APPLYING OF THE METHOD OF THE SERIES OF MATRIXES TO ESTABLISH THE CARDINAL OF THE SET OF ITERATIVE PSEUDO-BOOLEAN ALGEBRAS

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Abstract: A.I.Maltzev [1] proposed in 1966 the reseach of the iterative algebras of propositional logic functions. J.I.Yanov and A.A.Muchnic [2] constructed the exampls of continuous cardinal sets of the closed classes from functions of general 3-valued logic. Using our method of the series of matrices, we proved that the mention resultat [2] remains valued for the case of the functions of simplest non-classical logic. This logic (i.e. the logic of Jaśkowki's First Matrix [3]) from the functional point of view represents a much more narrow fragment of general 3-valued logic. From our result it follows the base result from [2], but no the reverse.

The system Σ of pseudo-Boolean functions (or formulas) is called the *basis* of an iterative algebra of similar functions, if Σ is independent and complete in the logic of this algebra.

Theorem 1. *There are the families of continuous cardinal of iterative algebras, distinct two by two, of pseudo-Boolean functions, and each of these algebras have an infinite basis.*

Indeed, we consider for analysis the series of functions

$$\{ \mathop{\&}_{i=1}^{m} ((\neg x_1 \& \dots \& \neg x_{i-1} \& \neg x_{i+1} \& \dots \& \neg x_m) \supset \perp x_i) \mid$$

$$\mid m = 2, 3, \dots \}. \qquad (1)$$

Noting these functions of 2, 3, ... arguments with the symbols C_2, C_3, ..., respectively, let's note some of their properties. For any $m = 2, 3, \dots$, the function C_m is symmetric (i.e. at any permutation of the arguments it remains equal to itself), and satisfies the conditions

$$C_m = \perp C_m ; \qquad (2)$$

$$C_m(x_i/1) = 1 \qquad (1 \leq i \leq m) \qquad (3)$$

$$C_m(x_i/\perp y, x_j/\perp z) = 1 \quad (1 \leq i < j \leq m) \qquad (4)$$

It is easy to see that for any $m = 2, 3, \dots$ the function C_m does not take the value 0. In addition, it is equal to τ if exactly one of the variables $x_1, \dots, x_m$ takes the value of τ and all others - the value 0; Cm takes the value 1 in all other cases of values of the variables.

Note by M_n ($n = 2, 3, \dots$) the following matrix, which contains n lines and $(n + 2^n - 1)$ columns:

$$\begin{pmatrix} 000 \dots 0\ \tau\tau\tau\tau\tau\tau \dots 11 \\ 000 \dots \tau\ 0\tau\tau\tau\tau\tau \dots 11 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 00\tau \dots 0\ 0\ \tau\tau\tau 11 \dots 11 \\ 0\tau 0 \dots 0\ 0\ \tau 11\ \tau\tau \dots 11 \\ \tau 00 \dots 0\ 0\ 1\tau 1\ \tau 1 \dots \tau 1 \end{pmatrix}$$

Considering the mention properties (2) - (4), it is not hard to see that the function C_m ($m = 2, 3, \dots$) preserves the matrix M_n then and only then when $m \neq n$. Indeed, taking $m = n$ and applying C_m on components over the first n columns of this matrix, we obtain the missing column in matrix, whose elements are equal to τ. In general, since C_m does not take the value 0, by applying the function C_m on some columns of the matrix M_n one obtains either one of the $2^n - 1$ right columns, or that column which is missing.

But to get the column that is made only from τ, C_m should be applied only to the first n columns of the matrix, otherwise it will appear the set containing 1. C_m also should be applied to all these columns (to avoid m-set consisting only of 0) and without repetitions (to avoid the repetition of τ in some set). But this operation is possible only for $m = n$. Therefore, for any $n = 2, 3, \ldots$ function C_n does not preserve matrix M_n and so it is not expressible through the other functions of the system (1). In other words, the system (1) is independent. And from this we can conclude that his own subsystems, proper and infinite, generates a lot of continuous cardinal of iterative distinct algebras, each of them having infinite basis. The theorem is proved.

In the next two theorems author will use an effecient syntactical specification for the notion of expressibility, proposed by A.V. Kuznețov [4]. First, let's remember that two formulas A and B are considered equivalent in a logic L, if their equivalence $(A{\sim}B)$ is valid in L. In this sense it is clear that the formulas are equivalent in the logic $L\mathfrak{I}$., if and only if, the pseudo-Boolean functions expressed by them, are (identical) equal. We say that the formula F is expressible in the logic L via the system of formulas Σ, if there is a finite sequence of formulas $F_1, F_2, \ldots, F_l$, called the expression of formula F in L through Σ, whose last term is F, and any term of it is either a variable or it belongs to the system Σ, or is obtained from previous terms constructed by applying one of the following two rules: 1) the weak rule of substitution, allowing to move from two formulas A and B to the result of substitution of B in A, everywhere where a variable will appear, 2) replacement rule of the equivalent in L, allowing to move from a given formula, to any other formula equivalent to it in L.

Let us admit the concept that a formula F preserves on the set $\{0, \tau, 1\}$ the predicate R, if and only if pseudo-Boolean function F preserves R. We say that formula F separates from formulas $G_1, \ldots, G_m$ (on $\{0, \tau, 1\}$ by the predicate R, if $G_1, \ldots, G_m$ preserve the predicate R, and F does not preserve this predicate.

Theorem 2. *There are iterative algebras of pseudo-Boolean 3-valued functions, which have no bases.*

Indeed, let us consider a system of pseudo-Boolean functions represented by the following system of formulas:

$$\{\perp x_1, \perp (x_1 \vee x_2), \ldots,$$

$$\perp (x_1 \vee x_2 \vee \ldots \vee x_n), \ldots\} \qquad (5)$$

We note the formulas $\perp x_1$, $\perp (x_1 \vee x_2)$,..., respectively, by symbols D_1, D_2,... we show that the algebra generated by this series, denoted by the symbol

$$[D_1, \ldots, Dn, \ldots], \qquad (6)$$

has no basis in $L\Im$.

Theorem 3.. *There are the multitudes of continuous cardinal of iterative algebras of 3-valued pseudo-Boolean functions, and none of these algebras have not any basis.*

Indeed, let note with the symbols

C_n $(n = 2, 3, \ldots)$ and D_n $(n = 1, 2, \ldots)$

respectively the formulas:

$$\mathop{\&}_{i=1}^{n} ((\neg x_1 \,\&\, \ldots \,\&\, \neg x_{i-1} \,\&$$

$$\&\neg x_{i+1} \,\&\, \ldots \,\&\, \neg x_n) \supset x_i),$$

$$\perp (x_1 \vee \ldots \vee x_n).$$

We introduce in the analysis the formulas system

$$\{C_2, C_3, \ldots, C_n, \ldots, D_1, D_2, \ldots, D_n, \ldots\}, \quad (7)$$

that is the reunion of the systems (1) and (5)

Let us consider for n = 2, 3, ... the matrix:

$$\begin{pmatrix} 00\ldots0\,\tau\tau\tau\tau\tau\tau\ldots11 \\ 00\ldots\tau\,0\tau\tau\tau\tau\tau\ldots11 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0\tau\ldots0\,0\,\tau\,\tau\tau11\ldots11 \\ \tau0\ldots0\,0\,\tau\,11\,\tau\tau\ldots11 \\ \tau\tau\ldots\tau\,\tau\,\tau\,\tau\,1\,\tau1\ldots\tau\,1 \end{pmatrix} \qquad (8)$$

in which the number of lines is equal to (n +1), the number of columns is equal to (n+2^{n+1}-1), each of the first n columns from the left contains exactly (n-1) times 0 and twice τ, and where among the columns containing only τ and 1 is missing only the column containing n times τ and ends with the element 1. It is not hard to see that any formula of the system (5) preserves the matrix (8). Comparing the matrix (8) with the M_n matrix, we can conclude that for any $n = 2, 3, \ldots$, formula C_n is separated by the matrix (8) of all other formulas of ensemble (7). Therefore, taking all possible sub-systems $\{C_{i1}, C_{i2}, \ldots, C_{in}, \ldots\}$ of the system (1), we get a multitude of cardinal continuous of distinct algebras of the form

$$[C_{i1}, C_{i2}, \ldots, C_{in}, \ldots, D_1, D_2, \ldots, D_n, \ldots], \quad (9)$$

that is closed with respect to expressibility.

Now consider an arbitrary algebra of the form (9) and an arbitrary system $\sum$ of formulas from this algebra, through which there are expressible in $L\Im$ all its formulas. One show that $\sum$ can not be a basis for (9), namely can not be independent in $L\Im$.

References

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