APPLYING OF PREDICATES FOR SETTLEMENT OF MODEL COMPLETENESS IN THE 3-VALUED EXTENSION OF PROVABILITY-INTUITIONISTIC LOGIC

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Abstract: The 3-valued extension of provability-intuitionistic logic is studied. It represents the logic L^{Δ} of 3-valued Δ-pseudo-boolean algebra $Z_3^{\Delta} =< \{0,\tau,1\};\&,\vee,\supset,\neg,\Delta >$ $(0<\tau<1,\ \Delta 0=\tau, \Delta\tau=\Delta 1=1)$. A model of a Boolean function $f(p_1,...,p_n)$ in the logic L^{Δ} we call any formula , which express a Δ-pseudo-boolean function $F(p_1,...,p_n)$ such that for every set $<\alpha_1,...,\alpha_n>$ of zeros and units the equality $f(\alpha_1,...,\alpha_n)=F[\alpha_1,...,\alpha_n]$ take place. The system Σ of formulas of the closed in L^{Δ} class *K* of formulas is called model complete in *K*, if for every Boolean function, which has a model in *K*, can be expressed via Σ at least one model of it in L^{Δ}. The criteria of model completeness in three classes of formulas [3,4], which model the Boolean function permutable with 0 and with 1 in the logic L^{Δ} are established.

The provability-intuitionistic calculus I^{Δ} [1] is based on formulas that are built, traditionally, from propositional variables, using logical operator symbols $\&,\vee,\supset,\neg,\Delta$ (unary). This calculus is defined by the axioms and inference rules of intuitionistic calculus and the following three Δ-axioms: $(p\supset\Delta p)$, $((\Delta p\supset p)\supset p)$, $(((p\supset q)\supset p)\supset(\Delta q\supset p))$. By the logic of a calculus we mean the set of formulas deductible in this calculus. This paper analyzes the logic of the calculus which is obtained from I^{Δ} by adding the new axiom: $(p\vee\neg q\vee(p\supset q))$. The logic of this calculus (we denote by L^{Δ}) is an extension of the provability-intuitionistic logic and coincides with the logic of the Δ-pseudo-Boolean algebra:

$$Z_3^{\Delta} =< \{0,\tau,1\};\&,\vee,\supset,\neg,\Delta >,$$
$$(0<\tau<1,\ \Delta 0=\tau, \Delta\tau=\Delta 1=1).$$

We say that a formula $F(p_1,...,p_n)$ preserves on the algebra A the predicate $R(x_1,...,x_m)$, if for any elements $\alpha_{ij}\in$ A $(i=1,...,m; j=1,...,n)$ from the truth of $R(\alpha_{11},...,\alpha_{m1}),...,\ R(\alpha_{1n},...,\alpha_{mn})$ result $R(F[\alpha_{11},...,\alpha_{1n}],...,F[\alpha_{m1},...,\alpha_{mn}])$. It is well known the meaning of this notion, when instead of the formula F and algebra A, consider the function $f(p_1,...,p_n)$ and a set when f is defined.

The set of functions of algebra Z_3^{Δ} coincides with the class U of 3-valued functions that preserve predicate $\neg x=\neg y$.

Let Q be the class of all formulas preserving on the algebra Z_3^{Δ} the predicate $x\in\{0,1\}$. We divide the class Q in a countable set of subclasses by a relation $R(x,y)$, such that two formulas $F(p_1,...,p_n)$ and $G(p_1,...,p_n)$ are in the relation R if and only if the equality $F[\alpha_1,...,\alpha_n]=G[\alpha_1,...,\alpha_n]$ holds for any set $<\alpha_1,...,\alpha_n>$ of zeros and units. The class Q with the relation R gives a one-to-one application of the factor Q/R on the set of all Boolean functions.

Thus, every formula of Q in some sense is a model of a Boolean function. In this context, we call model of Boolean function $f(p_1,...,p_n)$ in logic L^{Δ} a such formula $F(p_1,...,p_n)$ of this logic, that for any set $<\alpha_1,...,\alpha_n>$ of 0 and 1 the equality $f(\alpha_1,...,\alpha_n)=F[\alpha_1,...,\alpha_n]$ is true .

A formula F is expressible in the logic L through a system Σ of formulas , if F can be obtained from variables and formulas of Σ, using weak rule of substitution and replacement by equivalent rule in L. Let K be a class of formulas closed with respect to expressibility in logic L^{Δ}. The system Σ of formulas of the class K is called model complete in K, if for any Boolean function with a model in K, through Σ is expressible in L at least one model of this function. The system Σ of formulas of class K is called model pre-complete in K, if Σ is not model complete in K, but for every formula F that is not expressible in L^{Δ} through Σ, the system $\Sigma \cup \{F\}$ is model complete in K.

In [2] we showed that

Theorem 1. *The system Σ of formulas of the logic L^{Δ} is model complete, if and only if for any predicate*

$$x=0,\ x\neq 1,\ x\neq 0,\ \neg x\neq\neg y,$$
$$\neg\neg x\leq\neg\neg y,\ (x=\tau)\vee(x\leq\neg\neg y), \quad (1)$$
$$\neg x\sim\neg y=\neg z\sim\neg t,$$

there exists a formula in Σ which does not preserve this predicate.

We denote by U_0, $U_{0.1}$, U_1, U_2, U_3, $U_{3.1}$, U_4 the classes of formulas of the logic L^{Δ} that preserve on the algebra Z_3^{Δ} the predicates (1), respectively, and by T_0, T_1, S, M, L the classes of Boolean functions, preserving on the $\{0,1\}$ the predicates $x=0$, $x=1$, $x\neq y$, $x\leq y$, $x\sim y=z\sim t$, respectively. The last classes are all pre-complete classes of Boolean functions. Any Boolean function of the class T_0 has at least one model in the class U_0 and in class $U_{0.1}$, any Boolean function of the class T_1 has at least one model in the class U_1, and so on for Boolean function of S - in U_2, of M - in U_3 and in $U_{3.1}$, of L - in U_4.

The following three theorems are the basis of algorithms, which allow for any finite system of formulas to determine if in the logic L^{Δ} are expressible models for any Boolean function which preserves the constant 0 and for any Boolean function which preserves the constant 1.

Theorem 2. *The system Σ of formulas of class U_0 is model complete in U_0 if and only if for any predicate*

$$x\neq 0,\ \neg\neg x\leq\neg\neg y,\ (x=\tau)\vee(x\leq\neg\neg y),$$
$$\neg x\sim\neg y=\neg z\sim\neg t,\ x\,\&\,y=0, \quad (2)$$
$$x\,\&\,y\neq 1,\ (x\,\&\,y=0)\vee(x\vee y=\tau)$$

there exists a formula in Σ, which does not preserve this predicate.

The **necessity** arises from the fact that classes of formulas that preserve on the algebra Z_3^{Δ} the predicates (2), respectively, are closed with respect to expressibility in logic L^{Δ}, are incomparable with respect to inclusion and are not model complete in U_0.

The **sufficiency**. Let the formulas of

$$\{f_1,f_2,f_3,f_4,f_5,f_6,f_7\} \quad (3)$$

belong to the system Σ and not preserve predicates (2) respectively. We prove that the system (3) is model complete in U_0. Let all the formulas of the system (3) preserve on the algebra Z_3^{Δ} the predicate $x\in\{0,1\}$. Then even the subsystem $\{f_1,f_2,f_4,f_5\}$ is model complete in the class U_0.

Let us now assume that at least one formula $G(p_1,\ldots,p_n)$ of the system (3) does not preserve on the algebra Z_3^{Δ} the predicate $x\in\{0,1\}$. Through the formulas 0 and G is expressible a formula $G^*(p)$, satisfying the conditions $G^*[0]=0, G^*[1]=\tau$. Then the sufficiency follows from the next three lemmas.

Lemma 2.1. *The constant 0 is expressible in the logic L^{Δ} through the formula f_1.*

Lemma 2.2. *A model of the Boolean function $\neg p\,\&\,q$ is expressible in the logic L^{Δ} through the formulas $0, G^*, f_2, f_3, f_4$ and f_6.*

Lemma 2.3. *A model of the Boolean function $p\vee q$ is expressible in the logic L^{Δ} through the formulas $0, G^*, f_5, f_6, f_7$ and any model of Boolean function $\neg p\,\&\,q$.*

Theorem 3. *The system Σ of formulas of class $U_{0.1}$ is model complete in $U_{0.1}$ if and only if for any predicate*

$$x\neq 0,\ \neg x\neq\neg y,\ \neg\neg x\leq\neg\neg y, \quad (4)$$
$$(x=\tau)\vee(x\leq\neg\neg y),\ \neg x\sim\neg y=\neg z\sim\neg t,$$
$$x\,\&\,y=0,\ x\,\&\,y\neq 1,\ (x\,\&\,y=0)\vee(x\vee y=\tau)$$

there exists a formula in Σ, which does not preserve this predicate.

The **necessity** arises from the fact that classes of formulas that preserve on the algebra Z_3^{Δ} the predicates (4), respectively, are closed with respect to expressibility in the logic L^{Δ}, are incomparable with respect to inclusion and are not model complete in $U_{0.1}$.

The **sufficiency**. Let the formulas of

$$\{f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8\} \quad (5)$$

belong to the system Σ and not preserve predicates (4) respectively. We prove that this system is model complete in $U_{0.1}$, i.e. that through it are expressible in logic L^{Δ} the models of the Boolean functions $p \,\&\, q$ and $p+q$, which constitute a complete in T_0 system. Let all the formulas of the system (5) preserve on the algebra Z_3^{Δ} the predicate $x \in \{0,1\}$. Then even the subsystem $\{f_1, f_3, f_5, f_7\}$ is model complete in the class $U_{0.1}$.

Let us now assume that at least one formula $G(p_1,...,p_n)$ of the system (5) does not preserve on the algebra Z_3^{Δ} the predicate $x \in \{0,1\}$. Then the sufficiency follows from the next six lemmas.

Lemma 3.1. *The constant 0 is expressible in the logic L^{Δ} through the formulas f_1 and f_2 .*

Lemma 3.2. *Through the formulas* 0 *and* G *is expressible a formula $G^*(p)$, satisfying the condition* $G^*[\tau] = G^*[1] = \tau$.

Lemma 3.3. *Through the formulas $0, G^*, f_3, f_4$ and f_7 is expressible in the logic L^{Δ} the formula $p \oplus (q \vee \tau)$ or a model of one of the two Boolean functions $\neg p \,\&\, q$ or $p+q$.*

Lemma 3.4. *Let X be any model of one of two Boolean function $\neg p \,\&\, q$ or $p+q$, or the formula $p \oplus (q \vee \tau)$. Then through $0, G^*$ and X is expressible in logic L^{Δ} a formula $A(p)$, satisfying the conditions*

$$A[0] = 0, A[1] = \tau. \qquad (6)$$

Lemma 3.5. *Through the formulas $0, f_6, f_7, f_8$, any formula A, satisfying the conditions (6) and any model of the Boolean function $\neg p \,\&\, q$ is expressible in the logic L^{Δ} a model of the Boolean function $p+q$.*

Lemma 3.6. *Let Y is one of the formulas $p \oplus (q \vee \tau)$ or $p \oplus q$. Then through $0, f_5, Y$ and any formula A, satisfying the conditions (6), is expressible in logic L^{Δ} a model of the Boolean function $p \,\&\, q$.*

Theorem 4. *The system Σ of formulas of class U_1 is model complete in U_1 if and only if for any predicate*

$$\begin{aligned}
&x=0,\ x \neq 1,\ \neg\neg x \leq \neg\neg y, \\
&(x=\tau) \vee (x \leq \neg\neg y),\ x \vee y \neq 0, \\
&(x=0) \,\&\, (y=1) \vee (x \neq 0) \,\&\, (y=\tau), \\
&(\neg\neg x \,\&\, y \neq 1) \,\&\, (y \neq 0), \\
&(x \,\&\, y \neq 1) \,\&\, (y \neq 0), \\
&((x \neq 0) \vee (y \neq 0) \vee (z \neq 1)) \,\&\, (z \neq 0), \\
&(\neg x \sim \neg y = \neg z \sim \neg u) \,\&\, (u \neq 0), \qquad (7) \\
&((\neg x \sim \neg y = \neg z \sim \neg u) \,\&\, \\
&(u \neq 0)) \vee ((x \,\&\, y \,\&\, z = 0) \,\&\, (u = \tau)), \\
&(x=0) \,\&\, (u \neq 0) \vee (u = \tau) \vee \\
&(x \,\&\, y \,\&\, z \,\&\, u \neq 0), \\
&(x=0) \,\&\, (u \neq 0) \vee (x \,\&\, u = \tau) \vee \\
&(x \,\&\, y \,\&\, z \,\&\, u \neq 0)
\end{aligned}$$

there exists a formula in Σ, which does not preserve this predicate.

The **necessity** arises from the fact that classes of formulas that preserve on the algebra Z_3^{Δ} the predicates (7), respectively, are closed with respect to expressibility in logic L^{Δ}, are incomparable with respect to inclusion and are not model complete in U_1.

The **sufficiency**. Let the formulas of

$$\{f_1, f_2, ..., f_{13}\} \qquad (8)$$

belong to the system Σ and not preserve predicates (7) respectively. We prove that the system (8) is model complete in U_1. Let all the formulas of the system (8) preserve on the algebra Z_3^{Δ} the predicate $x \in \{0,1\}$. Then even the subsystem $\{f_1, f_3, f_5, f_{10}\}$ is model complete in the class U_1.

Let us now assume that at least one formula $G(p_1,...,p_n)$ of the system (8) does not preserve on the algebra Z_3^{Δ} the predicate $x \in \{0,1\}$. Then the sufficiency follows from the next four lemmas.

Lemma 4.1. *Through the formulas f_1, f_2, f_7 and f_8 is expressible in the logic L^{Δ} at least one of the four systems of formulas*

$$\{1\},\ \{\perp p\},\ \{p \supset \tau\},\ \{\neg p \vee \tau, H(p,q)\}, \quad (9)$$

where *H satisfies the conditions*

$$H[0,0] = H[\tau,1] = 1.$$

Lemma 4.2. *Through the formulas f_5, f_6, f_9 and any of the systems (9) is expressible in the logic L^{Δ}*

the formula $D(p,q)$, *satisfying the conditions*

$$D[0,1] = D[1,0] = 0, D[1,1] = 1. \qquad (10)$$

Lemma 4.3. A *model of the Boolean function* $p \& q$ *is expressible in the logic* L^{Δ} *through the formulas* f_{10}, f_{11} *and any formula* $D(p,q)$, *satisfying the conditions (10),*

Lemma 4.4. A *model of the Boolean function* $p \vee \neg q$ *is expressible in the logic* L^{Δ} *through the formulas* $G, f_3, f_4, f_6, f_{12}, f_{13}$, *any model of the Boolean function* $p \& q$ and any *of the systems (9).*

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