LIBERTAS MATHEMATICA, vol XXVIII (2008)

Some Fixed Point Theorems For Mappings Satisfying Implicit Relations In Symmetric Spaces

Marcelina MOCANU and Valeriu POPA

Abstract. We prove some fixed point theorems for mappings satisfying implicit relations in symmetric spaces. We reduce the problem of existence of fixed points for some mappings satisfying contractive conditions of integral type to a problem of existence of fixed points for mappings in symmetric spaces.

Keywords: Fixed point theorem, symmetric space, semimetric space, quasi-metric space, contractive condition of integral type, Banach orbital condition.

Mathematics Subject Classification (2000): 54H25, 54E25, 47H10.

1 Introduction

It is well-known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. However, it has been observed in [7] that some of the defining properties of the metric are not needed in the proof of certain metric theorems. Motivated by this fact, Hicks and Rhoades [7] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric.

Let X be a non-empty set. A *symmetric* on X is nonnegative real-valued function D on $X \times X$ such that

(i) $D(x, y) = 0$ if and only if $x = y$;

(ii) $D(x, y) = D(y, x)$ for every $x, y \in X$.

Let D be a symmetric on a set X, and for $r > 0$ and $x \in X$ let $B(x, r) = \{y \in X : D(x, y) < r\}$. The family $\tau(D)$, formed by the empty set and all the sets $U \subset X$ satisfying the condition that for each $x \in U$ there is $r_x > 0$ such that $B(x, r_x) \subset U$, is a topology.

Let D be a symmetric on X. The following axioms were introduced by Wilson [15].

(W3): Given a sequence $\{x_n\}$ in X and $x, y \in X$, $\lim\limits_{n\to\infty} d(x_n, x) = 0$ and $\lim\limits_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W4): Given the sequences $\{x_n\}$, $\{y_n\}$ in X and $x \in X$, $\lim\limits_{n\to\infty} d(x_n, x) = 0$ and $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim\limits_{n\to\infty} d(y_n, x) = 0$.

It is easy to see that (W3) holds in every symmetric space (X, D) for which the topology $\tau(D)$ is Hausdorff.

A symmetric D on X is said to be *continuous* [5] if $\lim\limits_{n\to\infty} D(x_n, x) = 0$ and $\lim\limits_{n\to\infty} D(y_n, y) = 0$ imply $\lim\limits_{n\to\infty} D(x_n, y_n) = D(x, y)$, respectively it is said to be $1-$*continuous* [5] if $\lim\limits_{n\to\infty} D(x_n, x) = 0$ implies $\lim\limits_{n\to\infty} D(x_n, y) = D(x, y)$, where $\{x_n\}$, $\{y_n\}$ are sequences in X and $x, y \in X$. Clearly, the continuity of a symmetric is a stronger property than $1-$continuity.

For every $x \in X$ and $A \subset X$ define $D(x, A) := \inf\{D(x, y) : y \in A\}$ and denote by $cl(A)$ the closure of A with respect to the topology $\tau(D)$. Then $A \subset X$ is closed with respect to $\tau(D)$ if and only if $\{x \in X : D(x, A) = 0\} \subset A$.

Definition 1.1. [5] A symmetric D is said to be a *semimetric* iff $cl(A) = \{x \in X : D(x, A) = 0\}$ for every $A \subset X$.

For a symmetric D the following properties are equivalent [9], [5]:

(S1) D is a semimetric;

(S2) For each $x \in X$, the family of all balls $B(x, \varepsilon)$, $\varepsilon > 0$, centered at x is a neighborhood base for x;

(S3) The operator $cl_D : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $cl_D(A) = \{x \in X : D(x, A) = 0\}$ is idempotent.

A sequence in a symmetric space (X, D) is said to be $D-$*Cauchy* if it satisfies the usual metric conditions. There are several concepts of completness in the setting of symmetric spaces (see [7]):

(i) (X, D) is said to be $S-$*complete* if for every $D-$Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that $\lim\limits_{n\to\infty} D(x_n, x) = 0$;

(ii) (X, D) is said to be $D-$*complete* if for every $D-$Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that $x_n \to x$ in the topology $\tau(D)$.

Remark 1.1. $S-$completness implies $D-$completness. In semimetric spaces $D-$convergence coincides with convergence in the topology $\tau(D)$ [5], in particular the notions of $S-$completness and $D-$completness coincide.

Some fixed point theorems in symmetric spaces are proved in [1], [2], [7], [8], [12]. On the other hand, in the recent paper [4] Branciari established the following theorem.

Theorem 1.1. [4] *Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f : X \to X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi : [0, +\infty) \to [0, +\infty]$ is a Lebesgue-measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, such that, for each $\varepsilon > 0$, $\int\limits_0^\varepsilon \varphi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim\limits_{n\to\infty} f^n x = z$.

Quite recently, Rhoades [11] generalized the above result of Branciari by proving the following theorem.

Theorem 1.2. [11] *Let (X,d) be a complete metric space, $k \in [0,1)$ and let $f : X \to X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt,$$

where φ is as in Theorem 1.1 and

$$m(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}.$$

Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Other fixed point theorems in metric spaces and symmetric spaces, for mappings satisfying contractive conditions of integral type, are proved in [3], [10], [11], [13], [14].

The purpose of this paper is to prove some fixed point theorems for mappings satisfying implicit relations in symmetric spaces and to reduce the problem of existence of fixed points for some mappings satisfying contractive conditions of integral type to a problem of existence of fixed points for mappings in symmetric spaces.

2 Integral symmetrics

Let (X,d) be a metric space and let $\varphi : [0,+\infty) \to [0,+\infty]$ be a Lebesgue-measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,+\infty)$, such that $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0$. In this section we consider the mapping $D : X \times X \to \mathbb{R}$ defined by $D(x,y) = \int_0^{d(x,y)} \varphi(t)dt$. Throughout the paper we denote $\Phi(u) := \int_0^u \varphi(t)dt$ for $u \in [0,+\infty)$.

Remark 2.1. If $\{a_n\}$ is a sequence in $[0,+\infty)$ that converges to some a, then $\lim_{n\to\infty} \int_0^{a_n} \varphi(t)dt = \int_0^a \varphi(t)dt$, by Lebesgue's dominated convergence theorem. Thus, the function $\Phi : [0,+\infty) \to [0,+\infty)$ defined by $\Phi(u) := \int_0^u \varphi(t)dt$ is continuous. Actually, Φ is absolutely continuous, in particular uniformly continuous.

Lemma 2.1. *If $\{a_n\}$ is a sequence in $[0,+\infty)$, then $\lim_{n\to\infty} \int_0^{a_n} \varphi(t)dt = 0$, if and only if $\{a_n\}$ is convergent to zero.*

Proof. *Necessity:* Let $A := \limsup_{n\to\infty} a_n$. There exists a subsequence $\{a_{n_k}\}$ such that $A = \lim_{k\to\infty} a_{n_k}$. If $A = +\infty$ we may find a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that the sequence $\{a_{n_{\sigma(k)}}\}$ is nondecreasing. By Lebesgue's monotone convergence theorem, $\int_0^A \varphi(t)dt = \lim_{k\to\infty} \int_0^{a_{n_{\sigma(k)}}} \varphi(t)dt = 0$, hence $\int_0^\varepsilon \varphi(t)dt = 0$ for every $\varepsilon > 0$, a contradiction. We proved that

A is finite, hence the sequence $\{a_n\}$ is bounded. By Lebesgue's dominated convergence theorem, $\int_0^A \varphi(t)dt = \lim_{n\to\infty}\int_0^{a_n}\varphi(t)dt = 0$, hence $A = 0$. Since $0 \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n = 0$, it follows that $\{a_n\}$ is convergent to zero.

Lemma 2.2 *D is a symmetric on X.*

Proof. D is a well-defined non-negative real function on $X \times X$, since d is non-negative and φ is a Lebesgue-measurable mapping, non-negative and summable on each compact subset of $[0, +\infty)$. Since $\int_0^{\varepsilon}\varphi(t)dt > 0$ for each $\varepsilon > 0$, we have $D(x,y) = 0$ if and only if $d(x,y) = 0$, i.e. $x = y$. The symmetry of d implies the symmetry of D.

Lemma 2.3 *The symmetric D satisfies the axioms (W3) and (W4).*

Proof. (W3): Assume that $\lim_{n\to\infty} D(x_n, x) = 0$ and $\lim_{n\to\infty} D(x_n, y) = 0$. By Lemma 2.1, it follows that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$. Since $0 \le d(x,y) \le d(x_n, x) + d(x_n, y)$ for each n, we have $d(x,y) = 0$, hence $x = y$.

(W4): Assume that $\lim_{n\to\infty} D(x_n, x) = 0$ and $\lim_{n\to\infty} D(x_n, y_n) = 0$. By Lemma 2.1, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$. Since $0 \le d(y_n, x) \le d(x_n, x) + d(x_n, y_n)$ for each n, we have $\lim_{n\to\infty} d(y_n, x) = 0$. By Lemma 2.1, this implies $\lim_{n\to\infty} D(y_n, x) = 0$.

Lemma 2.4 *(a) $\{x_n\}$ is $d-$Cauchy if and only if $\{x_n\}$ is $D-$Cauchy.*

(b) The metric space (X,d) is complete if and only if the symmetric space (X, D) is $S-$complete.

Proof. (a) *Necessity:* Assume that $\{x_n\}$ is $d-$Cauchy, i.e. for every $\delta > 0$ there exists a positive integer $N(\delta)$ so that $d(x_m, x_n) < \delta$ for all $m, n \ge N(\delta)$. Let $\varepsilon > 0$. By Lemma 2.1, $\lim_{u\to 0}\Phi(u) = 0$, hence there exists $\delta(\varepsilon) > 0$ such that $0 \le \Phi(u) < \varepsilon$ for every $u \in (0, \delta(\varepsilon))$. Then $D(x_m, x_n) = \Phi(d(x_n, x_n)) < \varepsilon$ for all $m, n \ge N(\delta(\varepsilon))$. It follows that $\{x_n\}$ is $D-$Cauchy.

Sufficiency: Assume that $\{x_n\}$ is not $d-$Cauchy, i.e. there exists $\delta_0 > 0$ such that for every positive integer N there exist $m_N, n_N \ge N$.so that $d(x_{m_N}, x_{n_N}) \ge \delta_0$. Then $D(x_{m_N}, x_{n_N}) \ge \varepsilon_0$, where $\varepsilon_0 := \int_0^{\delta_0}\varphi(t)dt > 0$, therefore $\{x_n\}$ is not $D-$Cauchy.

(b) Use (a) and Lemma 2.1.

Remark 2.2 Every subset of X which is $d-$bounded is also $D-$bounded.

Lemma 2.5 *The symmetric D is continuous.*

Proof. Assume that $\lim_{n\to\infty} D(x_n, x) = 0$ and $\lim_{n\to\infty} D(y_n, y) = 0$. By Lemma 2.1, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$. Since $|d(x_n, y_n) - d(x,y)| \le d(x_n, x) + d(y_n, y)$, it follows that $\lim_{n\to\infty} d(x_n, y_n) = d(x,y)$, hence $\lim_{n\to\infty} D(x_n, y_n) = D(x,y)$ by Remark 2.1.

Theorem 2.1 *For every metric space (X,d), the mapping $D : X \times X \to \mathbb{R}$ defined by $D(x,y) = \int_0^{d(x,y)}\varphi(t)dt$ is a continuous semimetric for which axioms (W3) and (W4) hold. The diameters of balls in (X, D) with radius r are equibounded for each $r > 0$. Moreover, the metric space(X,d) is complete if and only if semimetric space (X, D) is $D-$complete.*

Proof. By Lemma 2.2, Lemma 2.3 and Lemma 2.5 , D is a continuous symmetric for which axioms (W3) and (W4) hold.

To prove that D is a semimetric, it suffices to check that the operator cl_D is idempotent. Let A be an arbitrary non-empty subset of X. Since $D(x,A)=0$ (respectively, $d(x,A)=0$) if and only if there exist a sequence $\{a_n\}$ in A such that $\lim\limits_{n\to\infty} D(x,a_n)=0$ (respectively, $\lim\limits_{n\to\infty} d(x,a_n)=0$) and since $\lim\limits_{n\to\infty} D(x,a_n)=0$ if and only if $\lim\limits_{n\to\infty} d(x,a_n)=0$ (by Lemma 2.1), it follows that $cl_D(A)=Cl(A)$. Here we denoted by Cl the usual closure operator in the topology of the metric space (X,d). Since Cl is idempotent, the operator cl_D is idempotent.

Since D is a semimetric, $D-$completness and $S-$completness coincide in (X,D), hence the metric space (X,d) is complete if and only if the semimetric space (X,D) is $D-$complete by Lemma 2.4.

Let $M := \sup\limits_{u\geq 0}\Phi(u)$. If $M<\infty$, then the diameter of (X,D) is bounded from above by M and the proof is completed. Next assume that $M=\infty$. Consider the generalized inverse Φ^{-1} of Φ, defined by $\Phi^{-1}(v):=\inf\{u:\Phi(u)>v\}$. Since $M=\infty$, $\Phi^{-1}(v)$ is finite for each $v\geq 0$. Since Φ is non-decreasing, the function $\Phi^{-1}:[0,+\infty)\to[0,+\infty)$ is non-decreasing and $u\leq\Phi^{-1}(\Phi(u))$ for every $u\in[0,+\infty)$. Let $r>0$. For every $a\in X$ consider in (X,D) the ball centered at a, of radius r, $B(a,r):=\{x\in X: D(x,a)<r\}$. Let $x_1,x_2\in B(a,r)$. For each $k\in\{1,2\}$ we have $\Phi(d(x_k,a))<r$, hence $d(x_k,a)\leq\Phi^{-1}(\Phi(d(x_k,a))\leq\Phi^{-1}(r)$. By the triangle inequality, $d(x_1,x_2)\leq d(x_1,a)+d(x_2,a)\leq 2\Phi^{-1}(r)$, hence $D(x_1,x_2)=\Phi(d(x_1,x_2))\leq\Phi\left(2\Phi^{-1}(r)\right)$. We proved that the diameters of all balls in (X,D) having radius r are bounded from above by $\Phi\left(2\Phi^{-1}(r)\right)$.

3 A Banach orbital condition in symmetric spaces

In what follows denote by Ψ the family of all non-decreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\lim\limits_{n\to\infty}\psi^n t=0$ for every $t>0$. Note that each $\psi\in\Psi$ satisfies $\psi(t)<t$ for all $t>0$ and $\psi(0)=0$.

Definition 3.1. Let (X,D) be a symmetric space. A function $T:(X,D)\to(X,D)$ is said to *satisfy the Banach orbital condition determined by the function* $\psi:[0,\infty)\to[0,\infty)$ if

$$D(Tx,T^2x)\leq\psi(D(x,Tx)) \tag{1}$$

for every $x\in X$.

Lemma 3.1. *Let (X,D) be a symmetric space and let $T:(X,D)\to(X,D)$ satisfying the Banach orbital condition (1). If $\psi\in\Psi$, then $\lim\limits_{n\to\infty}D(T^nz,T^{n+1}z)=0$ for every $z\in X$.*

Proof. Fix an arbitrary $z\in X$. By (1), for every positive integer n we have

$$D(T^nz,T^{n+1}z)\leq\psi\left(D(T^{n-1}z,T^nz)\right)$$

It follows by induction over n that

$$D(T^nz,T^{n+1}z)\leq\psi^n\left(D(z,Tz)\right), \tag{2}$$

for every $n \in \mathbb{N}$, but $\lim\limits_{n\to\infty} \psi^n t = 0$ for every $t \geq 0$, hence $\lim\limits_{n\to\infty} D(T^n z, T^{n+1} z) = 0$.

Definition 3.2. [6] A symmetric D on X is called a *quasimetric* if there exists a constant $q \geq 1$ so that

$$D(x, y) \leq q\,[D(x, z) + D(z, y)] \tag{3}$$

for all $x, y, z \in X$. If (3) holds for all $x, y, z \in X$ we say that D is a $q-$*quasimetric* and (X, D) is a $q-$*quasimetric space*.

Quasimetric spaces are an important tool for harmonic analysis and for analysis in metric measure spaces [6].

Remark 3.1. Every metric space is a $1-$quasisymmetric space.

Lemma 3.2. *Let (X, d) be a metric space, let φ be as in Theorem 1.1 and let $D : X \times X \to R$ be defined by $D(x, y) = \int_0^{d(x,y)} \varphi(t)dt$. Then*

(1) D is a quasimetric if Φ satisfies the Δ_2-condition, i.e. there exists a constant $C_2 > 0$ such that $\Phi(2u) \leq C_2\Phi(u)$ for each $u \geq 0$.

(2) If φ is non-increasing, then D is a metric.

Proof. (1) D is a quasimetric if and only if $\Phi(d(x, y)) \leq q\,[\Phi(d(x, z)) + \Phi(d(z, y))]$ with some constant $q \geq 1$, for all $x, y, z \in X$. Since Φ is non-decreasing, using the triangle inequality it follows that the preceding inequality holds if $\Phi(u + v) \leq q\,[\Phi(u) + \Phi(v)]$ for all $u, v \in [0, \infty)$. If $\Phi(2u) \leq C_2\Phi(u)$ for each $u \geq 0$, then $\Phi(u + v) \leq \Phi(2\max\{u, v\}) \leq C_2\Phi(\max\{u, v\}) = C_2 \max\{\Phi(u), \Phi(v)\} \leq C_2\,[\Phi(u) + \Phi(v)]$ for all $u, v \in [0, \infty)$, hence D is a C_2-quasimetric.

(2) If φ is non-increasing, then $\Phi(u + v) \leq \Phi(u) + \Phi(v)$ for all $u, v \in [0, \infty)$. We may assume that $u \leq v$. We have $\Phi(u + v) - [\Phi(u) + \Phi(v)] = \int_v^{u+v} \varphi(t)dt - \int_0^u \varphi(t)dt = \int_0^u [\varphi(t + v) - \varphi(t)]\,dt \leq 0$, where we used the change of variables formula for the Lebesgue integral and the monotonicity of φ.

Theorem 3.1. *Let (X, D) be a $q-$quasimetric space, where $q \geq 1$. If $T : (X, D) \to (X, D)$ satisfies the Banach orbital condition determined by the function $\psi(t) = \lambda t$, $t \geq 0$, for some $\lambda \in [0, 1)$ and if $\lambda q < 1$, then for every $z \in X$ the sequence $\{T^n z\}$ is $D-$Cauchy.*

Proof. Using (2) with $\psi(t) = \lambda t$, $t \geq 0$, it follows that

$$D(T^n z, T^{n+1} z) \leq \lambda^n D(z, Tz) \tag{4}$$

for every $z \in X$. It follows by induction over p that for all integers $p \geq 1$ we have

$$D(T^n z, T^{n+p} z) \leq D(z, Tz) \sum_{k=1}^{p} q^k \lambda^{n-1+k}, \forall z \in X, \forall n \in \mathbb{N} \tag{5}$$

For $p = 1$ (5) writes as $D(T^n z, T^{n+1} z) \leq q\lambda^n d(z, Tz)$, which is true by (4), since $q \geq 1$. Assume that (5) holds. We prove that

$$D(T^n z, T^{n+p+1} z) \leq D(z, Tz) \sum_{k=1}^{p+1} q^k \lambda^{n-1+k}, \forall z \in X, \forall n \in \mathbb{N}. \tag{6}$$

Fix arbitrarily $z \in X$ and $n \in \mathbb{N}$. Since D is a $q-$quasimetric,

$$D(T^n z, T^{n+p+1}z) \leq q\left[D(T^n z, T^{n+1}z) + D(T^{n+1}z, T^{n+p+1}z)\right].$$

Using (5) with $n+1$ instead n it follows that

$$qD(T^{n+1}z, T^{n+p+1}z) \leq D(z,Tz)\sum_{k=1}^{p} q^{k+1}\lambda^{n+k} = D(z,Tz)\sum_{k=2}^{p+1} q^k\lambda^{n-1+k}.$$

By (4), $qD(T^n z, T^{n+1}z) \leq q\lambda^n D(z,Tz)$. The latter three inequalities imply (6).

For all integers $p, n \geq 1$, $\sum\limits_{k=1}^{p} q^k\lambda^{n-1+k} = \lambda^{n-1}\frac{\lambda q-(\lambda q)^{p+1}}{1-\lambda q} \leq \frac{q}{1-\lambda q}\lambda^n$, since $\lambda q \in [0,1)$, hence

$$D(T^n z, T^{n+p}z) \leq \frac{q}{1-\lambda q}\lambda^n.$$

Since $\lim\limits_{n\to\infty}\lambda^n = 0$, the above inequality shows that the sequence $\{T^n z\}$ is $D-$Cauchy.

4 Implicit relations and fixed points in symmetric spaces

Let $\mathcal{F}$ be the set of all functions $F : \mathbb{R}_+^6 \to \mathbb{R}$. Define $\mathcal{E}_1 := \{F \in \mathcal{F} : F(t,0,t,0,0,t) > 0$ for every $t > 0\}$, $\mathcal{E}_2 := \{F \in \mathcal{F} : F(t,0,t,0,0,0) > 0$ for every $t > 0\}$, $\mathcal{U}_1 := \{F \in \mathcal{F} : F(t,t,t,t,t,0) > 0$ for every $t > 0\}$ and $\mathcal{U}_2 := \{F \in \mathcal{F} : F(t,t,t,t,0,0) > 0$ for every $t > 0\}$.

In what follows, $t = (t_1, ..., t_6) \in \mathbb{R}_+^6$ and a, b, c, d, e, k are real constants.

Example 4.1. In what follows, $t = (t_1, ..., t_6) \in \mathbb{R}_+^6$ and a, b, c, d, e, k are real constants.

(1) Let $F(t) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$. We have $F \in \mathcal{E}_1$ iff $b + c < 1$, $F \in \mathcal{E}_2$ iff $b < 1$, $F \in \mathcal{U}_1$ iff $a + b + c + d < 1$ and $F \in \mathcal{U}_2$ iff $a + b + c < 1$.

(2) Let $F(t) = t_1 - k\max\{t_i : 2 \leq i \leq 6\}$. For $i \in \{1,2\}$, $F \in \mathcal{E}_i$ iff $k < 1$ and $F \in \mathcal{U}_i$ iff $k < 1$.

(3) Let $F(t) = t_1(t_1 - at_2 - bt_5 - ct_6) - dt_3t_4$. Note that $F \in \mathcal{E}_2$ for every a, b, c, d. We have $F \in \mathcal{E}_1$ iff $c < 1$, $F \in \mathcal{U}_1$ iff $a + b + d < 1$ and $F \in \mathcal{U}_2$ iff $a + d < 1$.

(4) Let $F(t) = t_1^2 - at_2^2 - \frac{bt_3t_4}{1+t_5+t_6}$. Note that $F \in \mathcal{E}_1 \cap \mathcal{E}_2$ for every a, b. We have $F \in \mathcal{U}_2$ iff $a + b < 1$. In addition, $F \in \mathcal{U}_1$ iff one of the following conditions holds:

(i) $b \geq 0$ and $a + b \leq 1$;

(ii) $b < 0$ and $a \leq 1$.

Let (X, D) be a symmetric space, $T : (X,D) \to (X,D)$ and $F \in \mathcal{F}$. For $x, y \in X$ consider the following implicit relations, denoted by (I_1), (I_2), (I_3) and (I_4), respectively:

$$F(D(Tx,Ty), D(x,y), D(x,Ty), D(y,Tx), D(y,T^2x), D(y,Ty)) \leq 0$$

$$F(D(Tx,Ty), D(x,y), D(x,Ty), D(y,Tx), D(x,Tx), D(Tx,T^2x)) \leq 0$$

$$\min\{F(D(Tx,Ty), D(x,y), D(x,Ty), D(y,Tx), D(x,Tx), D(y,Ty)),$$
$$F(D(Tx,Ty), D(x,y), D(x,Ty), D(y,Tx), D(Tx,T^2x), D(y,Ty))\} \leq 0$$

$$\min\{F(D(Tx,Ty),D(x,y),D(x,Ty),D(y,Tx),D(y,T^2x),D(x,Tx)),$$
$$F(D(Tx,Ty),D(x,y),D(x,Ty),D(y,Tx),D(y,T^2x),D(Tx,T^2x))\} \leq 0$$

Theorem 4.1. *Let (X,D) be a symmetric space and let $T : X \to X$ satisfying the implicit relation (I_j) for every $x, y \in X$, where $j \in \{1,2,3,4\}$.*

a) If $j \in \{1,4\}$ and $F \in \mathcal{U}_1$ or if $j \in \{2,3\}$ and $F \in \mathcal{U}_2$, then T has at most one fixed point.

b) Assume that (X,D) is $S-$complete, D is $1-$continuous and there exists $z \in X$ such that the sequence $\{T^n z\}$ is $D-$Cauchy.

If $j \in \{1,3\}$ and $F \in \mathcal{E}_1$ or if $j \in \{2,4\}$ and $F \in \mathcal{E}_2$, then T has at least one fixed point.

Proof. a) Assume that $u, v \in X$ are fixed points of T. Denote $t := D(u,v)$. Suppose that T satisfies (I_j) for every $x, y \in X$, where $j \in \{1,2,3,4\}$. Taking $x := u$ and $y := v$ in the corresponding implicit relation, we get $F(t,t,t,t,t,0) \leq 0$ when $j \in \{1,4\}$, respectively $F(t,t,t,t,0,0) \leq 0$ when $j \in \{2,3\}$. If $F \in \mathcal{U}_1$ when $j \in \{1,4\}$, or if $F \in \mathcal{U}_2$ when $j \in \{2,3\}$, it follows that $D(u,v) = 0$, hence $u = v$.

b) Let $z \in X$ be such that the sequence $\{T^n z\}$ is $D-$Cauchy. Denote $z_n := T^n z$, $n \in \mathbb{N}$. Since the symmetric space (X,D) is $S-$complete, there exists $w \in X$ such that $\lim_{n\to\infty} D(z_n, w) = 0$. We take $x := z_n$, $n \in \mathbb{N}$, and $y := w$ in the implicit relation (I_j) satisfied by T.

Denote for abbreviation

$$\delta_n := (D(z_{n+1},Tw), D(z_n,w), D(z_n,Tw), D(w,z_{n+1})) \in \mathbb{R}^4.$$

For each $n \in \mathbb{N}$ we have for $j = 1$, $j = 2$, $j = 3$ and $j = 4$, respectively:

$$F(\delta_n, D(w,z_{n+2}), D(w,Tw)) \leq 0 \tag{7}$$

$$F(\delta_n, D(z_n,z_{n+1}), D(z_{n+1},z_{n+2})) \leq 0 \tag{8}$$

$$\min\{F(\delta_n, D(z_n,z_{n+1}), D(w,Tw)), F(\delta_n, D(z_{n+1},z_{n+2}), D(w,Tw))\} \leq 0 \tag{9}$$

$$\min\{F(\delta_n, D(w,z_{n+2}), D(z_n,z_{n+1})), F(\delta_n, D(w,z_{n+2}), D(z_{n+1},z_{n+2}))\} \leq 0 \tag{10}$$

Since $\{z_n\}$ is a $D-$Cauchy sequence, $\lim_{n\to\infty} D(z_n,z_{n+1}) = \lim_{n\to\infty} D(z_{n+1},z_{n+2}) = 0$. Since $\lim_{n\to\infty} D(z_n,w) = 0$, it follows by the $1-$continuity of D that $\lim_{n\to\infty} D(z_n,Tw) = D(w,Tw)$. Denote $\tau := D(w,Tw)$. Letting $n \to \infty$ in the corresponding inequality of (7)- (10) we get $F(\tau,0,\tau,0,0,\tau) \leq 0$ if $j \in \{1,3\}$, respectively $F(\tau,0,\tau,0,0,0) \leq 0$ if $j \in \{2,4\}$. If $F \in \mathcal{E}_1$ when $j \in \{1,3\}$(respectively, if $F \in \mathcal{E}_2$ when $j \in \{2,4\}$), it follows that $D(w,Tw) = 0$, hence w is a fixed point of T.

Corollary 4.1. *Let (X,D) be a $q-$quasimetric space, where $q \geq 1$. Assume that (X,D) is $S-$complete and D is $1-$continuous. Let $T : (X,D) \to (X,D)$ satisfying the Banach orbital condition determined by the function $\psi(t) = \lambda t$, $t \geq 0$, for some $\lambda \in [0,1)$ with $\lambda q < 1$. Assume that $T : X \to X$ satisfies the implicit relation (I_j) for every $x, y \in X$, where $j \in \{1,2,3,4\}$. Then T has a unique fixed point provided that at least one of the following properties holds true:*

(i) $j = 1$ and $F \in \mathcal{E}_1 \cap \mathcal{U}_1$;

(ii) $j = 2$ *and* $F \in \mathcal{E}_2 \cap \mathcal{U}_2$;
(iii) $j = 3$ *and* $F \in \mathcal{E}_1 \cap \mathcal{U}_2$;
(iv) $j = 4$ *and* $F \in \mathcal{E}_2 \cap \mathcal{U}_1$.

Proof. Apply Theorem 4.1 and Theorem 3.1.

We prove a result similar to Theorem 4.1 for mappings satisfying an implicit relation with eight variables.

Theorem 4.2. *Let* (X, D) *be a symmetric space, let* $G : \mathbb{R}_+^6 \to \mathbb{R}$ *be continuous and let* $T : X \to X$ *satisfying the implicit relation*

$$G(D(Tx,Ty), D(x,y), D(x,Ty), D(y,Tx), \tag{11}$$

$$D(x,Tx), D(y,Ty), D(Tx,T^2x), D(y,T^2x)) \leq 0$$

for every $x, y \in X$.

a) If $G(t,t,t,t,0,0,0,t) > 0$ *for every* $t > 0$, *then* T *has at most one fixed point.*

b) Assume that (X, D) *is* $S-$*complete,* D *is* $1-$*continuous and there exists* $z \in X$ *such that the sequence* $\{T^n z\}$ *is* $D-$*Cauchy.*

If $G(t,0,t,0,0,t,0,0) > 0$ *for every* $t > 0$, *then* T *has at least one fixed point.*

Proof. a) Assume that $G(t,t,t,t,0,0,0,t) > 0$ for every $t > 0$. Let $u, v \in X$ be fixed points of T. Denote $t := D(u,v)$. Taking $x := u$ and $y := v$ in (11), we get $G(t,t,t,t,0,0,0,t) \leq 0$. It follows that $D(u,v) = 0$, hence $u = v$.

b) Assume that $G(t,0,t,0,0,t,0,0) > 0$ for every $t > 0$. Let $z \in X$ be such that the sequence $\{T^n z\}$ is $D-$Cauchy. Denote $z_n := T^n z$, $n \in \mathbb{N}$. Since the symmetric space (X, D) is $S-$complete, there exists $w \in X$ such that $\lim\limits_{n\to\infty} D(z_n, w) = 0$. Taking $x := z_n$, $n \in \mathbb{N}$, and $y := w$ in (11) we get

$$G(D(z_{n+1}, Tw), D(z_n, w), D(z_n, Tw), D(w, z_{n+1}), \tag{12}$$

$$D(z_n, z_{n+1}), D(w, Tw), D(z_{n+1}, z_{n+2}), D(w, z_{n+2})) \leq 0, n \in \mathbb{N}.$$

As in the proof of Theorem 4.1, $\lim\limits_{n\to\infty} D(z_n, z_{n+1}) = \lim\limits_{n\to\infty} D(z_{n+1}, z_{n+2}) = 0$ and $\lim\limits_{n\to\infty} D(z_n, Tw) = D(w, Tw)$. Letting $n \to \infty$ in (12) and using the continuity of G, we get $G(\tau, 0, \tau, 0, 0, \tau, 0, 0) \leq 0$, where we denoted $\tau := D(w, Tw)$. Then $\tau = 0$, hence w is a fixed point of T.

5 A fixed point theorem for mappings satisfying a general contractive condition of integral type

In what follows (X, d) is a metric space, $D(x,y) := \displaystyle\int_0^{d(x,y)} \varphi(t)dt$ for $x, y \in X$, φ is as in Theorem 1.1 and Ψ is the family of functions introduced in Section 3.

Lemma 5.1. *Let $\psi \in \Psi$. A function $T : (X, D) \to (X, D)$ satisfies the Banach orbital condition determined by ψ if the following inequality holds for every $x, y \in X$:*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \tag{13}$$

where

$$M(x,y) := \max\left\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, d(x,Tx), d(y,Ty), d(Tx,T^2x), d(y,T^2x)\right\}.$$

Proof. Taking an arbitrary $x \in X$ and $y := Tx$ in the inequality (13) we get $D(Tx,T^2x) \leq \psi\left(\int_0^{M(x,Tx)} \varphi(t)dt\right)$.

But

$$M(x,Tx) = \max\left\{d(x,Tx), \frac{d(x,T^2x)}{2}, d(Tx,T^2x)\right\} \leq$$

$$\max\left\{d(x,Tx), \frac{d(x,Tx)+d(Tx,T^2x)}{2}, d(Tx,T^2x)\right\} \leq \max\left\{d(x,Tx), d(Tx,T^2x)\right\}.$$

Then

$$\int_0^{M(x,Tx)} \varphi(t)dt \leq \int_0^{\max\{d(x,Tx),d(Tx,T^2x)\}} \varphi(t)dt = \max\left\{\int_0^{d(x,Tx)} \varphi(t)dt, \int_0^{d(Tx,T^2x)} \varphi(t)dt\right\}.$$

It follows that

$$D(Tx,T^2x) \leq \max\{\psi(D(x,Tx)), \psi(D(Tx,T^2x))\}. \tag{14}$$

In case $D(Tx,T^2x) \leq D(x,Tx)$, (14) implies $D(Tx,T^2x) \leq \psi(D(x,Tx))$. In case $D(Tx,T^2x) > D(x,Tx)$, (14) implies $D(Tx,T^2x) \leq \psi\left(D(Tx,T^2x)\right)$, hence $D(Tx,T^2x) = 0 > D(x,Tx)$, a contradiction. Then T satisfies the Banach orbital condition (1).

Theorem 5.1. *If the metric space (X, d) is complete and if $T : X \to X$ satisfies inequality (13) for every $x, y \in X$, where $\psi \in \Psi$ is right-continuous, then T has a unique fixed point $w \in X$ and $\lim_{n\to\infty} D(T^n z, w) = 0$ for every $z \in X$.*

Proof. The symmetric space (X, D) is $S-$complete, according to Lemma 2.4. By Lemma 2.5, the symmetric D is continuous, in particular it is $1-$continuous.

By Lemma 5.1, $T : (X, D) \to (X, D)$ satisfies the Banach orbital condition determined by ψ, hence, according to Lemma 3.1,

$$\lim_{n\to\infty} D(T^n z, T^{n+1}z) = 0 \tag{15}$$

for every $z \in X$.

For arbitrarily fixed $z \in X$, denote $z_n := T^n z$, $n \in \mathbb{N}$. We will prove later that the sequence $\{z_n\}$ is $d-$Cauchy, applying an argument used by Branciari [4] and by Rhoades [11].

Assume that $\{z_n\}$ is $d-$Cauchy. Then $\{z_n\}$ is $D-$Cauchy, by Lemma 2.4. Since (X, D) is $S-$complete, there exists $w \in X$ such that $\lim\limits_{n\to\infty} D(z_n, w) = 0$. Since

$$M(x,y) \leq \max\{d(x,y), d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty), d(Tx,T^2x), d(y,T^2x)\},$$

(13) yields

$$D(Tx,Ty) \leq \psi(\max\{D(x,y), D(x,Ty), D(y,Tx), D(x,Tx), D(y,Ty), D(Tx,T^2x), D(y,T^2x)\})$$

for every $x, y \in X$. Denoting $G(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) = t_1 - \psi(\max\{t_2,t_3,t_4,t_5,t_6,t_7,t_8\})$ the above inequality writes as (11) for every $x, y \in X$. We have $G(t,t,t,t,0,0,0,t) = G(t,0,t,0,0,t,0,0) = t - \psi(t) > 0$ for every $t > 0$. By Theorem 4.2, w is the unique fixed point of T.

Next we prove that the sequence $\{z_n\}$ is $d-$Cauchy.

Suppose that $\{z_n\}$ is not $d-$Cauchy. It follows that there exist a positive number ε_0 and two increasing sequences $\{m_p\}_{p\geq 1}$ and $\{n_p\}_{p\geq 1}$, with $m_p < n_p < m_{p+1}$ for each $p \geq 1$, such that

$$d(z_{m_p}, z_{n_p}) \geq \varepsilon_0 \text{ and } d(z_{m_p}, z_{n_p-1}) < \varepsilon_0, \tag{16}$$

for each $p \geq 1$. We take $x := z_{m_p-1}$ and $y := z_{n_p-1}$ in the inequality (13). Note that

$$\begin{aligned} M(z_{m_p-1}, z_{n_p-1}) = \max\{&d(z_{m_p-1}, z_{n_p-1}), \frac{1}{2}[d(z_{m_p-1}, z_{n_p}) + d(z_{m_p}, z_{n_p-1})],\\ &d(z_{m_p-1}, z_{m_p}), d(z_{n_p-1}, z_{n_p}), d(z_{m_p}, z_{m_p+1}), d(z_{m_p+1}, z_{n_p-1})\}. \end{aligned}$$

By the triangle inequality and using the second inequality in (16) we obtain the following relations:

$$d(z_{m_p-1}, z_{n_p-1}) < d(z_{m_p-1}, z_{m_p}) + \varepsilon_0,$$

$$\begin{aligned} d(z_{m_p-1}, z_{n_p}) &\leq d(z_{m_p-1}, z_{m_p}) + d(z_{m_p}, z_{n_p-1}) + d(z_{n_p-1}, z_{n_p})\\ &< d(z_{m_p-1}, z_{m_p}) + d(z_{n_p-1}, z_{n_p}) + \varepsilon_0, \end{aligned}$$

$$d(z_{m_p+1}, z_{n_p-1}) < d(z_{m_p}, z_{m_p+1}) + \varepsilon_0.$$

The last three inequalities imply

$$M(z_{m_p-1}, z_{n_p-1}) \leq \max\{d(z_{m_p-1}, z_{m_p}), d(z_{m_p}, z_{m_p+1}), d(z_{n_p-1}, z_{n_p})\} + \varepsilon_0.$$

By Remark 2.1, the function $\Phi : [0, +\infty) \to [0, +\infty)$ defined by $\Phi(u) := \int_0^u \varphi(t)dt$ is continuous. Using (13), the first inequality in (16) and the above inequality we get

$$\Phi(\varepsilon_0) \leq \psi(\Phi(\max\{d(z_{m_p-1}, z_{m_p}), d(z_{m_p}, z_{m_p+1}), d(z_{n_p-1}, z_{n_p})\} + \varepsilon_0))$$

for every $p \geq 1$. By (15) $\lim\limits_{p\to\infty} \max\{d(z_{m_p-1}, z_{m_p}), d(z_{m_p}, z_{m_p+1}), d(z_{n_p-1}, z_{n_p})\} = 0$. Then letting $p \to \infty$ in the above inequality and using the continuity of Φ at ε_0 and the right-continuity of ψ at $\Phi(\varepsilon_0)$, it follows that $\Phi(\varepsilon_0) \leq \psi(\Phi(\varepsilon_0))$. This implies $\Phi(\varepsilon_0) = 0$, a contradiction with the assumption that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$. We conclude that $\{z_n\}$ is $d-$Cauchy and the proof is completed.

Remark 5.1. Theorem 5.1 is a strong generalization of Theorem 1.2 of Rhoades. Denoting $T := f$ we see that $m(x,y) \leq M(x,y)$ for all $x, y \in X$. Then by Theorem 5.1 we get in the particular case $\psi(t) = ct$ a generalization of Theorem 1.2. Example 3 given by Rhoades [11] shows that in Theorem 1.2 one cannot replace $m(x,y)$ by $\max\{d(x,y), d(x,fy), d(y,fx), d(x,fx), d(y,fy)\}$. Consequently, we cannot weaken the assumption (13) of Theorem 5.1 by setting

$$\max\{d(x,y), d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty), d(Tx,T^2x), d(y,T^2x)\}$$

instead $M(x,y)$.

References

[1] M. Aamri and D. El-Moutawakil, *Common fixed points under contractive conditions in symmetric spaces*, Appl. Math. E-Notes, 3 (2003), 156-162.

[2] M. Aamri, A. Bassou and D. El-Moutawakil, *Common fixed points for weakly compatible maps in symmetric spaces with applications to probabilistic spaces*, Appl. Math. E-Notes, 5 (2005), 171-175.

[3] A. Aliouche, *A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type*, J. Math. Anal. Appl. 322 (2006), 796-802.

[4] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Internat. J. Math. Math. Sci. 29 (2) (2002), 531-536.

[5] F. Galvin and S.D. Shore, *Completeness in semimetric spaces*, Pacific J. of Math., vol. 113, 1(1984), 67-75.

[6] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer Verlag, 2001.

[7] T. L. Hicks and B. E. Rhoades, *Fixed point theorems in symmetric spaces with applications to probabilistic spaces*, Nonlinear Anal. 36 (1999), 331-344.

[8] M. Imdad, Javid Ali and Ladlay Khan, *Coincidence and fixed points in symmetric spaces under strict contractions*, J. Math. Anal. Appl. 320 (2006), 352-360.

[9] J. Jachymski, J. Matkowski and T. Swiatkowski, *Nonlinear contractions on semimetric spaces*, J. of Applied Analysis, vol.1, no. 2(1995), 125-134.

[10] J. K. Kohli and Sachiw Vashistha, *Common fixed point theorems for compatible and weakly compatible mappings satisfying general contractive type conditions*, Stud. Cerc. St. Ser. Mat. 16 (2006), 33-42.

[11] B. E. Rhoades, *Two fixed point theorems for mappings satisfying a general contractive condition of integral type*, Internat. J. Math. Math. Sci. 63 (2003), 4007-4013.

[12] S. L. Singh and B. Prasad, *Coincidence and fixed points of hybrid maps in symmetric spaces*, Austral. J. Math. Anal. Appl. 3,2 (2006), 1-10.

[13] D. Türkoglu and I. Altun, *A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation*, to appear.

[14] P. Viayaraju, B. E. Rhoades and R. Mohansaj, *A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type*, Internat. J. Math. Math. Sci. 15 (2005), 2359-2364.

[15] W.A. Wilson, *On semi-metric spaces*, Amer. J. Math. 53 (1931), 361-373.