\left\{\sum_{r=1}^{N}|u_r||\Delta u_r|\right\}^{\frac{p}{p-1}}\right)dx$$

$$\leq\frac{1}{p}D+\left(\frac{p-1}{p}\right)\int_H|\Delta g|^{-\frac{1}{p-1}}|g|^{\frac{p}{p-1}}\left\{\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{1}{2}}\cdot\left\{\sum_{r=1}^{N}|\Delta u_r|^2\right\}^{\frac{1}{2}}\right\}^{\frac{p}{p-1}}dx$$

$$=\frac{1}{p}D+\left(\frac{p-1}{p}\right)\int_H\left[|\Delta g|^{\frac{1}{2}}\left\{\sum_{r=1}^{N}|u_r|^2\right\}^{\frac{p}{2(p-1)}}\right]\times$$

$$\left[|\Delta g|^{-\left(\frac{p+1}{2(p-1)}\right)} |g|^{\frac{p}{p-1}} \left\{ \sum_{r=1}^{N} |\Delta u_r|^2 \right\}^{\frac{p}{2(p-1)}} \right] dx$$

$$\leq \frac{1}{p} D + \left(\frac{p-1}{p}\right) \left\{ \int_H |g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{(p-1)}} dx \right\}^{\frac{1}{2}} \times$$

$$\left\{ \int_H |\Delta g|^{-\left(\frac{p+1}{(p-1)}\right)} |g|^{\frac{2p}{p-1}} \left\{ \sum_{r=1}^{N} |\Delta u_r|^2 \right\}^{\frac{p}{p-1}} dx \right\}^{\frac{1}{2}}$$

$$= \frac{1}{p} D + \left(\frac{p-1}{p}\right) D^{\frac{1}{2}} A^{\frac{1}{2}} .$$

By rewriting I_2 and applying Young's inequality with indices p, $\frac{p}{p-1}$ we have

$$(15) \qquad I_2 = \int_H \left[|g|^{\frac{1}{p}} \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{1}{p-1}} \right] \left[|\Delta g|^{-\frac{1}{p}} |g| \sum_{r=1}^{N} |\nabla u_r|^2 \right] dx$$

$$\leq \int_H \left(\frac{1}{p} |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} \right.$$

$$\left. + \left(\frac{p-1}{p}\right) |\Delta g|^{-\frac{1}{p-1}} |g|^{\frac{p}{p-1}} \left\{ \sum_{r=1}^{N} |\nabla u_r|^2 \right\}^{\frac{p}{p-1}} \right) dx$$

$$= \frac{1}{p} D + \left(\frac{p-1}{p}\right) B.$$

Rewriting I_3 and applying Hölder's inequality with indices p, $\frac{p}{p-1}$ we have

(16) $$I_3 = \int_H \left[|\Delta g|^{\frac{1}{p}} \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{1}{p-1}} \right] \left[|\Delta g|^{-\frac{1}{p}} |g| \sum_{r=1}^{N} |\nabla u_r|^2 \right] dx$$

$$\le \left\{ \int_H |\Delta g| \left\{ \sum_{r=1}^{N} |u_r|^2 \right\}^{\frac{p}{p-1}} dx \right\}^{\frac{1}{p}} \times$$

$$\left\{ \int_H |\Delta g|^{-\frac{1}{p-1}} |g|^{\frac{p}{p-1}} \left\{ \sum_{r=1}^{N} |\nabla u_r|^2 \right\}^{\frac{p}{p-1}} dx \right\}^{\frac{p-1}{p}}$$

$$= D^{\frac{1}{p}} B^{\frac{p-1}{p}}.$$

Now using (14)-(16) in (13) and applying the elementary inequality $2ab \le a^2 + b^2$ (a,b reals) and Young's inequality with indices p, $\frac{p}{p-1}$ we observe that

(17) $$D \le \left(\frac{4}{p-1}\right) D + 2D^{\frac{1}{2}} A^{\frac{1}{2}} + 2B + \frac{4p}{(p-1)^2} D^{\frac{1}{p}} B^{\frac{p-1}{p}}$$

$$= \left(\frac{4}{p-1}\right) D + 2(\epsilon^{\frac{1}{2}} D^{\frac{1}{2}})(\epsilon^{-\frac{1}{2}} A^{\frac{1}{2}}) + 2B + \frac{4p}{(p-1)^2} (\epsilon^{\frac{1}{p}} D^{\frac{1}{p}})(\epsilon^{-\frac{1}{p}} B^{\frac{p-1}{p}})$$

$$\le \left(\frac{4}{p-1}\right) D + \epsilon D + \frac{1}{\epsilon} A + 2B + \frac{4p}{(p-1)^2} \left[\frac{1}{p} \epsilon D + \left(\frac{p-1}{p}\right) \epsilon^{-\frac{1}{p-1}} B \right]$$

$$= \left[\epsilon + \frac{4}{p-1} + \frac{4\epsilon}{(p-1)^2} \right] D + \frac{1}{\epsilon} A + \left[2 + \frac{4}{(p-1)} \epsilon^{-\frac{1}{p-1}} \right] B,$$

for $\epsilon > 0$. Now for any $\delta \ge 0$, from (2) we observe that

(18) $$\delta C - \delta \left(\frac{2p}{p-1}\right)^{-\left(\frac{2p}{p-1}\right)} D \ge 0.$$

Combining this fact with (17) we have

$$D \le \left[\epsilon + \frac{4[\epsilon+(p-1)]}{(p-1)^2}\right] D + \tfrac{1}{\epsilon}A + \left[2 + \frac{4}{(p-1)}\epsilon^{-\frac{1}{p-1}}\right] B \tag{19}$$

$$+ \delta C - \delta\left(\frac{2p}{p-1}\right)^{-\left(\frac{2p}{p-1}\right)} D,$$

for all $\epsilon > 0$, $\delta \geq 0$. Rewriting (19) we get the desired inequality in (10). The proof of the theorem is complete.

Remark 2. We note that in the special cases, when (i) $N=1$, $u_1 = u$ and (ii) $N=1$, $u_1 = u$ and $p=2$, the inequality (10) reduces to the inequalities which we believe are new to the literature. If we specialize the inequality (10) by taking $N=1$, $u_1 = u$ and then by putting $g = |x|^{\alpha+2}$, $\alpha \geq 0$ real constant and hence $|\nabla g|^2 = (\alpha+2)^2 |x|^{2\alpha+2}$, $\Delta g = (\alpha+n)(\alpha+2)|x|^{\alpha}$, we get an inequality similar to that of inequality given by Bennett in [3, p. 992].

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