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Optimization problems in classes of rearrangements for (p,q)-Laplace equations

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Abstract: This paper is concerned with maximization and minimization of a functional associated with solutions of (p,q)-Laplace equations depending on functions which belong to a class of rearrangements. We prove existence and uniqueness results, and present some features of optimal solutions.

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Dedicated to .....

1 Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, and let $1 < q < p$. We consider the boundary value problem

$$-\Delta_p u - \Delta_q u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The non-homogeneous differential operator $\Delta_p + \Delta_q$ is called (p,q)-Laplacian. As observed in [17], it stems from a wide range of important applications including biophysics [10], plasma physics [19], reaction-diffusion equations [7], as well as models of elementary particles [2]. In the last decades there has been a great interest in the investigation of these problems mainly concerning existence and multiplicity of solutions, eigenvalues, ground-state solutions [1, 6, 11].

In the present paper we consider the case $f(x,u) = g(x)|u|^{\alpha-1}$, where $1 \leq \alpha < q$ and $g(x)$ is a measurable bounded non-negative function which is positive in a subset with a positive measure. For $v \in H_0^{1,p}(\Omega)$ we define

$$I(v) = \int_\Omega \Big(\frac{1}{p}|\nabla v|^p + \frac{1}{q}|\nabla v|^q - \frac{1}{\alpha} g|v|^\alpha\Big) dx,$$

and consider the classical minimization problem

$$\inf_{v \in H_0^{1,p}(\Omega)} I(v). \tag{1.1}$$

The proof of the existence of a solution to this problem is standard. Let $v_i \in H_0^{1,p}(\Omega)$ be a sequence such that

$$\check{I} := \lim_{i \to \infty} I(v_i),$$

where $\check{I}$ is the value of the inferior of $I(v)$. By using Poincaré and Hölder inequalities we find

$$\int_\Omega g|v_i|^\alpha dx \le C \int_\Omega |\nabla v_i|^\alpha dx \le C\Big(\int_\Omega |\nabla v_i|^p dx\Big)^{\frac{\alpha}{p}}.$$

Here and in what follows we denote by C constants (possibly different) independent of i. It follows that

$$I(v_i) \ge \frac{1}{p}\int_\Omega |\nabla v_i|^p \, dx - C\Big(\int_\Omega |\nabla v_i|^p \, dx\Big)^{\frac{\alpha}{p}}.$$

By using the well known inequality

$$(\epsilon B)\frac{1}{\epsilon} \le \frac{\alpha}{p}(\epsilon B)^{\frac{p}{\alpha}} + \frac{p-\alpha}{p}\Big(\frac{1}{\epsilon}\Big)^{\frac{p}{p-\alpha}}$$

for $B = C\Big(\int_\Omega |\nabla v_i|^p dx\Big)^{\frac{\alpha}{p}}$ and for a suitable value of ϵ we find

$$C\Big(\int_\Omega |\nabla v_i|^p dx\Big)^{\frac{\alpha}{p}} \le \frac{1}{2p}\int_\Omega |\nabla v_i|^p dx + C.$$

(Clearly, the two constants C in above are different.) Hence, we have

$$I(v_i) \ge \frac{1}{2p}\int_\Omega |\nabla v_i|^p \, dx - C.$$

It follows that the value of $\check{I}$ is finite. We also find that

$$\int_\Omega |\nabla v_i|^p dx \le C.$$

Hence, a subsequence of $\{v_i\}$ (denoted again $\{v_i\}$) converges weakly in $H^{1,p}(\Omega)$ and in $H^{1,q}(\Omega)$ to some function $u \in H_0^{1,p}(\Omega)$. By Rellich's Theorem, $\{v_i\}$ converges to

u strongly in $L^p(\Omega)$ as well as in $L^q(\Omega)$ and in $L^\alpha(\Omega)$. Therefore, we have

$$\begin{aligned}\check{I} \leq I(u) &= \int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla u|^q - \frac{1}{\alpha}g|u|^\alpha\Big)dx \\ &\leq \liminf_{i\to\infty}\int_\Omega \frac{1}{p}|\nabla v_i|^p dx + \liminf_{i\to\infty}\int_\Omega \frac{1}{q}|\nabla v_i|^q dx - \lim_{i\to\infty}\int_\Omega \frac{1}{\alpha}g|v_i|^\alpha\, dx \\ &\leq \lim_{i\to\infty}\int_\Omega \Big(\frac{1}{p}|\nabla v_i|^p + \frac{1}{q}|\nabla v_i|^q - \frac{1}{\alpha}g|v_i|^\alpha\Big)dx = \check{I}.\end{aligned}$$

Hence, u is a minimizer in (1.1). Since $I(u) = I(|u|)$, we may assume that a non-negative minimizer exists.

We note that a minimizer u of the functional $I(v)$ is a maximizer of the functional

$$-I(v) = \int_\Omega \Big(\frac{1}{\alpha}g|v|^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx.$$

It is easy to show that a non-negative minimizer u of problem (1.1) is a solution to the following boundary value problem:

$$\int_\Omega \big((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla\phi - g u^{\alpha-1}\phi\big)dx = 0 \quad \forall\phi \in H_0^{1,p}(\Omega). \tag{1.2}$$

Problem (1.2) has been discussed in [15, 16]. In particular, since g is non-negative and bounded, any solution u belongs to $C^{1,\sigma}(\Omega)$ for some $0 < \sigma < 1$. Another important fact is that either $u(x) \equiv 0$ or $u(x) > 0$ in Ω (see [18], Theorem 2.5.1 page 74 and Corollary 7.1.3 page 163).

Under our assumptions (in particular $\alpha < q < p$), a minimizer of (1.1) is non trivial. Indeed, if $v \in H^{1,p}(\Omega)$, $v > 0$, and $\epsilon > 0$ small enough we have

$$I(\epsilon v) = \epsilon^\alpha \int_\Omega \Big(\frac{\epsilon^{p-\alpha}}{p}|\nabla v|^p + \frac{\epsilon^{q-\alpha}}{q}|\nabla v|^q - \frac{1}{\alpha}g(x)v^\alpha\Big)dx < 0.$$

The following uniqueness result is crucial for our purposes.

Theorem 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, let $g \in L^\infty_+(\Omega)$ and let $1 \leq \alpha < q < p$. If $u, v \in H_0^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ are positive solutions to problem* (1.2) *then $u(x) = v(x)$ in Ω.*

Proof. This proof is inspired by the proof of Theorem 3.2 in [9]. See also [13], page 160. Define $A = \{x \in \Omega : u(x) > v(x)\}$. If we prove that A is empty, the assertion of the theorem follows. We argue by contradiction, assuming A is not empty. For $\epsilon > 0$, define $u_\epsilon = u + \epsilon$ and $v_\epsilon = v + \epsilon$. Note that in A we have $u_\epsilon(x) > v_\epsilon(x)$. Using

$$\phi_1(x) = \max\Big[\frac{u_\epsilon^\alpha(x) - v_\epsilon^\alpha(x)}{u_\epsilon^{\alpha-1}(x)}, 0\Big]$$

as test function in the equation for u we obtain

$$\begin{aligned}&\int_A |\nabla u|^{p-2}\nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla u|^{q-2}\nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx\\ &= \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha)\Big(\frac{u}{u_\epsilon}\Big)^{\alpha-1}dx.\end{aligned}$$

Using

$$\phi_2(x) = \max\Big[\frac{u_\epsilon^\alpha(x) - v_\epsilon^\alpha(x)}{v_\epsilon^{\alpha-1}(x)}, 0\Big]$$

as test function in the equation for v we obtain

$$\begin{aligned}&\int_A |\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx\\ &= \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha)\Big(\frac{v}{v_\epsilon}\Big)^{\alpha-1}dx.\end{aligned}$$

Subtracting the latter equality from the first one we get

$$\begin{aligned}&\int_A |\nabla u|^{p-2}\nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla u|^{r-2}\nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx\\ &+ \int_A |\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{r-2}\nabla v \cdot \nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx = L_\epsilon,\end{aligned} \tag{1.3}$$

where

$$L_\epsilon = \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha)\Big(\frac{u}{u_\epsilon}\Big)^{\alpha-1}dx + \int_A g(x)(v_\epsilon^\alpha - u_\epsilon^\alpha)\Big(\frac{v}{v_\epsilon}\Big)^{\alpha-1}dx. \tag{1.4}$$

We claim that the sum of the second and the fourth integrals in the left hand side of (1.3) in non-negative. Indeed, since

$$\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big) = \nabla u + (\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha \nabla u - \alpha\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla v$$

and

$$\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big) = \nabla v + (\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha \nabla v - \alpha\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla u$$

we find

$$\begin{aligned}
&\int_A |\nabla u|^{q-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx\\
&= \int_A \Big\{|\nabla u|^q\Big(1+(\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla u|^{q-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v\Big\}dx \qquad (1.5)\\
&+ \int_A \Big\{|\nabla v|^q\Big(1+(\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla v|^{q-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u\Big\}dx.
\end{aligned}$$

Now, for $X, Y \in \mathbb{R}^N$, we recall the inequality

$$|X|^{q-2}X\cdot Y \le \frac{1}{s}|X|^q + \frac{1}{q}|Y|^q \quad\text{with}\quad \frac{1}{s}+\frac{1}{q}=1,$$

and replace the vector X by λX, where λ is a positive real number. We get

$$|X|^{q-2}\lambda^{q-1}X\cdot Y \le \frac{1}{s}\lambda^q|X|^q + \frac{1}{q}|Y|^q.$$

With $\lambda = \big(\frac{v_\epsilon}{u_\epsilon}\big)^{\frac{\alpha-1}{q-1}}$, $X=\nabla u$, $Y=\nabla v$ we find

$$|\nabla u|^{q-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v \le \frac{1}{s}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{s(\alpha-1)}|\nabla u|^q + \frac{1}{q}|\nabla v|^q.$$

Similarly, with $\lambda = \big(\frac{u_\epsilon}{v_\epsilon}\big)^{\frac{\alpha-1}{q-1}}$, $X=\nabla v$, $Y=\nabla u$ we find

$$|\nabla v|^{q-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u \le \frac{1}{s}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{s(\alpha-1)}|\nabla v|^q + \frac{1}{q}|\nabla u|^q.$$

In view of these inequalities, by (1.5) we get

$$\begin{aligned}
&\int_A |\nabla u|^{q-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx\\
&\ge \int_A \Big\{|\nabla u|^q\Big[1+(\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha - \frac{\alpha}{s}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{s(\alpha-1)} - \frac{\alpha}{q}\Big]\\
&+ |\nabla v|^q\Big[1+(\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha - \frac{\alpha}{s}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{s(\alpha-1)} - \frac{\alpha}{q}\Big]\Big\}dx.
\end{aligned}$$

Since the function

$$\psi(t) = 1+(\alpha-1)t^\alpha - \frac{\alpha}{s}t^{s(\alpha-1)} - \frac{\alpha}{q}$$

is non-negative for $t>0$ we find

$$\int_A |\nabla u|^{q-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx \ge 0$$

as claimed. Therefore, from (1.3) we find

$$\int_A |\nabla u|^{p-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx \le L_\epsilon, \tag{1.6}$$

where L_ϵ is defined as in (1.4). The left hand side of (1.6) can be rewritten as in (1.5) with p in place of q, that is

$$\begin{aligned}
&\int_A |\nabla u|^{p-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx\\
&= \int_A \Big\{|\nabla u|^p\Big(1+(\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla u|^{p-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v\Big\}dx\\
&+ \int_A \Big\{|\nabla v|^p\Big(1+(\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla v|^{p-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u\Big\}dx.
\end{aligned} \tag{1.7}$$

Using the inequality

$$|X|^{p-2}\lambda^{p-1}X\cdot Y \le \frac{1}{r}\lambda^p|X|^p + \frac{1}{p}|Y|^p, \quad \frac{1}{r}+\frac{1}{p}=1$$

we find

$$|\nabla u|^{p-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v \le \frac{1}{r}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{s(\alpha-1)}|\nabla u|^p + \frac{1}{p}|\nabla v|^p$$

and

$$|\nabla v|^{p-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u \le \frac{1}{r}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{s(\alpha-1)}|\nabla v|^p + \frac{1}{p}|\nabla u|^p.$$

Using these inequalities, from (1.6) and (1.7) we find

$$\begin{aligned}
&\int_A \Big\{|\nabla u|^p\Big[1+(\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha - \frac{\alpha}{r}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{q(\alpha-1)} - \frac{\alpha}{p}\Big]\\
&+ |\nabla v|^p\Big[1+(\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha - \frac{\alpha}{r}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{r(\alpha-1)} - \frac{\alpha}{p}\Big]\Big\}dx \le L_\epsilon.
\end{aligned}$$

Putting

$$\varphi(t) = 1+(\alpha-1)t^\alpha - \frac{\alpha}{r}t^{r(\alpha-1)} - \frac{\alpha}{p},$$

we have

$$\int_A \Big\{|\nabla u|^p\varphi\Big(\frac{v_\epsilon}{u_\epsilon}\Big) + |\nabla v|^p\varphi\Big(\frac{u_\epsilon}{v_\epsilon}\Big)\Big\}dx \le L_\epsilon. \tag{1.8}$$

Since

$$\lim_{\epsilon\to 0} u_\epsilon = u, \quad \lim_{\epsilon\to 0} v_\epsilon = v, \quad \lim_{\epsilon\to 0} L_\epsilon = 0,$$

as $\epsilon \to 0$, (1.8) yields

$$\int_A \Big\{ |\nabla u|^p \varphi\Big(\frac{v}{u}\Big) + |\nabla v|^p \varphi\Big(\frac{u}{v}\Big) \Big\} dx \le 0. \tag{1.9}$$

We have $\varphi(1) = 0$ and $\varphi'(t) = \alpha(\alpha-1)t^{\frac{\alpha p - 2p + 1}{p-1}}\big(t^{\frac{p-\alpha}{p-1}} - 1\big)$, hence $\varphi(t) > 0$ for $t \neq 1$. Since $\frac{u}{v} > 1$ in A, by (1.9) we must have $|\nabla u| = |\nabla v| = 0$ in A. Hence, $\nabla(u - v) = 0$ in A and $u - v = 0$ on ∂A. Then, $u(x) = v(x)$, contradicting the definition of A. The theorem follows. $\square$

If $F \subset \mathbb{R}^N$ is a measurable set we denote with $|F|$ its Lebesgue measure. We say that two measurable functions $g_1(x)$ and $g_2(x)$ defined in Ω have the same rearrangement if (see [3, 14])

$$|\{x \in \Omega : g_1(x) \ge t\}| = |\{x \in \Omega : g_2(x) \ge t\}| \;\; \forall t \in \mathbb{R}.$$

Let $g_0(x)$ be a non-negative bounded function defined in Ω. We assume that $g_0(x) > 0$ in a subset of positive measure. We denote by $\mathcal{G} = \mathcal{G}(g_0)$ the class of functions g which have the same rearrangement as g_0. Moreover, we denote by $\overline{\mathcal{G}}$ the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$.
With $g \in \overline{\mathcal{G}}$, we consider the functional

$$J(g) = \int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q \Big) dx, \tag{1.10}$$

where u is the variational positive solution to problem (1.2). Observe that this function u satisfies

$$\int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q \Big) dx = \sup_{v \in H_0^1(\Omega), v \ge 0} \int_\Omega \Big(\frac{1}{\alpha} g v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q \Big) dx,$$

and that, by Theorem 1.1, the superior is unique.
We are interested in the maximization and the minimization of the functional $J(g)$ for $g \in \mathcal{G}$.

2 Optimization

We make use of the following results.

Lemma 2.1. *Let $g : \Omega \to \mathbb{R}$ and $w : \Omega \to \mathbb{R}$ be measurable functions, and suppose that every level set of w has zero measure. Then there exists a non-decreasing function φ such that $\varphi(w)$ is a rearrangement of g. Furthermore, there exists a non-increasing function ψ such that $\psi(w)$ is a rearrangement of g.*

Proof. The first assertion follows from Lemma 2.9 of [4]. The second assertion follows applying the first one to $-w$. □

Recall that $\overline{\mathcal{G}}$ denotes the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$. It is well known that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact (see for example [4], Lemma 2.2).

Lemma 2.2. *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^\infty(\Omega)$, and let $w \in L^1(\Omega)$. If there is a non-decreasing function φ such that $\varphi(w) \in \mathcal{G}$ then*

$$\int_\Omega g\, w\, dx \le \int_\Omega \varphi(w)\, w\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\varphi(w)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a non-increasing function ψ such that $\psi(w) \in \mathcal{G}$ then

$$\int_\Omega g\, w\, dx \ge \int_\Omega \psi(w)\, w\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathcal{F}}$.

Proof. The first assertion follows from Lemma 2.4 of [4]. The second assertion follows from the first one putting $\psi(t) = \varphi(-t)$. □

Theorem 2.3. *Let Ω, g_0, $\mathcal{G}$ and $\overline{\mathcal{G}}$ be as in above. If $J(g)$ is defined as in* (1.10) *for $g \in \overline{\mathcal{G}}$ then:*
i) $J(g)$ is weakly continuous with respect to the weak topology of $L^\infty(\Omega)$;*
ii) $J(g)$ is Gâteaux differentiable with derivative $\frac{1}{\alpha}u^\alpha$ (here u is the variational positive solution to problem (1.2) *corresponding to g);*
iii) $J(g)$ is strictly convex on $\overline{\mathcal{G}}$.

Proof. Let $g_i \in \overline{\mathcal{G}}$, $g_i \rightharpoonup g$. We shall prove that

$$\lim_{i\to\infty} J(g_i) = J(g). \tag{2.1}$$

Indeed, if u_i is the solution to (1.2) corresponding to g_i, putting $\phi = u_i$ in that equation we find

$$\int_\Omega \big(|\nabla u_i|^p + |\nabla u_i|^q\big) dx = \int_\Omega g_i u_i^\alpha \, dx.$$

Since g_i is uniformly bounded, using Poincaré and Hölder inequalities we find

$$\int_\Omega |\nabla u_i|^p \, dx \le C \int_\Omega |\nabla u_i|^\alpha \le C\Big(\int_\Omega |\nabla u_i|^p \, dx\Big)^{\frac{\alpha}{p}}$$

and

$$\int_\Omega |\nabla u_i|^p \, dx \le C. \tag{2.2}$$

Recall that we denote by C constants independent of i. By (2.2), it follows that a subsequence (denoted again $\{u_i\}$) converges weakly in $H^{1,p}(\Omega)$ (as well as in $H^{1,q}(\Omega)$) and strongly in $L^\alpha(\Omega)$ to some function $z \in H_0^{1,p}(\Omega)$.

We claim that

$$\lim_{i\to\infty} \int_\Omega (g_i - g)(u_i^\alpha - z^\alpha)\, dx = 0.$$

Indeed, since

$$|(g_i - g)(u_i^\alpha - z^\alpha)| \le C\alpha |u_i - z| \cdot |u_i + z|^{\alpha-1}$$

we have

$$\Big|\int_\Omega (g_i - g)(u_i^\alpha - z^\alpha)\, dx\Big| \le C\alpha \|u_i - z\|_{L^\alpha(\Omega)} \|u_i + z\|^{\alpha-1}_{L^\alpha(\Omega)}.$$

Since $u_i \to z$ strongly in $L^\alpha(\Omega)$, the claim follows.

Since

$$\lim_{i\to\infty} \int_\Omega (g_i - g) z^\alpha \, dx = 0,$$

and since

$$0 = \lim_{i\to\infty} \int_\Omega (g_i - g)(u_i^\alpha - z^\alpha)\, dx = \lim_{i\to\infty} \int_\Omega (g_i - g) u_i^\alpha \, dx - \lim_{i\to\infty} \int_\Omega (g_i - g) z^\alpha \, dx$$

we find

$$\lim_{i\to\infty} \int_\Omega (g_i - g) u_i^\alpha \, dx = 0. \tag{2.3}$$

Obviously, if u is the positive solution to problem (1.2) corresponding to g we also have

$$\lim_{i\to\infty} \int_\Omega (g_i - g) u^\alpha \, dx = 0. \tag{2.4}$$

To conclude the proof of assertion i), we write

$$\begin{aligned}
&J(g) + \int_\Omega \frac{1}{\alpha}(g_i - g) u^\alpha \, dx \\
&= \int_\Omega \Big(\frac{1}{\alpha} g_i u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big) dx \\
&\le \int_\Omega \Big(\frac{1}{\alpha} g_i u_i^\alpha - \frac{1}{p}|\nabla u_i|^p - \frac{1}{q}|\nabla u_i|^q\Big) dx = J(g_i) \qquad (2.5)\\
&= \int_\Omega \frac{1}{\alpha}(g_i - g) u_i^\alpha \, dx + \int_\Omega \Big(\frac{1}{\alpha} g u_i^\alpha - \frac{1}{p}|\nabla u_i|^p - \frac{1}{q}|\nabla u_i|^q\Big) dx \\
&\le \int_\Omega \frac{1}{\alpha}(g_i - g) u_i^\alpha \, dx + J(g).
\end{aligned}$$

By (2.5), (2.3) and (2.4) we find (2.1), and assertion i) of the theorem is proved.

To prove assertion ii), we claim that the function z mentioned in above is equal to u, the variational positive solution to problem (1.2) corresponding to g. Indeed from

$$\int_\Omega |\nabla z|^p \, dx \le \liminf_{i\to\infty} \int_\Omega |\nabla u_i|^p \, dx,$$

$$\int_\Omega |\nabla z|^q \, dx \le \liminf_{i\to\infty} \int_\Omega |\nabla u_i|^q \, dx,$$

and (2.1) we find

$$\begin{aligned} J(g) &= \int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big) dx \\ &= \lim_{i\to\infty} J(g_i) = \lim_{i\to\infty} \int_\Omega \Big(\frac{1}{\alpha} g_i u_i^\alpha - \frac{1}{p}|\nabla u_i|^p - \frac{1}{q}|\nabla u_i|^q\Big) dx \\ &\le \lim_{i\to\infty} \int_\Omega \frac{1}{\alpha} g_i u_i^\alpha \, dx - \liminf_{i\to\infty} \int_\Omega \frac{1}{p}|\nabla u_i|^p dx - \liminf_{i\to\infty} \int_\Omega \frac{1}{q}|\nabla u_i|^q dx \\ &\le \int_\Omega \Big(\frac{1}{\alpha} g z^\alpha - \frac{1}{p}|\nabla z|^p - \frac{1}{q}|\nabla z|^q\Big) dx \le J(g). \end{aligned}$$

It follows that

$$\int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big) dx = \int_\Omega \Big(\frac{1}{\alpha} g z^\alpha - \frac{1}{p}|\nabla z|^p - \frac{1}{q}|\nabla z|^q\Big) dx.$$

By the uniqueness of the maximizer of

$$-I(v) = \int_\Omega \Big(\frac{1}{\alpha} g v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big) dx$$

for $v \ge 0$, $v \in H_0^{1,p}(\Omega)$, we must have $u = z$, as claimed.
In view of the latter result, (2.3) implies

$$\lim_{i\to\infty} \int_\Omega g_i u_i^\alpha \, dx = \lim_{i\to\infty} \int_\Omega g u_i^\alpha \, dx = \int_\Omega g u^\alpha dx.$$

Let $t_i > 0$ be a sequence of real numbers such that $t_i \to 0$ as $i \to \infty$. Let $h \in \overline{\mathcal{G}}$ and let $g_i = g + t_i(h - g)$. Then, by (2.5) we find

$$J(g) + t_i \int_\Omega \frac{1}{\alpha}(h-g)\, u^\alpha \, dx \le J(g + t_i(h-g)) \le J(g) + t_i \int_\Omega \frac{1}{\alpha}(h-g) u_i^\alpha \, dx.$$

It follows that

$$\int_\Omega \frac{1}{\alpha}(h-g)u^\alpha\, dx \le \frac{J(g+t_i(h-g))-J(g)}{t_i} \le \int_\Omega \frac{1}{\alpha}(h-g)u_i^\alpha\, dx. \tag{2.6}$$

Since $g_i \to g$ as $i \to \infty$, $u_i \to u$ in $L^\alpha(\Omega)$. Therefore,

$$\lim_{i\to\infty} \int_\Omega \frac{1}{\alpha}(h-g)u_i^\alpha\, dx = \int_\Omega \frac{1}{\alpha}(h-g)u^\alpha\, dx.$$

Since the sequence t_i is arbitrary, from the latter equation and (2.6) we find

$$\lim_{t\to 0^+} \frac{J(g+t(h-f))-J(g)}{t} = \int_\Omega \frac{1}{\alpha}(h-g)u^\alpha\, dx.$$

This proves assertion ii).

Let $0<t<1$. If $g_1,\ g_2 \in \overline{\mathcal{G}}$ and $g_t = tg_1+(1-t)g_2$, we have

$$\begin{aligned}&\int_\Omega \Big(\frac{1}{\alpha}g_t v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx\\ &= t\int_\Omega \Big(\frac{1}{\alpha}g_1 v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx\\ &+(1-t)\int_\Omega \Big(\frac{1}{\alpha}g_2 v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx.\end{aligned}$$

If we take the superior for $v \in H_0^{1,p}(\Omega)$ in both sides of this equation we find

$$J(g_t) \le tJ(g_1)+(1-t)J(g_2).$$

To prove strict convexity, assume equality holds in the latter inequality for some $0<t<1$. If u_t, u_1 and u_2 are the solutions corresponding to g_t, g_1 and g_2 respectively, then we have

$$\begin{aligned}&t\int_\Omega \Big(\frac{1}{\alpha}g_1 u_t^\alpha - \frac{1}{p}|\nabla u_t|^p - \frac{1}{q}|\nabla u_t|^q\Big)dx\\ &+(1-t)\int_\Omega \Big(\frac{1}{\alpha}g_2 u_t^\alpha - \frac{1}{p}|\nabla u_t|^p - \frac{1}{q}|\nabla u_t|^q\Big)dx\\ &= t\int_\Omega \Big(\frac{1}{\alpha}g_1 u_1^\alpha - \frac{1}{p}|\nabla u_1|^p - \frac{1}{q}|\nabla u_1|^q\Big)dx\\ &+(1-t)\int_\Omega \Big(\frac{1}{\alpha}g_2 u_2^\alpha - \frac{1}{p}|\nabla u_2|^p - \frac{1}{q}|\nabla u_2|^q\Big)dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_\Omega\Big(\frac{1}{\alpha}g_1u_t^\alpha-\frac{1}{p}|\nabla u_t|^p-\frac{1}{q}|\nabla u_t|^q\Big)dx\\ &=\int_\Omega\Big(\frac{1}{\alpha}g_1u_1^\alpha-\frac{1}{p}|\nabla u_1|^p-\frac{1}{q}|\nabla u_1|^q\Big)dx\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\Big(\frac{1}{\alpha}g_2u_t^\alpha-\frac{1}{p}|\nabla u_t|^p-\frac{1}{q}|\nabla u_t|^q\Big)dx\\ &=\int_\Omega\Big(\frac{1}{\alpha}g_2u_2^\alpha-\frac{1}{p}|\nabla u_2|^p-\frac{1}{q}|\nabla u_2|^q\Big)dx.\end{aligned}$$

By uniqueness we must have $u_t = u_1 = u_2$, and

$$-\Delta_p u_t - \Delta_q u_t = g_1\ u_t^{\alpha-1} \quad \text{a.e. in} \quad \Omega$$

and

$$-\Delta_p u_t - \Delta_q u_t = g_2\ u_t^{\alpha-1} \quad \text{a.e. in} \quad \Omega.$$

Since $g_1(x) \geq 0$ and $g_1(x) \not\equiv 0$, by the strong maximum principle $u_t(x) > 0$. It follows that $g_1(x) = g_2(x)$ a.e. in Ω, which yields strict convexity of $J(g)$.
The theorem is proved. $\square$

Theorem 2.4. *Let Ω, g_0, $\mathcal{G}$ and $\overline{\mathcal{G}}$ be as in above. Let $J(g)$ be defined as in* (1.10) *for $g \in \overline{\mathcal{G}}$.*
i) The problem

$$\max_{g\in\mathcal{G}} J(g)$$

has a solution $\hat{g}$. Moreover, if $\hat{g}$ is any solution and if $\hat{u} = u_{\hat{g}}$ then we have $\hat{g} = \varphi(\hat{u})$ where φ is some non-decreasing function.
ii) The problem

$$\min_{g\in\mathcal{G}} J(g)$$

has a unique solution $\check{g}$. Moreover, if $\check{u} = u_{\check{g}}$ we have $\check{g} = \psi(\check{u})$, where ψ is some non-increasing function.

Proof. By Theorem 2.1, $J(g)$ is weakly continuous and strictly convex. Therefore, assertion i) follows by Theorem 7 of [3].

Let us prove assertion ii). The uniqueness follows by the strict convexity of $J(g)$. To prove existence, let

$$\check{J} = \inf_{g\in\mathcal{G}} J(g),$$

and let $\{g_i\}$ be a sequence such that

$$\check{J} = \lim_{i\to\infty} J(g_i).$$

Since $\overline{\mathcal{G}}$ is weakly compact we can assume that for some subsequence of $\{g_i\}$ (again denoted $\{g_i\}$) there is $\check{g} \in \overline{\mathcal{G}}$ with $g_i \rightharpoonup \check{g}$ in the weak* topology of $L^\infty(\Omega)$. By Theorem 2.3, we have $\check{J} = J(\check{g})$. Let us show that $\check{g} \in \mathcal{G}$. If $g \in \overline{\mathcal{G}}$, if $0 < t < 1$ and if $g_t = tg + (1-t)\check{g}$, since $J(g)$ is Gâteaux differentiable (by Theorem 2.3) we have, for $u = u_{\check{g}}$,

$$J(\check{g}) \le J(g_t) = J(\check{g} + t(g-\check{g})) = J(\check{g}) + t\int_\Omega (g-\check{g})\frac{1}{\alpha}u^\alpha \, dx + o(t) \quad \text{as} \quad t \to 0.$$

Hence,

$$\int_\Omega (g-\check{g})\frac{1}{\alpha}u^\alpha \, dx \ge 0.$$

Equivalently,

$$\int_\Omega g\, u^\alpha \, dx \ge \int_\Omega \check{g}\, u^\alpha \, dx \quad \forall g \in \overline{\mathcal{G}}. \tag{2.7}$$

On the other hand, by equation (1.2) it follows that the function u (and the function u^α) cannot have flat zones in the set

$$E = \{x \in \Omega : \check{g}(x) > 0\}.$$

Consider first the case $|E| = |\Omega|$. Then, by Lemma 2.1 there is a non-increasing function ψ such that $\psi(u^\alpha)$ is a rearrangement of $\check{g}$. By (2.7) and Lemma 2.2 we must have $\check{g} = \psi(u^\alpha) \in \mathcal{G}$.

If $|E| < |\Omega|$, since $\check{g} \in \overline{\mathcal{G}}$, by Lemma 2.14 of [4] we have

$$|E| \ge |\{x \in \Omega : g_0(x) > 0\}|.$$

Then there is $g_1 \in \mathcal{G}$ such that its support is contained in E. By Lemma 2.1 there is a non-increasing function $\psi_1(t)$ such that $\psi_1(u^\alpha)$ is a rearrangement of g_1 on E. Define

$$m = \inf_{x\in\Omega\setminus E} u^\alpha(x).$$

By using (2.7) one proves that $u^\alpha(x) < m$ in E (see [5, 8, 9] for details). Now define

$$\tilde{\psi}(t) = \begin{cases} \psi_1(t) & \text{if } \ 0 \le t < m \\ 0 & \text{if } \ t \ge m. \end{cases}$$

The function $\psi(t)$ is non-increasing and $\psi(u^\alpha)$ is a rearrangement of $g_1(x)$ in Ω. Indeed, the functions g_1 and $\psi(u^\alpha)$ have the same rearrangement on E, and both vanish on $\Omega \setminus E$. By Lemma 2.2 we must have $\check{g} = \psi(u^\alpha) \in \mathcal{G}$. The theorem is proved. $\square$

References

[1] S. Aizicovici, N.S. Papageorgiou, and V. Staicu,Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints, *Memoirs Amer. Math. Soc.*, Vol **196**, No. 915 (2008).

[2] V. Benci, P. D'Avenia, D. Fortunato, and L. Pisani, Solitons in several space dimensions: Derrick's Problem and infinitely many solutions, *Arch. Rational Mech. Anal.*, **154** (2000), 297–324.

[3] G. R. Burton,Rearrangements of functions, maximization of convex functionals and vortex rings. *Math. Ann.*, **276** (1987), 225–253.

[4] G. R. Burton, Variational problems on classes of rearrangements and multiple configurations for steady vortices. *Ann. Inst. Henri Poincarè*, **6** (1989), 295–319.

[5] G.R. Burton, J.B. McLeod, Maximisation and minimisation on classes of rearrangements. *Proc. Roy. Soc. Edinburgh Sect. A,* **119** (1991), 287–300.

[6] P. Candito, S.A. Marano, and K. Perera, On a class of critical (p,q)-Laplacian problems, *NoDEA Nonlinear Differential Equations Appl.*, **22** (2015), 1959–1972.

[7] L. Cherfils and Y. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with (p,q)-Laplacian, *Comm. Pure Appl. Anal.*, **4** (2005), 9–22.

[8] F. Cuccu, B. Emamizadeh, G. Porru, Optimization of the first eigenvalue in problems involving the p-Laplacian, *Proc. Amer. Math. Soc.*, **137** (2009), 1677–1687

[9] F. Cuccu, G. Porru and S. Sakaguchi, Optimization problems on general classes of rearrangements, *Nonlinear Analysis*, **74** (2011), 5554–5565.

[10] P.C. Fife, Mathematical aspects of reacting and diffusing systems, *Lect. Notes in Biomath.* **28**, Springer, Berlin, 1979.

[11] G.M. Figueiredo and H.R. Quoirin, Ground states of elliptic problems involving non homogeneous operators, *Indiana Univ. Math. J.*, **65** (2016), 779–795.

[12] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, *Clarendon Press,* Oxford, New York, Tokyo, 1993.

[13] P. Lindqvist, On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, **109** (1990), 157–164. Addendum ibidem 116 (1992), 583–584.

[14] B. Kawohl, Rearrangements and Convexity of level sets in PDE's, *Springer, Lectures Notes in Mathematics,* n. **115**, 1985.

[15] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.*, **12** (1988), 1203–1219.

[16] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Comm. Partial Differential Equations*, **16** (1991), 311–361.

[17] S. A. Marano and N. S. Papageorgiou, Constant-sign and nodal solutions of coercive (p,q)-Laplacian problems, *Nonlinear Anal.*, **77** (2013), 118–129.

[18] P. Pucci, J. Serrin,The maximum principle, *Birkhäuser Verlag,* Basel, 2007.

[19] H. Wilhelmsson, Explosive instabilities of reaction-diffusion equations, *Phys. Rev. A* **36** (1987), 965–966.

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