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Brezis-Browder Principles In Separable Ordered Sets

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Abstract. The Brezis-Browder principle [Adv. Math., 21(1976), 355-364] may be viewed as a variant of the Zorn-Bourbaki maximality result for separable structures.

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1 Introduction

Let M be some nonempty set; and $(\leq)$, some *quasi-order* (i.e.: reflexive and transitive relation) over it. Further, let $x \vdash \varphi(x)$ stand for a function from M to $R_+ := [0, \infty[$. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w). \tag{1.1}$$

A basic result involving such points is the 1976 Brezis-Browder ordering principle [6]:

Theorem BB. *Suppose that*

$$\begin{array}{l} (M, \leq) \textit{ is sequentially inductive:} \\ \textit{each ascending sequence in } M \textit{ has an upper bound} \end{array} \tag{1.2}$$

$$\varphi \textit{ is } (\leq)\textit{-decreasing } (x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)). \tag{1.3}$$

Then, for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$.

This principle, including the well known Ekeland's [7,8], found some useful applications to convex and nonconvex analysis; we refer to the quoted papers for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of Theorem BB were proposed; see, for instance, Altman [2], Turinici [21], Anisiu [3] or Kang and Park [14]. Here, we shall concentrate on the *structural* way of enlargement. This, roughly speaking, consists of $(R_+, \geq)$ being substituted by an ordered structure $(P, \leq)$ endowed with countable regularity properties for its chains. Some basic results in this area were obtained by Gajek and Zagrodny [10]; see also Zhu, Fan and Zhang [24]. It is our aim in the following to show that a simplification of these facts is possible; details will be given in Section 4 (the transitive case) and Section 5 (the amorph case). The specific tool for deducing these is a variant of the Zorn-Bourbaki maximality principle for separable structures given in Section 3. All preliminary material involving such objects is presented in Section 2. Further discussions about these questions will be performed elsewhere.

2 Separable Ordered Sets

(A) Let W stand for the class of ordinal numbers, introduced in a "factorial" way; cf. Kuratowski and Mostowski [16, Ch 7, Sect 2]. Precisely, given a partially ordered structure $(P, \leq)$, call it *well ordered* if each (nonempty) part of P admits a first element. Given the couple $(P, \leq)$, $(Q, \leq)$ of such objects, put

$$(P, \leq) \equiv (Q, \leq) \text{ iff there exists a strictly increasing bijection:} P \to Q.$$

This is an equivalence relation; the order type of $(P, \leq)$ (denoted $\text{ord}(P, \leq)$) is just its equivalence class; also referred to as an *ordinal*.

Note that W is not a set, as results from the Burali-Forti paradox; cf. Sierpinski [19, Ch 14, Sect 2]. However, when one restricts to a *Grothendieck universe* $\mathcal{G}$ (taken as in Hasse and Michler [11, Ch 1, Sect 2]) this contradictory character is removed for the class $W(\mathcal{G})$ of all *admissible* (modulo $\mathcal{G}$) ordinals (generated by (non-contradictory) well ordered parts of $\mathcal{G}$). In the following, we drop any reference to $\mathcal{G}$, for simplicity. So, by an *ordinal* in W one actually means a $\mathcal{G}$-admissible ordinal with respect to a "sufficiently large" Grothendieck universe $\mathcal{G}$. Clearly,

$$\xi\text{=admissible ordinal and } \eta \leq \xi \text{ imply } \eta\text{=admissible ordinal.}$$

Hence, in the formulae

$$W(\alpha) = \{\xi \in W; \xi < \alpha\}, \quad W[\alpha] = \{\xi \in W; \xi \leq \alpha\},$$

the symbol W in the brackets is the "absolute" class of all ordinals.

Now, an enumeration of W is realized via the immediate successor map of a subset $M \subseteq W$

$$\text{suc}(M) = \min\{\xi \in W; M < \xi\} \ \text{ (hence } \text{suc}(\alpha) = \alpha + 1, \ \forall \alpha \in W).$$

(Here, $M < \xi$ means: $\lambda < \xi, \forall \lambda \in M$). It begins with the natural numbers $0, 1, ...$; the set of all these is denoted by N. Their immediate successor is $\omega = \text{suc}(N)$ (the first transfinite ordinal); the next in this enumeration is $\omega + 1$, and so on.

In parallel to this, we may (construct and) enumerate the class of all admissible cardinals. Let P and Q be nonempty sets; we put

$$P \preceq Q (P \sim Q) \text{ iff there exists an injection (bijection):} P \to Q.$$

The former is a quasi-order; while the latter is an equivalence. Denote also

$$P \prec Q \text{ if and only if } P \preceq Q \text{ and } \neg(P \sim Q).$$

This relation is *irreflexive* ($\neg(P \prec P)$, for each P) and *transitive*; hence a *strict order*. Let $\alpha > 0$ be an (admissible) ordinal; we say that it is an (admissible) *cardinal* if

$$W(\xi) \prec W(\alpha), \quad \text{for each } \xi < \alpha.$$

The class of all these will be denoted by Z. Now, the enumeration we are looking for is realized via the immediate successor (in Z) map

$$\text{SUC}(M) = \min\{\eta \in Z; M < \eta\}, \quad M \subseteq Z.$$

Precisely, this begins with the natural numbers $0, 1, \ldots$. The immediate successor (in Z) of all these is $\omega = \text{SUC}(N)$ (the first transfinite cardinal). To describe the remaining ones, we may introduce via transfinite recursion the function $\lambda \vdash \aleph_\lambda$ from W to Z as

$$\begin{aligned}
&\aleph_0 = \omega; \quad \text{and, for each } \lambda > 0,\\
&\aleph_\lambda = \text{SUC}(\aleph_{\lambda-1}), \quad \text{if } \lambda - 1 \text{ exists}\\
&\aleph_\lambda = \text{SUC}\{\aleph_\xi; \xi < \lambda\}, \quad \text{if } \lambda - 1 \text{ does not exist.}
\end{aligned}$$

Note that, in such a case, the order structure of $Z(\omega, \leq) = \{\xi \in Z; \omega \leq \xi\}$ is completely reducible to the one of W; further details may be found in Sierpinski [op. cit., Ch 15, Sect 7].

Any nonempty part P with $P \prec W(\omega)$ ($P \sim W(\omega)$) is termed *finite* (*effectively countable*); the union of these ($P \preceq W(\omega)$) is referred to as P is *countable*. When $P = W(\xi)$, all such properties will be transferred to ξ.

Now, the immediate successor in Z of $\omega = \aleph_0$ is $\Omega = \aleph_1$ (the first uncountable ordinal). The motivation of our convention comes from

$$\xi \text{ is countable, for each } \xi < \Omega; \quad \text{but } \Omega \text{ is not countable.} \tag{2.1}$$

A basic consequence of this is precised in the statement below (to be found, e.g., in Alexandrov [1, Ch 3, Sect 4]):

Proposition 1. *The following are valid:*

i) *The ordinal Ω cannot be attained via sequential limits of countable ordinals. That is: if (α_n) is an ascending sequence of countable ordinals then*

$$\alpha = \sup_n(\alpha_n) (= \lim_n(\alpha_n)) \tag{2.2}$$

is countable too.

ii) *Each second kind countable ordinal is attainable via such sequences. In other words: if $\alpha < \Omega$ is of second kind then, there exists a strictly ascending sequence (α_n) of countable ordinals with the property (2.2).*

(B) Let M be a nonempty set; and $(\leq)$, some *order* (=antisymmetric quasi-order) on it. By a $(\leq)$-*chain* of M we shall mean any (nonempty) part A of M with $(A, \leq)$ being well ordered (see above). Note that any such object may be written as $A = \{a_\xi; \xi < \lambda\}$, where the net $\xi \vdash a_\xi$ is strictly ascending ($\xi < \eta \Longrightarrow a_\xi < a_\eta$); the uniquely determined ordinal λ is just $\text{ord}(A, \leq)$. Now, by the remark above

$$A \text{ is countable} \Longleftrightarrow \text{ord}(A, \leq) < \Omega.$$

If, moreover, $\mathrm{ord}(A, \leq) \leq \omega$, we say that A is *normally countable*. The following characterization of this concept is almost immediate.

Proposition 2. *The* $(\leq)$*-chain* A *is normally countable if and only if*

$$A = \{b_n; n < \omega\},\ \textit{where}\ n \vdash b_n\ \textit{is ascending}\ (p < q \Longrightarrow b_p \leq b_q). \tag{2.3}$$

Let P, Q be nonempty parts with $P \supseteq Q$. We say that P is *majorized* by Q (and write $P \propto Q$) provided

$$Q \text{ is cofinal in } P\ (\forall x \in P, \exists y \in Q \text{ with } x \leq y).$$

The $(\leq)$-chain $S \subseteq M$ is called *upper countable* in case

$$S \propto T, \text{ for some normally countable } (\leq)\text{-chain } T \subseteq S. \tag{2.4}$$

Clearly, this happens if S is normally countable. As a completion, we have

Proposition 3. *The generic relation holds*

$$(\forall (\leq)\textit{-chain})\ \textit{countable} \Longrightarrow \textit{upper countable}. \tag{2.5}$$

Hence, the $(\leq)$*-chain* $S \subseteq M$ *is upper countable if and only if*

$$S \propto T,\ \textit{for some countable}\ (\leq)\textit{-chain}\ T \subseteq S. \tag{2.6}$$

Proof. Let $S = \{s(\xi); \xi < \lambda\}$ be the representation of this $(\leq)$-chain where $\lambda := \mathrm{ord}(S, \leq) < \Omega$. If λ is a first kind ordinal, we are done; because $T = \{s(\lambda - 1)\}$ is then cofinal in S. Assume now λ is a second kind ordinal. By Proposition 1 there exists a strictly ascending sequence of ordinals (λ_n) with $\lambda = \sup_n(\lambda_n)$. But then, $T = \{s(\lambda_n); n < \omega\}$ is a normally countable $(\leq)$-chain (of S) cofinal in S; i.e., we are again done. ■

Remark. The reciprocal of (2.5) is not in general true; just take any $(\leq)$-chain S of M with $\Omega \leq \mathrm{ord}(S, \leq)=$ first kind ordinal.

(C) Let us now return to our initial setting. We say that the order structure $(M, \leq)$ is *separable* if (cf. Zhu, Fan and Zhang [24])

$$\text{any } (\leq)\text{-chain of } M \text{ is upper countable.} \tag{2.7}$$

For example, this holds (under (2.5)) whenever

$$(M, \leq) \text{ is } \textit{strongly separable}\text{: each } (\leq)\text{-chain of } M \text{ is countable.} \tag{2.8}$$

In fact, the reciprocal holds too; so that, we may formulate

Proposition 4. *Under these conventions,*

$$(\forall (M, \leq)\textit{=ordered structure})\ \textit{separable} \Longleftrightarrow \textit{strongly separable}. \tag{2.9}$$

Proof. Assume that $(M, \leq)$ is separable; and let $S = \{s(\xi); \xi < \lambda\}$ be some $(\leq)$-chain of M; where $\lambda := \mathrm{ord}(S, \leq)$. If, by absurd, S is not countable, we must have $\lambda \geq \Omega$. The initial segment (of S) $U = \{s(\xi); \xi < \Omega\}$ is not countable too; cf. (2.1). On the other hand, by hypothesis, U is upper countable; so, there exists a strictly ascending sequence $(\xi_n; n < \omega)$ of ranks in $W(\Omega)$ with

$$U \propto \{s(\xi_n); n < \omega\}; \quad \text{hence } \Omega = \lim_n(\xi_n).$$

This, however, cannot be accepted, in view of Proposition 1. Hence, S is countable; and the proof is complete. ■

In the following, we shall give some useful examples of such structures.

c1) Let $\mathcal{I}(M) := \{(x,x); x \in M\}$ stand for the identical relation over M. By an *almost uniformity* (on M) we shall mean any family $\mathcal{U}$ of parts in $M \times M$ with

$$\mathcal{I}(M) \subseteq U, \text{ for each } U \in \mathcal{U} \quad \text{(i.e.: } \mathcal{I}(M) \subseteq \cap\mathcal{U}).$$

Suppose that we fixed such an object. Call the (ascending) net $(a_\xi; \xi < \lambda)$, *$\mathcal{U}$-Cauchy*, when

$$\forall U \in \mathcal{U}, \exists \mu = \mu(U), \text{ such that } \mu \le \xi \le \eta \Longrightarrow (a_\xi, a_\eta) \in U.$$

Likewise, call the (ascending) sequence $(b_n; n < \omega)$, *$\mathcal{U}$-asymptotic*, in case

$$\forall U \in \mathcal{U}, \exists k = k(U), \text{ such that } n \ge k \Longrightarrow (b_n, b_{n+1}) \in U.$$

It is not hard to see that the global conditions below are equivalent

$$\text{each ascending net is } \mathcal{U}\text{-Cauchy} \tag{2.10}$$

$$\text{each ascending sequence is } \mathcal{U}\text{-asymptotic.} \tag{2.11}$$

By definition, either of these will be referred to as $\mathcal{U}$ is (strongly) *regular*.

Lemma 1. *Assume the almost uniformity $\mathcal{U}$ is (strongly) regular and*

$$\begin{array}{l}\mathcal{U} \textit{ is pseudo-metrizable: there exists a countable subfamily} \\ \mathcal{V} \subseteq \mathcal{U}, \textit{ cofinal in } (\mathcal{U}, \supseteq) \ (\forall U \in \mathcal{U}, \exists V \in \mathcal{V}: \ U \supseteq V)\end{array} \tag{2.12}$$

$$\mathcal{U} \textit{ is sufficient} \quad (\cap\mathcal{U} = \mathcal{I}(M)). \tag{2.13}$$

Then, $(M, \le)$ is (strongly) separable.

Proof. Without loss, one may assume $\mathcal{U}$ itself is countable; for, otherwise, we simply replace $\mathcal{U}$ by $\mathcal{V}$. The case of $\mathcal{U}$ being finite is clear; so, it remains to discuss the alternative of $\mathcal{U}$ being effectively countable ($\mathcal{U} = (U_n; n < \omega)$). Let S be some $(\le)$-chain in M. If there exists a last element $s = \max(S)$, we are done; so, without loss, one may assume that

$$\text{for each } x \in S \text{ there exists } y \in S \text{ with } x < y. \tag{2.14}$$

By the (strong) regularity of $\mathcal{U}$, it is not hard to see that (cf. Turinici [22])

$$\begin{array}{l}\forall x \in S, \forall U \in \mathcal{U}, \text{ there exists } y = y(x,U) \in S(x, \le) \\ \text{such that, for each } p, q \in S: \ y \le p \le q \Longrightarrow (p,q) \in U.\end{array}$$

[Here, for each $P \subseteq M$ and each relation $(*)$ over M we put

$$P(a,*) = \{x \in P; a * x\}, \ a \in P \quad \text{(the } a\text{-section of } (*) \text{ in } P)].$$

This, in turn, yields an ascending sequence $(x_n; n < \omega)$ in S with

$$\text{(for each } n\text{)} \quad p, q \in S,\ x_n \leq p \leq q \Longrightarrow (p, q) \in U_n. \tag{2.15}$$

We claim that $S \propto \{x_n; n < \omega\}$. In fact, if this were not true, $(x_n; n < \omega)$ admits an upper bound $u \in S$; and, by (2.14), there must be $v \in S$ with $u < v$ (hence $u \neq v$). But then, (2.15) yields (via (2.13)) $(u, v) \in \mathcal{I}(M)$; hence $u = v$, contradiction. This proves our claim; and completes the argument. ■

In particular, let $d : M \times M \to R_+$ be a *pseudometric* over M (in the sense: $d(x, x) = 0, \forall x \in M$). Then, the family $\mathcal{U}(d) = \{U_\varepsilon; \varepsilon > 0\}$, where

$$U_\varepsilon = \{(x, y) \in M \times M; d(x, y) < \varepsilon\}, \quad \varepsilon > 0,$$

is a pseudo-metrizable almost uniformity over M. In addition,

$$\mathcal{U} \text{ is sufficient iff } d \text{ is sufficient } (d(x, y) = 0 \Longrightarrow x = y).$$

A translation of Lemma 1 in terms of d is immediate; we do not give details.

c2) By a *topology* over M we mean, as usually, any family $\mathcal{T} \supseteq \{\emptyset, M\}$ of parts in M, invariant to arbitrary unions and finite intersections. Assume that we fixed such an object; and let "cl" stand for the associated *closure* operator. Any subfamily $\mathcal{B} \subseteq \mathcal{T}$ with the property that each $D \in \mathcal{T}$ is a union of members in $\mathcal{B}$, will be referred to as a *basis* for $\mathcal{T}$. If, in addition, $\mathcal{B}$ is countable, then $\mathcal{T}$ will be called *second countable.* Finally, term the ambient order $(\leq)$, *closed from the left* provided $M(x, \geq)$ is closed, for each $x \in M$.

Lemma 2. *Assume that $\mathcal{T}$ is second countable and $(\leq)$ is closed from the left. Then, $(M, \leq)$ is (strongly) separable.*

Proof. Let $\mathcal{B} = \{B_n; n < \omega\}$ stand for a countable basis of $\mathcal{T}$. Further, take some choice function "Ch" of the nonempty parts in M [$\text{Ch}(X) \in X$, for each $X \subseteq M, X \neq \emptyset$]. Given the arbitrary fixed $(\leq)$-chain S of M, denote $T = \{\text{Ch}(B \cap S); B \in \mathcal{B}\}$ (hence $T \subseteq S$). For the moment, T is countable (because $T \preceq \mathcal{B}$). In addition, we claim that $\text{cl}(T) \supseteq S$ [wherefrom, T is dense in S]. In fact, let s be some point of S; and U stand for an open neighborhood of it. By the definition of $\mathcal{B}$, U may be written as a union of members in this family; so

$$U \supseteq B \ni s \text{ (hence } U \ni \text{Ch}(B \cap S)), \quad \text{for some } B \in \mathcal{B};$$

and our claim follows. If T is cofinal in S, we are done (cf. Proposition 3). Otherwise, there must be some $s \in S$ with $T \subseteq M(s, \geq)$; wherefrom

$$S \subseteq \text{cl}(T) \subseteq \text{cl}(M(s, \geq)) = M(s, \geq);$$

i.e., $\{s\}$ is cofinal in S. The proof is thereby complete. ■

It remains now to establish under which conditions is $\mathcal{T}$, second countable. An appropriate answer is to be given in a metrizable context:

there exists a metric $d : M \times M \to R_+$
whose associated topology is just $\mathcal{T}$.

Then, e.g., the condition below yields the desired property for $\mathcal{T}$:

$$M \text{ has a countable dense subset } P \text{ (in the sense: } \mathrm{cl}(P) = M). \tag{2.16}$$

The proof is to be found in Bourbaki [5, Ch 9, Sect 2.8]; see also Alexandrov [op. cit., Ch 4, Sect 4]. Some related aspects may be found in Zhu, Fan and Zhang [24].

c3) Let R stand for the real axis. Denote by $(\leq, d)$ the usual order and metric. Take any (nonempty) part M of R with

$$M \text{ is bounded above} \quad (M \leq v, \text{ for some } v \in R). \tag{2.17}$$

The structure $(M, \leq)$ fulfills conditions of both Lemma 1 (with respect to the uniformity $\mathcal{U}(d)$) and Lemma 2 (with respect to the associated with d topology); wherefrom, $(M, \leq)$ is (strongly) separable. A similar conclusion is valid for the dual order $(\geq)$. Precisely, for each $M \subseteq R$ with

$$M \text{ is bounded below} \quad (M \geq u, \text{ for some } u \in R), \tag{2.18}$$

one has (by the same reasoning) that $(M, \geq)$ is (strongly) separable. This will be useful for our future developments.

3 Zorn-Bourbaki Principles

(A) Let M be a nonempty set; and $(\leq)$, some *order* (antisymmetric quasi-order) on it. Call the point $z \in M$, $(\leq)$-*maximal* in case

$$w \in M, z \leq w \Longrightarrow z = w; \text{ i.e.: } z < x \text{ is false, for each } x \in M. \tag{3.1}$$

(Here, $(<)$ is the strict order attached to $(\leq)$). Sufficient conditions for the existence of such elements may be obtained as follows. Call the (nonempty) part A of M, a *linear* $(\leq)$-*chain* provided $(A, \leq)$ is linearly ordered [$\forall x, y \in A$: either $x \leq y$ or $y \leq x$]; and a *(natural)* $(\leq)$-*chain*, when $(A, \leq)$ is well ordered [cf. Section 2].

Theorem ZB. *Suppose that one of the conditions below holds*

$$\textit{each linear } (\leq)\textit{-chain (of } M\textit{) is bounded above} \tag{3.2}$$

$$\textit{each } (\leq)\textit{-chain (of } M\textit{) is bounded above.} \tag{3.3}$$

Then, $(\leq)$ is a normal order, in the sense: for each $u \in M$ there exists a $(\leq)$-maximal $v \in M$ with $u \leq v$.

Some remarks are in order. The first explicit formulation of Theorem ZB in terms of (3.2) was given in 1914 by Hausdorff [12, Ch 6, Sect 1]; a slight different version of it was obtained in 1922 by Kuratowski [15]. Note that the quoted authors regarded Theorem ZB only as a handy tool in solving various existence problems in the setting of (AC)(= the Axiom of Choice). Finally, again under the lines of (3.2), we must mention the 1935 contribution due to Zorn [25]; who regarded Theorem ZB as an axiom. The version of this result involving (3.3) was stated in Bourbaki [4]; who also established its equivalence with the Well Ordering Principle in Zermelo [23] (equivalent with (AC)). For this reason, it is natural that Theorem ZB be referred to as the Zorn-Bourbaki principle. Note that, in the context of (AC),

$$(3.3) \Longrightarrow (3.2) \quad \text{(hence } (3.3) \Longleftrightarrow (3.2));$$

see also Felgner [9]. Further historical aspects may be found in Moore [17, Ch 4, Sect 4] and the references therein.

(B) Now, as results from the developments in Section 2, the verification of (3.3) for countable chains only will suffice (for its validity) in many concrete cases with a practical relevance. This suggests us considering maximality principles over (abstract) ordered structures with such regularity properties. So, let $(M, \leq)$ be a (partially) ordered set. Assume firstly that

$$\begin{array}{l} (M, \leq) \text{ is sequentially inductive: each normally countable} \\ (\leq)\text{-chain of } M \text{ is bounded from above (modulo } (\leq)). \end{array} \tag{3.4}$$

Note that, by Proposition 2, this notion is identical with the one of (1.2). Moreover, by Proposition 3, it may be also written as

$$\text{each countable } (\leq)\text{-chain of } M \text{ is bounded above (modulo } (\leq)). \tag{3.5}$$

Secondly, assume that (cf. Section 2)

$$\begin{array}{l} (M, \leq) \text{ is (strongly) separable: each } (\leq)\text{-chain } S \subseteq M \\ \text{is majorized by some countable } (\leq)\text{-chain } T \subseteq S. \end{array} \tag{3.6}$$

Remember that, by Proposition 4, this also reads

$$\text{each } (\leq)\text{-chain of } M \text{ is countable.} \tag{3.7}$$

Theorem 1. *Assume that (3.4)+(3.6) hold. Then, $(\leq)$ is a normal order (in the sense above).*

Proof. By the remarks involving (3.5)+(3.7), it is clear that Theorem ZB applies to these data; and, from this, we are done. ■

(C) Remember that, the regularity conditions in Theorem ZB are logically minimal so that its conclusion be retainable. (See the quoted papers for details). So, it is natural to ask whether this is also true for the conditions in Theorem 1. Two situations may occur.

i) Assume that in Theorem 1 condition (3.4) does not hold. By definition, there exists a strictly ascending sequence $K = \{x_n; n < \omega\}$ which is not bounded above in M. As a consequence, $(K, \leq)$ is not sequentially inductive; but it is (strongly) separable. This, added to $(K, \leq)$ having no $(\leq)$-maximal elements, proves the logical minimality of (3.4).

ii) Assume that, in Theorem 1, condition (3.6) does not hold. By definition, there must be a nonempty $(\leq)$-chain $L \subseteq M$ fulfiling (cf. Proposition 3)

$$L \propto Q \text{ is false, for each countable } (\leq)\text{-chain } Q \subseteq L.$$

As a consequence, the structure $(L, \leq)$ is sequentially inductive; but not (strongly) separable. This, added to $(L, \leq)$ having no $(\leq)$-maximal elements proves the logical minimality of (3.6).

Summing up, we proved

Proposition 5. *Either of the regularity conditions (3.4) and (3.6) in Theorem 1 is logically minimal for the conclusions given there to hold.*

4 Main Results

With these informations at hand, we may now return to the questions in Section 1. The natural setting for discussing them is the one of *transitive* relations. This, apart from giving us new useful forms of Theorem BB, allows a direct transition to the quasi-order and amorph cases.

(A) Let (M, ∇) and (P, ∇) be transitive structures. The relation over M

$$(x, y \in M) \;\; x < y \text{ iff } x\nabla y \text{ and } \neg(y\nabla x) \tag{4.1}$$

is *irreflexive* $(\neg(x < x), \forall x \in M)$ and transitive; hence a strict order. As a consequence, its completion

$$(x, y \in M) \;\; x\overline{<}y \text{ iff either } x < y \text{ or } x = y \tag{4.2}$$

is an order on M. Denote in the same way the strict/standard order on P attached to (∇). Further, let $\varphi : M \to P$ be some (∇, ∇)-*increasing* function

$$x\nabla y \Longrightarrow \varphi(x)\nabla\varphi(y) \text{ [equivalently: } \neg(\varphi(x)\nabla\varphi(y)) \Longrightarrow \neg(x\nabla y)]. \tag{4.3}$$

This allows us introducing the relation (in M)

$$(x, y \in M) \; x \sqsubset y \text{ iff } x\nabla y \text{ and } \neg(\varphi(y)\nabla\varphi(x)). \tag{4.4}$$

By the remark involving (4.3), one has

$$x \sqsubset y \text{ iff } x < y \text{ and } \varphi(x) < \varphi(y); \tag{4.5}$$

wherefrom, $(\sqsubset)$ is a strict order on M. Let $(\sqsubseteq)$ stand for the associated (by (4.2)) order in M. Note that, by the very definitions (and remarks) above

$$[(x \sqsubset y, y\nabla z) \text{ or } (x\nabla y, y \sqsubset z)] \text{ imply } x \sqsubset z. \tag{4.6}$$

In fact, assume e.g. that the former of these alternatives holds. As $x \sqsubset y \Longrightarrow x\nabla y$, one gets for the moment $x\nabla z$. If, by absurd, $\varphi(z)\nabla\varphi(x)$ then, combining with $\varphi(y)\nabla\varphi(z)$ (deductible via (4.3) and $y\nabla z$) gives $\varphi(y)\nabla\varphi(x)$; in contradiction with $x \sqsubset y$. Hence, $\neg(\varphi(z)\nabla\varphi(x))$; wherefrom, $x \sqsubset z$. The latter of these alternatives is handled in a similar way; and the claim follows.

Having these precised, call the point $z \in M$, $(\nabla, \nabla; \varphi)$-*maximal*, when

$$\text{for each } w \in M\text{: } z\nabla w \text{ implies } \varphi(w)\nabla\varphi(z). \tag{4.7}$$

Note that, if (M, ∇) is identical with (P, ∇), and (∇) is an order (on M) this concept reduces to the one in Section 3 (when φ=identity). Hence, maximality results of this type are not without interest for us. The basic step in deducing these is a characterization of our concept in terms of $(\sqsubseteq)$.

Lemma 3. *The generic relation is available*

$$(\forall z \in M): \quad (\nabla, \nabla; \varphi)\text{-maximal} \Longleftrightarrow (\sqsubseteq)\text{-maximal}. \tag{4.8}$$

Proof. It will suffice verifying that

$$z \text{ is not } (\nabla, \nabla; \varphi)\text{-maximal} \Longleftrightarrow z \text{ is not } (\sqsubseteq)\text{-maximal}.$$

The left part of this equivalence means

$$\exists w \in M \text{ such that: } z\nabla w, \neg(\varphi(w)\nabla\varphi(z)); \text{ hence } z \sqsubset w.$$

And, from this, the claim follows. ■

Now, as $(\sqsubseteq)$ is an order, the developments of Section 3 apply to $(M, \sqsubseteq)$; and this yields the following maximality principle to be used further.

Theorem 2. *Assume that*

$$(M, \sqsubseteq) \text{ is sequentially inductive (cf. (3.4))} \tag{4.9}$$

$$(M, \sqsubseteq) \text{ is (strongly) separable (cf. (3.6)).} \tag{4.10}$$

Then, for each $u \in M$ there exists $v \in M$ with

$$v \text{ is } (\nabla, \nabla; \varphi)\text{-maximal (cf. (4.7))} \tag{4.11}$$

in such a way that

$$u = v \text{ (hence } u \text{ is } (\nabla, \nabla; \varphi)\text{-maximal), whenever } M(u, \nabla) = \emptyset \tag{4.12}$$

$$u\nabla v, \quad \text{whenever } M(u, \nabla) \neq \emptyset. \tag{4.13}$$

Proof. By Theorem 1 (applicable, via (4.9)+(4.10)) it follows that, for each $u \in M$ there exists $v \in M$ with

$$u \sqsubseteq v \text{ (i.e.: either } u \sqsubset v \text{ or } u = v); \text{ and } v \text{ is } (\sqsubseteq)\text{-maximal.} \tag{4.14}$$

The latter of these yields (4.11), if one takes Lemma 3 into account. And the former of these gives the couple of alternatives (4.12)/(4.13). In fact, $M(u, \nabla) = \emptyset$ implies (4.12), in view of $[u \sqsubset v \Longrightarrow u\nabla v]$. Moreover, $M(u, \nabla) \neq \emptyset$ gives (4.13); for, in such a case, (4.14) holds with some $u_1 \in M(u, \nabla)$ in place of u. The proof is thereby complete. ■

It remains now to give sufficient conditions (involving our initial data) under which (4.9)+(4.10) be fulfilled. This necessitates further conventions and auxiliary facts. Let (x_n) be a sequence in M; we call it *ascending* (modulo (∇)) when $x_n\nabla x_m$, for $n < m$. Also, let us say that $u \in M$ is an *upper bound* (modulo (∇)) of (x_n) when

$$x_n\nabla u, \text{ for all } n \text{ (written as: } (x_n)\nabla u).$$

If u is generic in this convention, we say that (x_n) is *bounded from above* (modulo (∇)). Finally, we call (M, ∇), *sequentially inductive* provided

each ascending (modulo (∇)) sequence
is bounded from above (modulo (∇)).

The following auxiliary statement will be useful for us.

Lemma 4. *Under these conventions, one has*

$$(M, \nabla) \text{ sequentially inductive} \Longrightarrow (M, \sqsubseteq) \text{ sequentially inductive.} \tag{4.15}$$

Proof. Assume that (M, ∇) is sequentially inductive; and let K be some normally countable $(\sqsubseteq)$-chain of M. By definition, it may be represented as a θ-net (with $\theta < \omega$) $K = \{a_n; n < \theta\}$; where $n \vdash a_n$ is strictly ascending (modulo $(\sqsubseteq)$). The case $\theta < \omega$ is clear; so, without loss, one may assume $\theta = \omega$. By (4.4) (and the choice of (a_n))

$$p < q \Longrightarrow a_p \sqsubset a_q \Longrightarrow a_p \nabla a_q. \tag{4.16}$$

This sequence is therefore ascending (modulo (∇)); wherefrom (by hypothesis) $(a_n)\nabla v$, for some $v \in M$. This, along with (4.6), gives (via (4.16)) $(a_n) \sqsubset v$; hence $(a_n) \sqsubseteq v$; and the conclusion follows. ■

We are now in position to get an appropriate answer to the posed question.

Theorem 3. *Suppose that*

$$(M, \nabla) \text{ is sequentially inductive} \tag{4.17}$$

$$(P, \overline{<}) \text{ is (strongly) separable.} \tag{4.18}$$

Then, conclusions of Theorem 2 are retainable.

Proof. By Lemma 4, condition (4.9) holds via (4.17). We claim that (4.10) holds too (from (4.18)); and this will complete the argument. Let S be some $(\sqsubseteq)$-chain of M; and put $V = \varphi(S)$. Clearly,

$$V \text{ is a } (\overline{<})\text{-chain in } P; \quad \text{(cf. (4.5))};$$

so, in view of (4.18), V is countable (in P). On the other hand, the same relation (4.5) shows that φ is an order isomorphism between $(S, \sqsubseteq)$ and $(V, \overline{<})$; wherefrom, S is countable too; and the claim follows. ■

(B) In particular, assume that the (transitive) relation (∇) is a quasi-order $(\leq)$ in both M and P. By Theorem 3 we then derive (under (4.3))

Theorem 4. *Assume (4.18) is true, as well as*

$$(M, \leq) \text{ is sequentially inductive (in the sense of (1.2)).} \tag{4.19}$$

Then, for each $u \in M$, there exists $v \in M$, with

$$[u \leq v] \text{ and } \quad [v \leq w \Longrightarrow \varphi(w) \leq \varphi(v)]. \tag{4.20}$$

Note that, if $(P, \leq)$ is identical with $(R_+, \geq)$, the regularity condition (4.18) holds (cf. Section 2). In this case, Theorem 1 is nothing but the Brezis-Browder ordering principle [6] (subsumed to Theorem BB). So, it is natural that Theorem 4 be also referred to in this way. On the other hand, if $(M, \leq)$ is identical with $(P, \leq)$ and φ=the identity, Theorem 4 is just Theorem 1 (when $(\leq)$ is an order). Summing up, we get the logical implications

$$\text{Th 1} \Longrightarrow \text{Th 3} \Longrightarrow \text{Th 4} \Longrightarrow \text{Th 1}; \tag{4.21}$$

hence, all these are mutually equivalent. As a consequence of this, the Brezis-Browder ordering principle (Theorem BB) is deductible from the "separable" version of the Zorn-Bourbaki maximality principle (Theorem 1). The question of the reciprocal inclusion being also true remains open; we conjecture that the answer is positive.

(C) Let us return to our initial framework. The basic hypothesis used in all these developments is (4.3). So, the question arises of what can be said about such results when (4.3) is no longer available. To this end, put

$$(x, y \in M) \quad x \bigtriangleup y \text{ iff } x \nabla y \text{ and } \varphi(x) \nabla \varphi(y). \tag{4.22}$$

This is a transitive relation over M; and condition (4.3) holds with $(\bigtriangleup, \nabla)$ in place of (∇, ∇). An application of Theorem 3 to these data yields an appropriate answer to the problem we deal with.

Theorem 5. *Assume that (4.18) holds, as well as*

$$(M, \bigtriangleup) \text{ is sequentially inductive.} \tag{4.23}$$

Then, for each $u \in M$, there exists $v \in M$ with

$$v \text{ is } (\bigtriangleup, \nabla; \varphi)\text{-maximal (cf. (4.7))} \tag{4.24}$$

in such a way that

$$u = v \text{ (hence } u \text{ is } (\bigtriangleup, \nabla; \varphi)\text{-maximal), whenever } M(u, \bigtriangleup) = \emptyset \tag{4.25}$$

$$u \bigtriangleup v, \quad \text{whenever } M(u, \bigtriangleup) \neq \emptyset. \tag{4.26}$$

A quasi-order version of this (under the lines of Theorem 4) is immediately obtainable; we do not give details. In particular, when $(P, \leq)$ is identical with $(R_+, \geq)$, this (quasi-order) statement covers a related one in Kada, Suzuki and Takahashi [13]. Further aspects will be discussed in a separate paper.

5 Some Amorph Versions

A slight extension of these facts is to be reached when the relation (∇) over M is no longer transitive. Further aspects occasioned by the obtained results are then discussed.

(A) Let $(\perp)$ stand for an *amorph* relation over M. Denote by (∇) the transitive relation (over the same) attached to $(\perp)$

$$\begin{array}{l}(x, y \in M)\ x \nabla y \text{ iff } x = u_1 \perp ... \perp u_k = y \text{ (in the sense:} \\ u_i \perp u_{i+1}, \forall i \in \{1, ..., k-1\}), \text{ for some } k \geq 2 \text{ and } u_1, ..., u_k \in M.\end{array} \tag{5.1}$$

Take a transitive relation $(\bigtriangleup)$ over P; as well as a function $\varphi : M \to P$ with

$$\varphi \text{ is } (\perp, \bigtriangleup)\text{-increasing: } x \perp y \Longrightarrow \varphi(x) \bigtriangleup \varphi(y). \tag{5.2}$$

Note that, under (5.1) above, one gets

$$\varphi \text{ is } (\nabla, \bigtriangleup)\text{-increasing (in the sense of (4.3)).}$$

Given $z \in M$, we say that it is $(\perp, \triangle; \varphi)$-*maximal*, if

$$\text{(for each } w \in M\text{)}: \quad z \perp w \Longrightarrow \varphi(w) \triangle \varphi(z). \tag{5.3}$$

Again by (5.1), one gets the generic relation

$$\text{(for each } z \in M\text{)}: (\nabla, \triangle; \varphi)\text{-maximal} \Longrightarrow (\perp, \triangle; \varphi)\text{-maximal}. \tag{5.4}$$

So, existence results involving such points are deductible from Theorem 3 above. The only aspect to be clarified is that of expressing (4.17) in terms of $(\perp)$. This will necessitate a lot of new conventions. Let (x_n) be a sequence in M; we term it *ascending* (modulo $(\perp)$) when

$$x_n \perp x_{n+1}, \quad \text{for all ranks } n. \tag{5.5}$$

Further, let us call $u \in M$, an *asymptotic upper bound* (modulo $(\perp)$) of (x_n) (written as: $(x_n) \perp\perp u$) provided

$$\begin{array}{l}\forall n, \exists m \geq n \text{ with } x_m \perp u; \quad \text{or, equivalently:} \\ \text{there exists a subsequence } (y_n = x_{i(n)}) \text{ of } (x_n) \text{ with } (y_n) \perp u.\end{array} \tag{5.6}$$

When u is generic with such a property, we say that (x_n) is *asymptotic bounded above* (modulo $(\perp)$). Finally, call the structure $(M, \perp)$, *sequentially inductive* if

$$\begin{array}{l}\text{each ascending (modulo } (\perp)\text{) sequence} \\ \text{is asymptotic bounded above (modulo } (\perp)\text{).}\end{array} \tag{5.7}$$

The following auxiliary fact is useful for us.

Lemma 5. *Under these conventions,*

$$(M, \perp) \text{ sequentially inductive} \Longrightarrow (M, \nabla) \text{ sequentially inductive.} \tag{5.8}$$

Proof. Assume that $(M, \perp)$ is sequentially inductive; and let (x_n) be an ascending (modulo (∇)) sequence in M. By the very definition (5.1) of (∇), there must be a sequence (z_n) in M with

$$(z_n)\text{=ascending (modulo } (\perp)\text{)}; \ (x_n)\text{=subsequence of } (z_n).$$

This, by the accepted hypothesis, yields

$$(z_n) \perp\perp u \text{ (wherefrom } (x_n)\nabla u\text{)}, \ \text{for some } u \in M;$$

and the conclusion follows. ∎

Now, by simply adding this to Theorem 3, one gets

Theorem 6. *Assume (4.18) holds, as well as*

$$(M, \perp) \text{ is sequentially inductive.} \tag{5.9}$$

Then, for each $u \in M$, there exists $v \in M$ with

$$v \text{ is } (\perp, \triangle; \varphi)\text{-maximal (cf. (5.3))} \tag{5.10}$$

in such a way that, either (4.13) is retainable, or else

$$u = v \text{ (hence } u \text{ is } (\perp, \triangle; \varphi)\text{-maximal) whenever } M(u, \nabla) = \emptyset. \tag{5.11}$$

(Here, (∇) *is the transitive relation given by (5.1)).*

In particular, when $(\perp)$ is a transitive relation over M, this statement reduces to Theorem 3. Since the opposite inclusion also holds, we get

$$\text{Theorem 6} \Longleftrightarrow \text{Theorem 3} \ (\Longleftrightarrow \text{Theorem 1}). \tag{5.12}$$

Hence, this extension is technical in nature.

(B) Now, the basic assumption used here is (5.2). So, we may ask of what happens when such a condition is no longer true. To this end, put

$$(x, y \in M) \quad x \top y \text{ iff } x \perp y \text{ and } \varphi(x) \bigtriangleup \varphi(y). \tag{5.13}$$

This is an amorph relation over M; and condition (5.2) holds with $(\top, \bigtriangleup)$ in place of $(\perp, \bigtriangleup)$. An application of Theorem 6 to these data gives:

Theorem 7. *Assume (4.18) holds, as well as*

$$(M, \top) \text{ is sequentially inductive.} \tag{5.14}$$

Then, for each $u \in M$ *there exists* $v \in M$ *with*

$$v \text{ is } (\top, \bigtriangleup; \varphi)\text{-maximal (cf. (5.3))} \tag{5.15}$$

in such a way that, either (4.13) is retainable, or else

$$u = v \text{ (hence } u \text{ is } (\top, \bigtriangleup; \varphi)\text{-maximal) whenever } M(u, \nabla) = \emptyset. \tag{5.16}$$

(Here, $(\top)$ *is the amorph relation of (5.13); and* (∇)*, its associated by (5.1) transitive relation).*

(C) The following completion of these is to be made. Let $(Q, \preceq)$ be a well ordered structure with $Q \subseteq P$; we say that it is a *chain* subordinated to (P, ∇), when

$$x, y \in Q, x \preceq y, x \neq y \text{ imply } x \nabla y \tag{5.17}$$

The initial structure $(P, \bigtriangleup)$ will be termed *countably orderable* provided

$$\begin{array}{l} \text{for any chain } (Q, \preceq) \text{ subordinated to } (P, \nabla), \\ \text{the support set } Q \text{ is countable.} \end{array} \tag{5.18}$$

It is not hard to see that the regularity condition

$$(P, \nabla) \text{ is countably orderable} \tag{5.19}$$

is a particular case of (4.18). Hence, the following Brezis-Browder type statement is deductible from Theorem 7 above.

Theorem 8. *Assume that (5.14) is true, as well as (5.19). Then, conclusions of Theorem 7 are retainable.*

This result was obtained in 1992 by Gajek and Zagrodny [10]. The argument developed there - based essentially on the maximal principle in H. Rubin and J. E. Rubin [18, Sect 4] - is rather involved. Hence, the proposed (via Theorem 7) argument for this may be viewed as a simplification of the original one. Further aspects were delineated in Sonntag and Zalinescu [20].

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