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# Multiplicity of positive solutions for nonlinear singular Neumann problems

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**Abstract**: We consider a nonlinear Neumann problem driven by the $p-$Laplacian and a reaction which consists of a singular term plus a $(p-1)$ - linear perturbation which is resonant at $+\infty$ with respect to the principal eigenvalue. Using variational methods together with suitable truncation, comparison and approximation techniques, we show that the problem admits two positive smooth solutions.

**Keywords**: Singular term, resonance, nonlinear regularity, truncations, nonlinear maximum principle, local minimizer.

**MSC2010**: 35J92, 35J25, 35J67

*Dedicated to Professor Constantin Corduneanu on the occasion of his 90th birthday*

## 1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^2-$ boundary $\partial\Omega$. In this paper we study the following nonlinear singular Neumann problem

$$\left\{ \begin{array}{l} -\triangle_p u\left(z\right) + \xi\left(z\right) u\left(z\right)^{p-1} = u\left(z\right)^{-\mu} + f\left(z, u\left(z\right)\right) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ u > 0. \end{array} \right. \tag{1.1}$$

In this problem $\xi \in L^{\infty}\left(\Omega\right)$ and $\triangle_p$ denotes the $p-$Laplace differential operator defined by

$$\triangle_p u = div\left(\left|Du\right|^{p-2} Du\right), \text{ for all } u \in W^{1,p}\left(\Omega\right),\ 1 < p < \infty,$$

where $|.|$ designates the $\mathbb{R}^N$ norm. The term $u^{-\mu}$ is the singular contribution in the reaction and we assume that $\mu \in (0,1)$. The perturbation $f\left(z,x\right)$ is a Carathéodory function, that is, for all $x \in \mathbb{R}$, $z \to f\left(z,x\right)$ is measurable and for a.a. $z \in \Omega$, $x \to f\left(z,x\right)$ is continuous. We assume that $f\left(z,.\right)$ exhibits $(p-1)$-linear growth near to

$+\infty$ and we may have resonance with respect to $\widehat{\lambda}_1 > 0$, the principal eigenvalue of the operator $u \to -\triangle_p u + \xi(z)|u|^{p-2}u$ with Neumann boundary condition. The resonance occurs from the right of $\widehat{\lambda}_1 > 0$, . This makes the energy functional of (1.1) noncoercive, and so the direct method of the calculus of variations is not directly applicable to the problem.

In the past, singular problems were investigated primarily in the context of Dirichlet problems. We mention the works of Hirano-Sacon-Shioji [8], Papageorgiou-Radulescu [12], Sun-Wu-Long [19] (semilinear Dirichlet problems) and Giacomoni-Schindler-Takac [7], Papageorgiou-Radulescu-Repovs [15], Papageorgiou-Smyrlis [16], [17], Perera-Zhang [18], Zhao-He-Zhao [20] (nonlinear Dirichlet problems). The study of singular Neumann problems is lagging behind. We mention the works of Chabrowski [4] (semilinear equations) and Aizicovici-Papageorgiou-Staicu [2] (nonlinear equations). In both papers, the problem is parametric and the presence of the parameter $\lambda > 0$ permits a more precise control of the reaction terms for all small values of $\lambda > 0$. Here no such parameter appears in the reaction.

## 2 Mathematical Background and Hypotheses

In this section we recall the main mathematical tools which will be used in the study of problem (1.1). We will also introduce our hypotheses on the data of the problem.

Let $(X, \|.\|)$ be a Banach space and $X^*$ be its topological dual. By $\langle .,. \rangle$ we denote the duality brackets for the pair $(X^*, X)$ and by $\xrightarrow{w}$ we denote the weak convergence in $X$.

Given $\varphi \in C^1(X, \mathbb{R})$ and a real number $c$, we say that *$c$ is a critical value of $\varphi$* if there exists $u^* \in X$ such that $\varphi'(u^*) = 0$ and $\varphi(u^*) = c$.

We say that $\varphi$ satisfies the *Cerami condition* (the *$C$- condition*, for short), if the following condition holds:
"every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\,\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence."

This is a compactness-type condition on the functional $\varphi$, which leads to a deformation lemma from which one can derive the minimax theory of the critical values of $\varphi$. One of the main results in this theory is the so called "mountain pass theorem" of Ambrosetti-Rabinowitz [3], which we state here in a slightly more general form (see Gasinski-Papageorgiou [5]).

**Theorem 2.1.** *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the $C$- condition, $u_0, u_1 \in X$ and $\rho > 0$ are such that $\|u_1 - u_0\| > \rho$,*

$$max\{\varphi(u_0),\ \varphi(u_1)\} < \inf\{\varphi(u)\ :\ \|u - u_0\| = \rho\} =: m_\rho,$$

*and*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$$

*with*

$$\Gamma = \{\gamma \in C([0,1],\ X)\ :\ \gamma(0) = u_0,\ \gamma(1) = u_1\},$$

*then $c \geq m_\rho$ and $c$ is a critical value of $\varphi$.*

In the analysis of problem (1.1) , we will use the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. By $\|.\|$ we will denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega).$$

where $\|.\|_p$ stands for the $L^p$-norm. The Banach space $C^1\left(\overline{\Omega}\right)$ is an ordered Banach space with a positive (order) cone given by

$$C_+ = \left\{u \in C^1\left(\overline{\Omega}\right) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior, which contains the open set

$$D_+ = \left\{u \in C^1\left(\overline{\Omega}\right) : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

In fact, $D_+$ is the interior of $C_+$ when $C^1\left(\overline{\Omega}\right)$ is furnished with the relative $C\left(\overline{\Omega}\right)-$ norm topology.

We impose the following condition on the potential function $\xi(.)$.

**H** $(\xi)$ : $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure.

We consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\triangle_p u(z) + \xi(z)|u(z)|^{p-2} = \widehat{\lambda}|u(z)|^{p-2} u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue for problem (2.1) , if there exists a nontrivial solution $\widehat{u} \in W^{1,p}(\Omega)$, known as an eigenfunction corresponding to $\widehat{\lambda}$. From Lieberman [9] (Theorem 2) we know that every eigenfunction $\widehat{u} \in C^1\left(\overline{\Omega}\right)$. Invoking Mugnai-Papageorgiou [11], we conclude that there exists a smallest eigenvalue $\widehat{\lambda}_1$ of (2.1) with the following properties:

- $\widehat{\lambda}_1 > 0$ and it is isolated (that is, if $\widehat{\sigma}(p)$ denotes the spectrum of (2.1) , then there exists $\varepsilon > 0$ such that $\left(\widehat{\lambda}_1, \widehat{\lambda}_1 + \varepsilon\right) \cap \widehat{\sigma}(p) = \varnothing$);

- $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}$, $\widehat{v}$ are two eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = t\widehat{v}$ with $t \in \mathbb{R} \setminus \{0\}$),

- If $\gamma(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p dz$ for all $u \in W^{1,p}(\Omega)$ then

$$\widehat{\lambda}_1 = \inf\left\{\frac{\gamma(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\ u \neq 0\right\}. \tag{2.2}$$

From the above properties it follows that the eigenfunctions corresponding to $\widehat{\lambda}_1$ have constant sign.

By $\widehat{u}_1$ we denote the positive $L^p-$ normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction for $\widehat{\lambda}_1$. We have $\widehat{u}_1 \in C_+ \setminus \{0\}$. Moreover the nonlinear strong maximum principle (see, for example, Gasinski-Papageorgiou [5], p.738) implies that $\widehat{u}_1 \in D_+$.

Using the Ljusternik-Shnirelmann minimax scheme, we infer that $\widehat{\sigma}(p)$ contains a sequence $\left\{\widehat{\lambda}_k\right\}_{k\geq 1}$ of eigenvalues such that $\widehat{\lambda}_k \to +\infty$. All the eigenfunctions corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1$ are nodal (that is, sign changing).

Similar results also hold for a weighted version of problem (2.1). Namely, let $\eta \in L^\infty(\Omega)$, $\eta(z) \geq 0$ for a.a. $z \in \Omega$, $\eta \not\equiv 0$ and consider the nonlinear eigenvalue problem:

$$\left\{\begin{array}{l} -\triangle_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widetilde{\lambda}\eta(z)|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{array}\right. \tag{2.3}$$

Again there is a smallest eigenvalue $\widetilde{\lambda}_1(\eta)$ of (2.3) such that:

- $\widetilde{\lambda}_1(\eta) > 0$ and it is isolated;

- $\widetilde{\lambda}_1(\eta)$ is simple;

- $$\widetilde{\lambda}_1(\eta) = \inf\left\{\frac{\gamma(u)}{\int_\Omega \eta(z)|u(z)|^p dz} : u \in W^{1,p}(\Omega),\ u \neq 0\right\}$$

Also, $\widehat{u}_1(\eta)$ is the positive $L^p-$ normalized principal eigenfunction, which satisfies $\widehat{u}_1(\eta) \in D_+$. These properties lead to the following strict monotonicity property of the map $\eta \to \widetilde{\lambda}_1(\eta)$ :
"*If* $\eta_1, \eta_2 \in L^\infty(\Omega)$, $0 \leq \eta_1(z) \leq \eta_2(z)$, $\eta_1 \not\equiv 0$, $\eta_2 \not\equiv \eta_1$, *then*

$$\widetilde{\lambda}_1(\eta_2) < \widetilde{\lambda}_1(\eta_1)."$$

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,x)| \leq a_0(z)\left(1 + |x|^{r-1}\right) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)_+$, and $1 < r \leq p^*$, where $p^*$ is the critical Sobolev exponent, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases}$$

We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the $C^1-$functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\gamma(u) - \int_\Omega F_0(z,u)\,dz \text{ for all } u \in W^{1,p}(\Omega). \tag{2.4}$$

The next result can be found in Papageorgiou-Radulescu ([14], Proposition 8).

**Proposition 2.2.** *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1\left(\overline{\Omega}\right)-$ minimizer of $\varphi_0$, that is, there exists $\delta > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1\left(\overline{\Omega}\right) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \delta$$

*then $u_0 \in C_0^{1,\eta}\left(\overline{\Omega}\right)$ for some $\eta \in (0,1)$ and it is also a local $W^{1,p}(\Omega)-$ minimizer of $\varphi_0$, that is, there exists $\delta' > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \leq \delta'.$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\,dz \text{ for all } u,\, h \in W^{1,p}(\Omega).$$

This map has the following properties (see Motreanu-Motreanu-Papa-georgiou ([10], p.40).

**Proposition 2.3.** *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (2.4) is monotone, continuous (hence maximal monotone, too), bounded (that is, maps bounded sets to bounded sets) and of type $(S)_+$, that is, for every sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that $u_n \xrightarrow{w} u$ and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0,$$

*one has*

$$u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

Also, we recall the following two inequalities. For every $y,\ y' \in \mathbb{R}^N$, we have

$$\left(|y|^{p-2}y - |y'|^{p-2}y', y - y'\right)_{\mathbb{R}^N} \geq \begin{cases} \widehat{c}\frac{|y-y'|^2}{(1+|y|+|y'|)^{2-p}} & \text{if} \quad 1 < p < 2 \\ \widehat{c}\,|y-y'|^p & \text{if} \quad\ \ 2 \leq p \end{cases} \tag{2.5}$$

with $\widehat{c} > 0$ (see Gasinski-Papageorgiou [5]).

Finally, we fix some basic notations. If $x \in \mathbb{R}$, then $x^{\pm} = \max\{\pm x, 0\}$. Given $u \in W^{1,p}(\Omega)$, we set $u^{\pm}(.) = u(.)^{\pm}$. We have

$$u^{\pm} \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

By $|.|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

The hypotheses on the perturbation term $f(z,\ x)$ are the following:

$\mathbf{H}(f)$ : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(z,\ 0) = 0$ for a.a. $z \in \Omega$ and:

$(i)$ for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z,\ x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leq x \leq \rho,$$

and there exists a function $w \in C\left(\overline{\Omega}\right) \cap W^{1,\infty}(\Omega)$ such that

$$w(z) \geq \widehat{C} > 0 \text{ for all } z \in \Omega,\ A(w) \geq 0 \text{ in } W^{1,p}(\Omega)^*,$$
$$w(z)^{-\mu} + f(z,\ w(z)) \leq -C_w < 0 \text{ for a.a. } z \in \Omega,$$

$(ii)$ If $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$\widehat{\lambda}_1 \leq \liminf_{x\to\infty} \frac{f(z,x)}{x^{p-1}} \leq \limsup_{x\to\infty} \frac{f(z,x)}{x^{p-1}} \leq \widehat{C}_0$$
$$\text{uniformly for a.a. } z \in \Omega,$$
$$f(z,x)x - pF(z,x) \to -\infty \text{ as } x \to \infty$$
$$\text{uniformly for a.a. } z \in \Omega;$$

$(iii)$ there exists $\delta_0 \in \left(0, \widehat{C}\right)$ such that

$$f(z,x) \geq C_m > 0 \text{ for all } z \in \Omega,\ \text{all } 0 \leq m \leq x \leq \delta_0;$$

$(iv)$ for every $\rho > 0$, there exists $\widehat{\theta}_\rho > 0$ such that for a.a. $z \in \Omega, x \to x^{-\mu} + f(z,\ x) + \widehat{\theta}_\rho x^{p-1}$ is nondecreasing on the interval $[\min\{\rho^{-\mu}, \rho\}, \max\{\rho^{-\mu}, \rho\}]$.

**Remarks:** Since we are interested in positive solutions and all of the above hypotheses concern only the nonnegative semiaxis $\mathbb{R}_+ = [0,\ +\infty)$, without any loss of generality, we may (and will) assume that

$$f(z,\ x) = 0 \text{ for a.a.} z \in \Omega,\ \text{ all } x \leq 0. \tag{2.6}$$

In hypothesis $\mathbf{H}(f)\,(i)$, the condition $A\,(w) \geq 0$ in $W^{1,p}\,(\Omega)^*$ means that

$$\langle A\,(w)\,, h\rangle = \int_\Omega |Dw|^{p-2}\,(Dw, Dh)_{\mathbb{R}^N}\, dz \geq 0 \text{ for all } h \in W^{1,p}\,(\Omega)\,,\ h \geq 0.$$

Hypothesis $\mathbf{H}(f)\,(ii)$ permits resonance with respect to the principal eigenvalue $\widehat{\lambda}_1 > 0$. The second asymptotic condition in hypothesis $\mathbf{H}(f)\,(ii)$ implies that the resonance is from the right of $\widehat{\lambda}_1$ in the sense that

$$\widehat{\lambda}_1 x^p - pF\,(z, x) \to -\infty \text{ as } x \to +\infty,\ \text{uniformly for a.a. } z \in \Omega.$$

Hypothesis $\mathbf{H}(f)\,(iv)$ is satisfied if for example for a.a. $z \in \Omega$, $f\,(z, .)$ is differentiable on $(0, \infty)$ and for every $\rho > 0$, there exists $\widehat{a}_\rho > 0$ such that

$$f_x^{'}(z,\ x)x \geq -\widehat{a}_\rho x^{p-1} \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leq x \leq \rho.$$

**Example:** The following function satisfies hypotheses $\mathbf{H}(f)$ (for the sake of simplicity we drop the $z-$ dependence):

$$f\,(x) = \begin{cases} x^{p-1} - 2x^{r-1} & \text{if } \ 0 \leq x \leq 1 \\ \eta x^{p-1} + x^{\tau-1} - (2+\eta)\, x^{q-1} & \text{if } \quad 1 < x, \end{cases}$$

with $\eta \geq \widehat{\lambda}_1$ and $1 < q < \tau < p < r < \infty$.

We mention that by a solution of problem (1.1) we mean a function $u \in W^{1,p}\,(\Omega)$ such that $u > 0$ and

$$\begin{aligned} &\int_\Omega |Du|^{p-2}\,(Du, Dh)_{\mathbb{R}^N}\, dz + \int_\Omega \xi\,(z)\, u^{p-1} h dz \\ &= \int_\Omega u^{-\mu} h dz + \int_\Omega f\,(z, u)\, h dz \text{ for all } h \in W^{1,p}\,(\Omega)\,. \end{aligned}$$

In the next section we prove the existence of two positive smooth solutions.

# 3 Positive solutions

We start by considering the following auxiliary singular Neumann problem

$$\begin{cases} -\triangle_p u(z) + \xi(z) u(z)^{p-1} = u(z)^{-\mu} \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u > 0. \end{cases} \tag{3.1}$$

**Proposition 3.1.** *If hypothesis* $\mathbf{H}(\xi)$ *holds and* $\mu \in (0,1)$ *then problem* (3.1) *admits a unique positive solution* $\widetilde{u} \in D_+$.

*Proof.* Let $\varepsilon > 0$ and consider the $C^1-$functional $\psi_\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\varepsilon(u) = \frac{1}{p}\gamma(u) - \frac{1}{1-\mu}\int_\Omega \left[(u^+)^p + \varepsilon\right]^{\frac{1-\mu}{p}} dz \text{ for all } u \in W^{1,p}(\Omega).$$

Since

$$\psi_\varepsilon(u) \geq \frac{C_1}{p}\|u\|^p - \frac{1}{1-\mu}\int_\Omega \left(u^+\right)^{1-\mu} dz - C_2$$

for some $C_1$, $C_2 > 0$, all $u \in W^{1,p}(\Omega)$ (see Mugnai-Papageorgiou [11], Lemma 4.11), it follows that $\psi_\varepsilon(.)$ is coercive.

Moreover, via the Sobolev embedding theorem, we see that $\psi_\varepsilon(.)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem we can find $u_\varepsilon \in W^{1,p}(\Omega)$ such that

$$\psi_\varepsilon(u_\varepsilon) = \inf\left\{\psi_\varepsilon(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.2}$$

Let $\tau \in (0,1)$. Then

$$\psi_\varepsilon(\tau) < \frac{\tau^p}{p}\|\xi\|_1 - \frac{\tau^{1-\mu}}{1-\mu}|\Omega|_N < \frac{\tau^p}{p}\|\xi\|_1 + \frac{1}{1-\mu}\left[\varepsilon^{\frac{1-\mu}{p}} - \tau^{1-\mu}\right]|\Omega|_N. \tag{3.3}$$

For $\tau > 2\varepsilon^{\frac{1}{p}}$, we have

$$\frac{\tau^p}{p}\|\xi\|_1 + \frac{1}{1-\mu}\left[\varepsilon^{\frac{1-\mu}{p}} - \tau^{1-\mu}\right]|\Omega|_N < \frac{\tau^p}{p}\|\xi\|_1 - \frac{\tau^{1-\mu}}{1-\mu}\left[1 - \left(\frac{1}{2}\right)^{1-\mu}\right]|\Omega|_N. \tag{3.4}$$

Since $\tau \in (0,1)$ and $0 < 1-\mu < 1 < p$, we can find $\tau_0 \in (0,1)$ small such that

$$\frac{\tau_0^p}{p}\|\xi\|_1 - \frac{\tau_0^{1-\mu}}{1-\mu}\left[1 - \left(\frac{1}{2}\right)^{1-\mu}\right]|\Omega|_N < 0. \tag{3.5}$$

From (3.3), (3.4) and (3.5) it follows that for $\varepsilon \in \left(0, \left(\frac{\tau_0}{2}\right)^p\right)$ we have

$$\psi_\varepsilon(\tau_0) < \psi_\varepsilon(0) = -\frac{1}{1-\mu}\varepsilon^{\frac{1-\mu}{p}}|\Omega|_N,$$

hence

$$\psi_\varepsilon(u_\varepsilon) < \psi_\varepsilon(0) \text{ (see (3.2))},$$

therefore $u_\varepsilon \neq 0$. From (3.2) we have

$$\psi'_\varepsilon(u_\varepsilon) = 0,$$

hence

$$\begin{aligned} &\langle A(u_\varepsilon), h\rangle + \textstyle\int_\Omega \xi(z) |u_\varepsilon(z)|^{p-2} u_\varepsilon(z) h dz \\ &= \textstyle\int_\Omega (u_\varepsilon^+)^{p-1} \left[(u_\varepsilon^+)^p + \varepsilon\right]^{\frac{1-(\mu+p)}{p}} h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.6}$$

In (3.6) we choose $h = -u_\varepsilon^- \in W^{1,p}(\Omega)$. Then we have

$$\left\|Du_\varepsilon^-\right\|_p^p + \int_\Omega \xi(z) \left(u_\varepsilon^-\right)^p dz = 0,$$

hence

$$\widehat{\lambda}_1 \left\|u_\varepsilon^-\right\|_p^p \leq 0 \text{ (see (2.2))},$$

therefore

$$u_\varepsilon \geq 0, \ u_\varepsilon \neq 0.$$

From (3.6) we infer that

$$\begin{cases} -\triangle_p u_\varepsilon(z) + \xi(z) u_\varepsilon(z)^{p-1} = u_\varepsilon(z)^{p-1} \left[u_\varepsilon(z)^p + \varepsilon\right] u(z)^{\frac{1-(\mu+p)}{p}} \\ \hfill \text{for a.a. } z \in \Omega, \\ \frac{\partial u_\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega \end{cases} \tag{3.7}$$

(see Papageorgiou-Radulescu [13]). From (3.7) and Papageorgiou-Radulescu [14] we have

$$u_\varepsilon \in L^\infty(\Omega).$$

So, we can apply Theorem 2 of Lieberman [9] and deduce that

$$u_\varepsilon \in C_+ \setminus \{0\}.$$

From (3.7) and hypothesis $\mathbf{H}(\xi)$ we have

$$\triangle_p u_\varepsilon(z) \leq \|\xi\|_\infty u_\varepsilon(z)^{p-1} \text{ for a.a. } z \in \Omega,$$

therefore

$$u_\varepsilon \in D_+$$

by the nonlinear strong maximum principle (see Gasinski-Papageorgiou [5], p. 738).
**Claim:** $\left\{u_\varepsilon : 0 < \varepsilon < \left(\frac{\tau_0}{2}\right)^p\right\} \subseteq W^{1,p}(\Omega)$ is bounded.

To prove the Claim, we argue by contradiction. So, suppose that the Claim is not true. Then, we can find $\{\varepsilon_n\}_{n\geq 1} \subseteq \left(0, \left(\frac{\tau_0}{2}\right)^p\right)$ and $\{u_n = u_{\varepsilon_n}\}_{n\geq 1} \subseteq D_+$ such that

$$\|u_n\| \to +\infty \text{ as } n \to \infty. \tag{3.8}$$

Let

$$y_n := \frac{u_n}{\|u_n\|},\ n\in \mathbb{N}.$$

Then

$$\|y_n\| = 1 \text{ and } y_n \geq 0 \text{ for all } n\in \mathbb{N}. \tag{3.9}$$

From (3.6) we have

$$\begin{gathered}\langle A\left(y_n\right), h\rangle + \textstyle\int_\Omega \xi\left(z\right) y_n\left(z\right)^{p-1} hdz \\ = \textstyle\int_\Omega y_n^{p-1}\left[u_n^p + \varepsilon_n\right]^{\frac{1-(\mu+p)}{p}} hdz \text{ for all } h \in W^{1,p}\left(\Omega\right), \text{ all } n \in \mathbb{N}.\end{gathered} \tag{3.10}$$

In (3.10) we choose $h = y_n \in W^{1,p}\left(\Omega\right)$ and obtain

$$\|Dy_n\|_p^p + \int_\Omega \xi\left(z\right) y_n^p dz = \int_\Omega \frac{y_n^p}{\left[u_n^p + \varepsilon_n\right]^{\frac{p+\mu-1}{p}}} dz \text{ for all } n \in \mathbb{N}. \tag{3.11}$$

Also, from the first part of the proof, we have

$$\begin{aligned} p\psi_{\varepsilon_n}\left(u_n\right) &= \|Du_n\|_p^p + \int_\Omega \xi\left(z\right) u_n^p dz - \frac{p}{1-\mu}\int_\Omega \left[u_n^p + \varepsilon_n\right]^{\frac{1-\mu}{p}} dz \\ &< 0 \text{ for all } n \in \mathbb{N}, \end{aligned}$$

hence

$$\begin{gathered}\|Dy_n\|_p^p + \textstyle\int_\Omega \xi\left(z\right) y_n^p dz - \frac{p}{1-\mu}\int_\Omega \frac{\left[u_n^p+\varepsilon_n\right]^{\frac{1-\mu}{p}}}{\|u_n\|^p} dz \\ < 0 \text{ for all } n \in \mathbb{N}.\end{gathered} \tag{3.12}$$

Using (3.11) in (3.12), we arrive at

$$\begin{aligned} &\int_\Omega \frac{y_n^p}{\left[u_n^p + \varepsilon_n\right]^{\frac{p+\mu-1}{p}}} dz < \frac{p}{1-\mu}\int_\Omega \frac{\left[u_n^p + \varepsilon_n\right]^{\frac{1-\mu}{p}}}{\|u_n\|^p} dz \\ &\leq \frac{p}{1-\mu}\int_\Omega \frac{u_n^{1-\mu} + \varepsilon_n^{\frac{1-\mu}{p}}}{\|u_n\|^p} dz \to 0 \text{ as } n \to \infty \text{ (see (3.8))}, \end{aligned}$$

hence

$$\|y_n\| \to 0 \text{ as } n \to \infty$$

(see (3.11) and Mugnai-Papageorgiou [11], Lemma 4.11).

But this contradicts (3.9) .Therefore the Claim is true.

Now consider a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \left(0, \left(\frac{\tau_0}{2}\right)^p\right)$ such that $\varepsilon_n \to 0^+$. On account of the Claim and by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} \widetilde{u} \text{ in } W^{1,p}(\Omega), \ u_n \to \widetilde{u} \text{ in } L^p(\Omega), \ \widetilde{u} \geq 0. \tag{3.13}$$

For all $n \in \mathbb{N}$ we have:

$$-\|Du_n\|_p^p - \int_\Omega \xi(z) u_n^p dz + \int_\Omega \frac{u_n^p}{[u_n^p + \varepsilon_n]^{\frac{p+\mu-1}{p}}} dz = 0 \tag{3.14}$$

(see (3.6) with $h = u_n \in W^{1,p}(\Omega)$), and

$$\|Du_n\|_p^p + \int_\Omega \xi(z) u_n^p dz - \frac{p}{1-\mu} \int_\Omega [u_n^p + \varepsilon_n]^{\frac{1-\mu}{p}} dz \leq -C_3 < 0 \tag{3.15}$$

for some $C_3 > 0$ (see (3.3) , (3.5)). Adding (3.14) and (3.15) we obtain

$$\begin{aligned} 0 &\leq \textstyle\int_\Omega \frac{u_n^p}{[u_n^p+\varepsilon_n]^{\frac{p+\mu-1}{p}}} dz \leq -C_3 + \frac{p}{1-\mu} \int_\Omega [u_n^p + \varepsilon_n]^{\frac{1-\mu}{p}} dz \\ &\leq -C_3 + \textstyle\frac{p}{1-\mu} \int_\Omega \left[u_n^{1-\mu} + \varepsilon_n^{\frac{1-\mu}{p}}\right] dz \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{3.16}$$

If $\widetilde{u} = 0$ (see (3.13)), then

$$\int_\Omega \left[u_n^{1-\mu} + \varepsilon_n^{\frac{1-\mu}{p}}\right] dz \to 0 \text{ as } n \to \infty.$$

This convergence and (3.16) above lead to a contradiction. Therefore

$$\widetilde{u} \neq 0.$$

By (3.13) and by passing to a further subsequence if necessary, we can say that there exists a function $\eta \in L^p(\Omega)$ such that

$$\begin{aligned} &0 \leq u_n(z) \leq \eta(z) \ \text{ for a.a. } z \in \Omega, \ \text{all } n \in \mathbb{N} \text{ and} \\ &u_n(z) \to \widetilde{u}(z) \ \text{ for a.a. } z \in \Omega. \end{aligned} \tag{3.17}$$

We can always assume that

$$1 \leq \eta(z) \ \text{ for a.a. } z \in \Omega. \tag{3.18}$$

For all $n \in \mathbb{N}$, let

$$\Omega_+^n = \{z \in \Omega : u_n(z) - \widetilde{u}(z) > 0\}$$

and

$$\Omega_-^n = \{z \in \Omega : u_n(z) - \widetilde{u}(z) \le 0\}.$$

We have

$$\begin{aligned} &\int_\Omega \frac{u_n^{p-1}}{\left[u_n^p+\varepsilon_n\right]^{\frac{p+\mu-1}{p}}}(u_n-\widetilde{u})\,dz \\ &= \int_{\Omega_+^n} \frac{u_n^{p-1}}{\left[u_n^p+\varepsilon_n\right]^{\frac{p+\mu-1}{p}}}(u_n-\widetilde{u})\,dz + \int_{\Omega_-^n} \frac{u_n^{p-1}}{\left[u_n^p+\varepsilon_n\right]^{\frac{p+\mu-1}{p}}}(u_n-\widetilde{u})\,dz \\ &\le \int_{\Omega_+^n} \frac{u_n-\widetilde{u}}{u_n^\mu}dz + \int_{\Omega_-^n} \frac{1}{2\eta^\mu}\left(\frac{u_n}{\eta}\right)^{p-1}(u_n-\widetilde{u})\,dz \text{ for all } n \in \mathbb{N} \end{aligned} \tag{3.19}$$

(see (3.17) and (3.18)). From (3.17), we have

$$0 \le \widetilde{u}(z) \le \eta(z) \text{ for a.a. } z \in \Omega, \tag{3.20}$$

$$-u_n(z)^{-\mu} \le -\eta(z)^{-\mu} \text{ for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}. \tag{3.21}$$

From (3.20) and (3.21), we have

$$-\widetilde{u}(z)\,u_n(z)^{-\mu} \le -\eta(z)^{1-\mu} \text{ for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}. \tag{3.22}$$

Then for all $n \in \mathbb{N}$

$$\int_{\Omega_+^n} \frac{u_n-\widetilde{u}}{u_n^\mu}dz = \int_{\Omega_+^n}\left[u_n^{1-\mu} - \widetilde{u}u_n^{-\mu}\right]dz \le \int_{\Omega_+^n}\left[\eta^{1-\mu}-\eta^{1-\mu}\right]dz = 0$$

(see (3.17) and (3.22)), hence

$$\limsup_{n\to\infty} \int_{\Omega_+^n} \frac{u_n-\widetilde{u}}{u_n^\mu}dz \le 0. \tag{3.23}$$

Also,

$$\int_{\Omega_-^n} \frac{1}{2\eta^\mu}\left(\frac{u_n}{\eta}\right)^{p-1}(u_n-\widetilde{u})\,dz \to 0 \text{ as } n\to\infty \tag{3.24}$$

(see (3.17) and (3.13)).

Returning to (3.19), passing to the limit as $n \to \infty$ and using (3.23) and (3.24), we obtain

$$\limsup_{n\to\infty} \int_{\Omega_+^n} \frac{u_n^{p-1}}{\left[u_n^p+\varepsilon_n\right]^{\frac{p+\mu-1}{p}}}(u_n-\widetilde{u})\,dz \le 0. \tag{3.25}$$

In (3.6) we choose $h = u_n - \widetilde{u} \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(u_n), u_n-\widetilde{u}\rangle + \int_\Omega \xi(z)\,|u_n(z)|^{p-1}(u_n-\widetilde{u})\,dz \\ &= \int_\Omega \frac{u_n^{p-1}}{\left[u_n^p+\varepsilon_n\right]^{\frac{p+\mu-1}{p}}}(u_n-\widetilde{u})\,dz \text{ for all } n \in \mathbb{N}, \end{aligned}$$

hence

$$\limsup_{n\to\infty}\left\langle A\left(u_n\right),u_n-\widetilde{u}\right\rangle\leq 0$$

(see (3.13) and (3.25)), therefore

$$u_n\to\widetilde{u}\text{ in }W^{1,p}\left(\Omega\right),\,\widetilde{u}\geq 0,\,\widetilde{u}\neq 0. \tag{3.26}$$

(see Proposition 2.3).

Now in (3.6) with $\varepsilon_n$ in place of $\varepsilon$, pass to the limit as $n\to\infty$ and use (3.26) and the fact that $\varepsilon_n\to 0^+$. We obtain

$$\left\langle A\left(\widetilde{u}\right),h\right\rangle+\int_\Omega\xi\left(z\right)\widetilde{u}^{p-1}hdz=\int_\Omega\widetilde{u}^{-\mu}hdz\text{ for all }h\in W^{1,p}\left(\Omega\right). \tag{3.27}$$

First in (3.27) we choose $h=\frac{1}{\left[\widetilde{u}^p+\varepsilon\right]^{\frac{p-1}{p}}}\in W^{1,p}\left(\Omega\right)$. Then

$$\int_\Omega\xi\left(z\right)\frac{\widetilde{u}^{p-1}}{\left[\widetilde{u}^p+\varepsilon\right]^{\frac{p-1}{p}}}dz\geq\int_\Omega\frac{\widetilde{u}^{-\mu}}{\left[\widetilde{u}^p+\varepsilon\right]^{\frac{p-1}{p}}}dz.$$

We let $\varepsilon\to 0$ and use Fatou's lemma. We obtain

$$\int_\Omega\frac{dz}{\widetilde{u}^{p+\mu-1}}dz\leq\left\|\xi\right\|_\infty\left|\Omega\right|_N. \tag{3.28}$$

Then we choose $h=\frac{1}{\left[\widetilde{u}^p+\varepsilon\right]^{\frac{2(p-1)+\mu}{p}}}\in W^{1,p}\left(\Omega\right)$. As above we obtain

$$\int_\Omega\xi\left(z\right)\frac{\widetilde{u}^{p-1}}{\left[\widetilde{u}^p+\varepsilon\right]^{2(p-1)+\mu}}dz\geq\int_\Omega\frac{\widetilde{u}^{-\mu}}{\widetilde{u}^{2(p-1)+\mu}}dz,$$

hence

$$\left\|\xi\right\|_\infty^2\left|\Omega\right|_N\geq\int_\Omega\frac{dz}{\widetilde{u}^{2(p+\mu-1)}}dz\text{ (see (3.28))}.$$

Continuing this way we obtain

$$\int_\Omega\frac{dz}{\widetilde{u}^{k(p+\mu-1)}}dz\leq\left\|\xi\right\|_\infty^k\left|\Omega\right|_N\text{ for all }k\in\mathbb{N}. \tag{3.29}$$

From (3.29) it follows that $\widetilde{u}^{-k(p+\mu-1)}\in L^q\left(\Omega\right)$ for all $q\geq 1$ and also

$$\limsup_{q\to\infty}\left\|\widetilde{u}^{-k(p+\mu-1)}\right\|_q<\infty.$$

Therefore $\widetilde{u}^{-(p+\mu-1)} \in L^{\infty}(\Omega)$ (see Gasinski-Papageorgiou [6], Problem 3.104, p. 477). Note that

$$\widetilde{u}^{-\mu} = \widetilde{u}^{-(p+\mu-1)}\widetilde{u}^{(p-1)}$$

so, from (3.27) and Papageorgiou-Radulescu [14] we have $\widetilde{u} \in L^{\infty}(\Omega)$. Then Theorem 2 of Lieberman [9] implies that $\widetilde{u} \in C_{+} \setminus \{0\}$.

We have

$$\begin{cases} -\triangle_p \widetilde{u}(z) + \xi(z)\widetilde{u}(z)^{p-1} = \widetilde{u}(z)^{-\mu} \text{ for a.a. } z \in \Omega, \\ \frac{\partial \widetilde{u}}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases}$$

hence

$$\triangle_p \widetilde{u}(z) \leq \|\xi\|_{\infty} \widetilde{u}(z)^{(p-1)} \text{ for a.a. } z \in \Omega,$$

therefore

$$\widetilde{u} \in D_{+}$$

by the nonlinear strong maximum principle (see Gasinski-Papageorgiou [5], p.738).

Finally we show that this positive solution is unique. So, suppose that $\widetilde{u}_0 \in W^{1,p}(\Omega)$ is another positive solution of (3.1). Again we have $\widetilde{u}_0 \in D_{+}$ and

$$\begin{aligned} 0 &\leq \langle A(\widetilde{u}) - A(\widetilde{u}_0), \widetilde{u} - \widetilde{u}_0 \rangle + \int_{\Omega} \xi(z) \left( \widetilde{u}^{p-1} - \widetilde{u}_0^{p-1} \right) (\widetilde{u} - \widetilde{u}_0)\, dz \\ &= \int_{\Omega} \left( \widetilde{u}^{-\mu} - \widetilde{u}_0^{-\mu} \right) (\widetilde{u} - \widetilde{u}_0)\, dz \leq 0. \end{aligned}$$

Therefore $\widetilde{u} = \widetilde{u}_0$ (see (2.5)). This proves the uniqueness of the positive solution $\widetilde{u} \in D_{+}$. □

Since $\widetilde{u} \in D_{+}$, we can find $t \in (0,1)$ small such that

$$\underline{u}(z) := t\widetilde{u}(z) \in (0, \delta_0] \text{ for all } z \in \overline{\Omega}, \tag{3.30}$$

where $\delta_0 > 0$ is as postulated by hypothesis $\mathbf{H}(f)(iii)$.

We have

$$\begin{aligned} &-\triangle_p \underline{u}(z) + \xi(z)\underline{u}(z)^{p-1} = t^{p-1}\left[ -\triangle_p \widetilde{u}(z) + \xi(z)\widetilde{u}(z)^{p-1} \right] \\ &= t^{p-1}\widetilde{u}(z)^{-\mu} \text{ (see Proposition 3.1)} \\ &\leq \underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) \text{ for all } z \in \Omega. \end{aligned} \tag{3.31}$$

(Recall that $t \in (0,1)$ and see (3.30) and hypothesis $\mathbf{H}(f)(iii)$).

Using $\underline{u} \in D_+$ and $w \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ from hypothesis $\mathbf{H}(f)(i)$, we introduce the following truncation of the reaction in problem (1.1) :

$$\widehat{f}(z,x) = \begin{cases} \underline{u}(z)^{-\mu} + f(z,\underline{u}(z)) & \text{if} \quad x < \underline{u}(z) \\ x^{-\mu} + f(z,x) & \text{if} \quad \underline{u}(z) \le x \le w(z) \\ w(z)^{-\mu} + f(z,w(z)) & \text{if} \quad w(z) < x. \end{cases} \tag{3.32}$$

Evidently this is a Carathéodory function.

In what follows, by $[\underline{u}, w]$ we denote the order interval in $W^{1,p}(\Omega)$ defined by

$$[\underline{u}, w] = \left\{ u \in W^{1,p}(\Omega) : \underline{u}(z) \le u(z) \le w(z) \text{ for all } z \in \Omega \right\}.$$

Also, by $int_{C^1(\overline{\Omega})}[\underline{u}, w]$ we denote the interior in $C^1(\overline{\Omega})$ of $[\underline{u}, w] \cap C^1(\overline{\Omega})$.

**Proposition 3.2.** *If hypotheses* $\mathbf{H}(\xi)$, $\mathbf{H}(f)$ *hold, then problem* (1.1) *has a positive solution*

$$u_0 \in int_{C^1(\overline{\Omega})}[\underline{u}, w].$$

*Proof.* We set $\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,ds$ and consider the $C^1$–functional $\widehat{\varphi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz - \int_\Omega \widehat{F}(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (3.32) it is clear that $\widehat{\varphi}(.)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\widehat{\varphi}(.)$ is sequentially weakly lower semicontinuous. So, by Weierstrass-Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}(u_0) = \inf\left\{\widehat{\varphi}(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.33}$$

From (3.33) it follows

$$\widehat{\varphi}'(u_0) = 0,$$

hence

$$\langle A(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h\,dz = \int_\Omega \widehat{f}(z,u_0)\,h\,dz \text{ for all } h \in W^{1,p}(\Omega), \tag{3.34}$$

therefore

$$\begin{cases} -\triangle_p u_0(z) + \xi(z)|u_0(z)|^{p-2}u_0 = \widehat{f}(z,u_0(z)) \text{ for a.a. } z \in \Omega, \\ \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.35}$$

(see Papageorgiou-Radulescu [13]). From (3.35) and Papageorgiou-Radulescu [14], we have $u_0 \in L^{\infty}(\Omega)$ and then Theorem 2 of Lieberman [9] implies that

$$u_0 \in C^1\left(\overline{\Omega}\right). \tag{3.36}$$

Now in (3.34) first we choose $h = (\underline{u} - u_0)^+ \in W^{1,p}(\Omega)$. Then, we have

$$\begin{aligned}
&\left\langle A(u_0), (\underline{u}-u_0)^+\right\rangle + \int_\Omega \xi(z)\, |u_0|^{p-2} u_0 (\underline{u}-u_0)^+ dz \\
&= \int_\Omega \left[\underline{u}^{-\mu} + \widehat{f}(z,\underline{u})\right](\underline{u}-u_0)^+ dz \text{ (see (3.32))} \\
&\geq \left\langle A(\underline{u}), (\underline{u}-u_0)^+\right\rangle + \int_\Omega \xi(z)\, \underline{u}^{p-1}(\underline{u}-u_0)^+ dz \text{ (see (3.31))},
\end{aligned}$$

therefore

$$\left\langle A(\underline{u}) - A(u_0), (\underline{u}-u_0)^+\right\rangle + \int_\Omega \xi(z)\left[\underline{u}^{p-1} - |u_0|^{p-2} u_0\right](\underline{u}-u_0)^+ dz \leq 0. \tag{3.37}$$

If $1 < p < 2$, then for $\eta = \max\left\{\|\underline{u}\|_{C^1(\overline{\Omega})}, \|u_0\|_{C^1(\overline{\Omega})}\right\}$ we have

$$\frac{\widehat{C}}{(1+2\eta)^{2-p}}\left\|D(\underline{u}-u_0)^+\right\|_2^2 + \int_\Omega \xi(z)\left((\underline{u}-u_0)^+\right)^2 dz \leq 0$$

(see (2.5)), hence

$$\frac{C_4}{(1+2\eta)^{2-p}}\left\|(\underline{u}-u_0)^+\right\|^2 \leq 0 \text{ for some } C_4 > 0,$$

therefore

$$\underline{u} \leq u_0.$$

If $2 \leq p$, then from (3.37) we have

$$\widehat{C}\left[\left\|D(\underline{u}-u_0)^+\right\|_p^p + \int_\Omega \xi(z)\,|\underline{u}-u_0|^p dz\right] \leq 0 \text{ (see (2.5))}$$

hence

$$C_5\left\|(\underline{u}-u_0)^+\right\|^p \leq 0 \text{ for some } C_5 > 0,$$

therefore

$$\underline{u} \leq u_0.$$

Next in (3.34) we choose $h = (u_0 - w)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\left\langle A(u_0), (u_0 - w)^+ \right\rangle + \int_\Omega \xi(z)\, u_0^{p-1} (u_0 - w)^+ \, dz \\
&= \int_\Omega \left[ w^{-\mu} + f(z, w) \right] (u_0 - w)^+ \, dz \text{ (see (3.32))} \\
&\le \int_\Omega (-C_w)(u_0 - w)^+ \, dz \text{ (see hypothesis } \mathbf{H}(f)(i)) \\
&\le \int_\Omega \left\langle A(w), (u_0 - w)^+ \right\rangle + \int_\Omega \xi(z)\, w^{p-1} (u_0 - w)^+ \, dz \text{ (see } \mathbf{H}(f)(i) \text{ and } \mathbf{H}(\xi)),
\end{aligned}$$

therefore

$$\left\langle A(u_0) - A(w), (u_0 - w)^+ \right\rangle + \int_\Omega \xi(z) \left[ u_0^{p-1} - w^{p-1} \right] (u_0 - w)^+ \, dz \le 0.$$

From this, as before, using (2.5), we obtain

$$u_0 \le w.$$

Therefore

$$u_0 \in [\underline{u}, w] \cap D_+ \text{ (see (3.36)).}$$

Now let $\rho = \|w\|$ and let $\widehat{\theta}_\rho > 0$ be as postulated by hypothesis $\mathbf{H}(f)(iv)$. For $\delta > 0$, let $\underline{u}^\delta = \underline{u} + \delta \in D_+$..

Then (see (3.31))

$$\begin{aligned}
&-\triangle_p \underline{u}^\delta(z) + \left( \xi(z) + \widehat{\theta}_\rho \right) + \left( \underline{u}^\delta \right)^{p-1} \\
&\le -\triangle_p \underline{u}(z) + \left( \xi(z) + \widehat{\theta}_\rho \right) \underline{u}^{p-1} + k(\delta) \text{ with } k(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
&\le \underline{u}(z)^{-\mu} + \widehat{\theta}_\rho \underline{u}^{p-1} + k(\delta)
\end{aligned} \tag{3.38}$$

Let $m = \min \left\{ \underline{u}(z) : z \in \overline{\Omega} \right\} > 0$. Recall that $\underline{u}(z) \in (0, \delta_0]$ for all $z \in \overline{\Omega}$. Then hypothesis $\mathbf{H}(f)(iii)$ implies that

$$0 < C_m \le f(z, \underline{u}(z)) \text{ for a.a. } z \in \Omega.$$

Since $k(\delta) \to 0^+$ as $\delta \to 0^+$, for $\delta > 0$ small we have

$$C_m - k(\delta) = \widehat{\eta} > 0. \tag{3.39}$$

Returning to (3.38) and using (3.39) we obtain

$$\begin{aligned}
&-\triangle_p \underline{u}^\delta(z) + \left( \xi(z) + \widehat{\theta}_\rho \right) \left( \underline{u}^\delta \right)^{p-1} \\
&\le \underline{u}^{-\mu} + f(z, \underline{u}) - \widehat{\eta} + \widehat{\theta}_\rho \underline{u}^{p-1} \text{ for } \delta > 0 \text{ small (see (3.39))} \\
&< u_0^{-\mu} + f(z, u_0) + \widehat{\theta}_\rho u_0^{p-1} \text{ (see hypothesis } \mathbf{H}(f)(iv)) \\
&= -\triangle_p u_0(z) + \left( \xi(z) + \widehat{\theta}_\rho \right) u_0^{p-1} \text{ for a.a. } z \in \Omega.
\end{aligned} \tag{3.40}$$

From (3.40) via the nonlinear Green's identity (see Gasinski-Papageorgiou [5], p.211), we get

$$\left\langle A\left(\underline{u}^{\delta}\right)-A\left(u_0\right),\left(\underline{u}^{\delta}-u_0\right)^{+}\right\rangle$$
$$+\int_{\Omega}\left[\xi(z)+\widehat{\theta}_{\rho}\right]\left[\left(\underline{u}^{\delta}\right)^{p-1}-u_0^{p-1}\right]\left(\underline{u}^{\delta}-u_0\right)^{+}dz\leq 0,$$

hence

$$\underline{u}^{\delta}\leq u_0 \text{ for } \delta>0 \text{ small},$$

therefore

$$u_0-\underline{u}\in D_{+}. \tag{3.41}$$

Next, for $\delta>0$, we set $u_0^{\delta}=u_0+\delta\in D_{+}.$. We have

$$\begin{aligned}&-\triangle_p u_0^{\delta}(z)+\left(\xi(z)+\widehat{\theta}_{\rho}\right)\left(u_0^{\delta}\right)^{p-1}\\&\leq u_0^{-\mu}+f\left(z,u_0\right)+\widehat{\theta}_{\rho}u_0^{p-1}+\chi(\delta)\ \text{ with } \chi(\delta)\to 0^{+} \text{ as } \delta\to 0^{+}\\&< w^{-\mu}+f(z,w)+\widehat{\theta}_{\rho}w^{p-1}+\chi(\delta)\ \text{ (see hypothesis } \mathbf{H}(f)\,(iv))\\&\leq -C_w+\widehat{\theta}_{\rho}w^{p-1}+\chi(\delta)\ \text{ (see hypothesis } \mathbf{H}(f)\,(i)\,).\end{aligned} \tag{3.42}$$

Since $\chi(\delta)\to 0^{+}$ as $\delta\to 0^{+}$, for $\delta>0$ small, we have

$$0\leq C_w-\chi(\delta).$$

Then, by (3.42) and using hypothesis $\mathbf{H}(f)\,(i))$ we arrive at

$$\begin{aligned}&\left\langle A\left(u_0^{\delta}\right),\left(u_0^{\delta}-w\right)^{+}\right\rangle+\int_{\Omega}\left(\xi(z)+\widehat{\theta}_{\rho}\right)\left(u_0^{\delta}\right)^{p-1}\left(u_0^{\delta}-w\right)^{+}dz\\&\leq\left\langle A(w),\left(u_0^{\delta}-w\right)^{+}\right\rangle+\int_{\Omega}\left(\xi(z)+\widehat{\theta}_{\rho}\right)w^{p-1}\left(u_0^{\delta}-w\right)^{+}dz\end{aligned}$$

hence

$$u_0^{\delta}\leq w \text{ for } \delta>0 \text{ small},$$

therefore

$$\left(w-u_0^{\delta}\right)(z)>0 \text{ for all } z\in\overline{\Omega}. \tag{3.43}$$

Then from (3.41) and (3.43) it follows that

$$u_0\in int_{C^1\left(\overline{\Omega}\right)}\left[\underline{u},w\right].$$

□

Next we will produce a second positive solution for problem (3.43), distinct from $u_0\in D_{+}$.

**Proposition 3.3.** *If hypotheses* $\mathbf{H}(\xi)$, $\mathbf{H}(f)$ *hold, then problem* (1.1) *admits a second positive solution*

$$\widehat{u} \in D_+.$$

*Proof.* We consider the following Carathéodory function

$$g(z,x) = \begin{cases} \underline{u}(z)^{-\mu} + f(z,\underline{u}(z)) & \text{if} \quad x < \underline{u}(z) \\ x^{-\mu} + f(z,x) & \text{if} \quad \underline{u}(z) \le x. \end{cases} \tag{3.44}$$

We set $G(z,x) = \int_0^x g(z,s)\,ds$ and consider the $C^1-$ functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\gamma(u) - \int_\Omega G(z,u(z))\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

**Claim:** $\varphi_0(.)$ satisfies the $C-$condition.

Let $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_0(u_n)| \le M_1 \text{ for some } M_1 > 0, \text{ all } n \in \mathbb{N} \tag{3.45}$$

and

$$(1 + \|u_n\|)\,\varphi_0'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ (as } n \to \infty). \tag{3.46}$$

From (3.46) we have

$$\begin{aligned} &\left|\langle A(u_n), h\rangle + \textstyle\int_\Omega \xi(z)|u_n|^{p-2}u_n h dz - \int_\Omega g(z,u_n)\,h dz\right| \le \tfrac{\varepsilon_n\|h\|}{1+\|u_n\|} \\ &\qquad\qquad \text{for all } h \in W^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+ \text{ as } n \to \infty. \end{aligned} \tag{3.47}$$

In (3.47) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\gamma(u_n^-) \le \int_\Omega \left(\underline{u}(z)^{-\mu} + f(z,\underline{u}(z))\right)\left(-u_n^-\right)dz \text{ (see (3.44))},$$

hence

$$C_6\left\|u_n^-\right\|^p \le C_7\left\|u_n^-\right\| \text{ for some } C_6,\ C_7 > 0, \text{ all } n \in \mathbb{N}$$

(see Mugnai-Papageorgiou [11], Lemma 4.11), therefore

$$\left\{u_n^-\right\}_{n\ge1} \subseteq W^{1,p}(\Omega) \text{ is bounded (recall that } p > 1). \tag{3.48}$$

We will show that $\{u_n^+\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded, too. Arguing by contradiction, suppose that

$$\left\|u_n^+\right\| \to \infty \text{ as } n \to \infty. \tag{3.49}$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then

$$\|y_n\| = 1 \text{ and } y_n \geq 0 \text{ for all } n \in \mathbb{N}.$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega), \ y_n \to y \text{ in } L^p(\Omega). \tag{3.50}$$

From $(3.47)$ and $(3.48)$, we have

$$\left| \left\langle A\left(u_n^+\right), h\right\rangle + \int_\Omega \xi(z)\left(u_n^+\right)^{p-1} h dz - \int_\Omega g\left(z, u_n^+\right) h dz \right| \leq C_8 \|h\|$$

for some $C_8 > 0$, all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$, hence

$$\left| \left\langle A(y_n), h\right\rangle + \int_\Omega \xi(z) y_n^{p-1} h dz - \int_\Omega \frac{g\left(z, u_n^+\right)}{\left\|u_n^+\right\|^{p-1}} h dz \right| \leq \frac{C_8\|h\|}{\left\|u_n^+\right\|^{p-1}}, \tag{3.51}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$. Hypotheses $\mathbf{H}(f)(i),(ii)$ imply that

$$|f(z,x)| \leq C_9\left[1 + x^{p-1}\right] \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ some } C_9 > 0.$$

This growth estimate together with (3.44), (3.49) and (3.50) implies that

$$\left\{ \frac{g\left(\cdot, u_n^+(\cdot)\right)}{\left\|u_n^+\right\|^{p-1}} \right\}_{n \geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded } \left(\frac{1}{p} + \frac{1}{p'} = 1\right). \tag{3.52}$$

So, by passing to a subsequence if necessary and using hypothesis $\mathbf{H}(f)(ii)$ we have

$$\begin{gathered} \frac{g\left(\cdot, u_n^+(\cdot)\right)}{\left\|u_n^+\right\|^{p-1}} \xrightarrow{w} \eta_0(z) y^{p-1} \text{ in } L^{p'}(\Omega) \text{ as } n \to \infty \\ \widehat{\lambda}_1 \leq \eta_0(z) \leq \widehat{C}_0 \text{ for a.a. } z \in \Omega. \end{gathered} \tag{3.53}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

In $(3.51)$ we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use $(3.50)$ and $(3.52)$. Then

$$\lim_{n \to \infty} \left\langle A(y_n), y_n - y\right\rangle = 0,$$

and in view of Proposition 2.3 we have

$$y_n \to y \text{ in } W^{1,p}(\Omega), \text{ hence } \|y\| = 1, y \geq 0. \tag{3.54}$$

Therefore, if in (3.51) we pass to the limit as $n \to \infty$ and use (3.54) and (3.53), then

$$\langle A(y), h\rangle + \int_\Omega \xi(z)\, y^{p-1} h dz = \int_\Omega \eta_0(z)\, y^{p-1} h dz \text{ for all } h \in W^{1,p}(\Omega).$$

Therefore

$$\begin{cases} -\triangle_p y(z) + \xi(z)\, y(z)^{p-1} = \eta_0(z)\, y(z)^{p-1} \text{ for a.a. } z \in \Omega, \\ \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.55}$$

First assume that $\eta_0 \not\equiv \widehat{\lambda}_1$ (see (3.53)). Then

$$\widetilde{\lambda}_1(\eta_0) < \widetilde{\lambda}_1\left(\widehat{\lambda}_1\right) = 1 \text{ (see (3.53))}. \tag{3.56}$$

By (3.55), (3.56) and since $y \not\equiv 0$ (see (3.54)), we infer that

$$y(\cdot) \text{ is nodal (that is, sign changing),}$$

which contradicts (3.54).

Next assume that $\eta_0(z) = \widehat{\lambda}_1$ for a.a. $z \in \Omega$. Then by (3.55) (cf. also (2.3)) and since $y \neq 0$ it follows that

$$y = \theta \widehat{u}_1 \text{ for some } \theta > 0 \text{ (see (3.54))}.$$

We have that $y \in D_+$ and so $y(z) > 0$ for all $z \in \overline{\Omega}$. Hence

$$u_n^+(z) \to \infty \text{ for all } z \in \overline{\Omega} \text{ as } n \to \infty,$$

hence

$$f\left(z, u_n^+(z)\right) u_n^+(z) - pF\left(z, u_n^+(z)\right) \to -\infty \text{ for a.a. } z \in \Omega \text{ as } n \to \infty$$

(see hypothesis $\mathbf{H}(f)(ii)$), therefore

$$\int_\Omega \left[f\left(z, u_n^+(z)\right) u_n^+(z) - pF\left(z, u_n^+(z)\right)\right] dz \to -\infty \text{ as } n \to \infty \tag{3.57}$$

(by Fatou's lemma). In (3.47) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-\left\|Du_n^+\right\|_p^p - \int_\Omega \xi(z)\left(u_n^+\right)^p dz + \int_\Omega g\left(z, u_n^+\right) u_n^+ dz \geq -\varepsilon_n \text{ for all } n \in \mathbb{N} \tag{3.58}$$

Also, from (3.45) and (3.48) it follows that

$$\left\|Du_n^+\right\|_p^p + \int_\Omega \xi(z)\left(u_n^+\right)^p dz - \int_\Omega pG\left(z, u_n^+\right) dz \geq -M_2 \tag{3.59}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$. Adding (3.58) and (3.59) we obtain

$$\int_\Omega \left[g\left(z, u_n^+\right)u_n^+ - pG\left(z, u_n^+\right)\right] dz \geq -M_3 \text{ for some } M_3 > 0, \text{ all } n \in \mathbb{N}$$

hence

$$\int_\Omega \left[f\left(z, u_n^+\right)u_n^+ - pF\left(z, u_n^+\right)\right] dz \geq -M_4 \text{ for some } M_4 > 0, \text{ all } n \in \mathbb{N} \tag{3.60}$$

(see (3.44)). Comparing (3.57) and (3.60), we have a contradiction.

Therefore $\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded, and then $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded (see (3.48)).

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega),\ u_n \to u \text{ in } L^p(\Omega). \tag{3.61}$$

In (3.47) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use the convergence in (3.61). We obtain

$$\lim_{n\to\infty} \left\langle A\left(u_n^+\right), u_n - u\right\rangle = 0,$$

therefore

$$u_n \to u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.3).}$$

Therefore $\varphi_0(\cdot)$ satisfies the $C-$condition. This proves the Claim.

From (3.32) and (3.44) we see that

$$\widehat{\varphi}|_{[\underline{u},w]} = \varphi_0|_{[\underline{u},w]}. \tag{3.62}$$

Recall that $u_0 \in D_+$ is a minimizer of $\widehat{\varphi}$ (see the proof of Proposition 3.2). Also recall that

$$u_0 \in int_{C^1(\overline{\Omega})}[\underline{u}, w]$$

(see Proposition 3.2).

This fact and (3.62) imply that $u_0$ is a local $C^1(\overline{\Omega})$ minimizer of $\varphi_0$ hence

$$u_0 \text{ is a local } W^{1,p}(\Omega) \text{ minimizer of } \varphi_0 \tag{3.63}$$

(see Proposition 2.2).

Let $K_{\varphi_0}$ denote the set of critical points of $\varphi_0$, that is,

$$K_{\varphi_0} = \left\{u \in W^{1,p}(\Omega) : \varphi_0'(u) = 0\right\}.$$

Using (3.44) we can easily check that

$$K_{\varphi_0} \subseteq [\underline{u}) \cap D_+ = \left\{u \in D_+ : \underline{u}(z) \leq u(z) \text{ for all } z \in \overline{\Omega}\right\}$$

(by the nonlinear regularity theory). Therefore, on account of (3.44), we see that we may assume that $K_{\varphi_0}$ is finite, or otherwise we already have an infinity of positive smooth solutions of (1.1), and so we are done.

Then (3.63) implies that there exists $\rho \in (0,1)$ small, such that

$$\varphi_0(u_0) < \inf\{\varphi_0(u) : \|u - u_0\| = \rho\} = m_0 \tag{3.64}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

Hypothesis $\mathbf{H}(f)(ii)$ implies that given any $\tau > 0$, we can find $M_5 = M_5(\tau) > 0$ such that

$$f(z,x)x - pF(z,x) \leq -\tau \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_5. \tag{3.65}$$

We have

$$\begin{aligned} \frac{d}{dx}\left[\frac{F(z,x)}{x^p}\right] &= \frac{f(z,x)x^p - pF(z,x)x^{p-1}}{x^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \\ &\leq \frac{-\tau}{x^{p+1}} \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_5 \text{ (see (3.65))}. \end{aligned}$$

Therefore

$$\frac{F(z,x)}{x^p} - \frac{F(z,v)}{v^p} \leq \frac{\tau}{p}\left[x^{-p} - v^{-p}\right] \text{ for a.a. } z \in \Omega, \text{ all } x \geq v \geq M_5. \tag{3.66}$$

Note that hypothesis $\mathbf{H}(f)(ii)$ implies that

$$\widehat{\lambda}_1 \leq \liminf_{x\to+\infty} \frac{pF(z,x)}{x^p} \leq \limsup_{x\to\infty} \frac{pF(z,x)}{x^p} \leq \widehat{C}_0 \text{ uniformly for a.a. } z \in \Omega. \tag{3.67}$$

So, if in (3.66) we let $x \to \infty$ and use (3.67), then

$$\widehat{\lambda}_1 v^p - pF(z,v) \leq -\tau \text{ for a.a. } z \in \Omega, \text{ all } v \geq M_5,$$

hence

$$\widehat{\lambda}_1 v^p - pF(z,v) \to -\infty \text{ as } v \to +\infty, \text{uniformly for a.a. } z \in \Omega,$$

therefore

$$\widehat{\lambda}_1 v^p - pG(z,v) \to -\infty \text{ as } v \to +\infty, \text{uniformly for a.a. } z \in \Omega, \tag{3.68}$$

(see (3.44)). For $t > 0$, we have

$$p\varphi_0(t\widehat{u}_1) = t^p \widehat{\lambda}_1 \|\widehat{u}_1\|_p^p - p\int_\Omega pG(z, t\widehat{u}_1)\,dz = \int_\Omega \left[\widehat{\lambda}_1 t^p - pG(z, t\widehat{u}_1)\right] dz,$$

hence

$$\varphi_0(t\widehat{u}_1) \to -\infty \text{ as } t \to +\infty \tag{3.69}$$

(see (3.68) and use Fatou's lemma). Then the Claim, (3.64) and (3.69) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\varphi_0} \subseteq [\underline{u}) \cap D_+ \text{ and } \varphi_0(u_0) < m_0 \leq \varphi_0(\widehat{u})$$

(see (3.64)). It follows that $\widehat{u} \in D_+$ is a second positive solution of (1.1), distinct from $u_0$. □

Concluding, we can state the following multiplicity theorem for problem (1.1).

**Theorem 3.4.** *If hypotheses* $\mathbf{H}(\xi)$, $\mathbf{H}(f)$ *hold, then problem* (1.1) *admits two positive solutions*

$$u_0,\ \widehat{u} \in D_+.$$

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