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# Chaotic Dynamics in a Class of Switched Liénard/Rayleigh Systems with Relativistic Acceleration

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**Abstract**: We consider a class of time-periodic switched systems, which are obtained as a perturbation of a planar autonomous reversible system by a periodic forcing term. The model is motivated by an extension of the classical Liénard and Rayleigh equations due to the presence of a relativistic acceleration. Using recent results from the theory of topological horseshoes, we provide a new result of existence of infinitely many subharmonic solutions, as well as more complex dynamics, as illustrated by some numerical examples.

**Keywords**: Reversible systems, Quadratic Liénard equations, Rayleigh equations, relativistic acceleration, switched systems, periodic solutions, complex dynamics.

**MSC2010**: 34C25, 34C28

To Professor Andrea Bacciotti (in memoriam)

## 1 Introduction

The present paper deals with the investigation of complex dynamics in a class of time-periodic switched systems obtained as a perturbation of planar reversible systems which, in turn, are related to some classes of Rayleigh and Liénard equations.

The switched systems that we are going to consider are (piecewise) continuous in time with isolated discrete switching events [49]. Actually, following Liberzon [49, §1.1.2], we suppose that there is a finite set $\mathcal{N} = \{1, \dots, m\}$ and, for every $j \in \mathcal{N}$, a locally Lipschitz continuous vector field $\vec{Z}_j : \mathbb{R}^2 \to \mathbb{R}^2$. To define a switched system generated by the above family, a *signal* is introduced as a piecewise constant surjective function $\sigma : \mathbb{R} \to \mathcal{N}$ which has a finite number of discontinuities, called *switched times*, in any bounded time-intervals and takes a constant value on every interval between two consecutive switching times. Then, the *switched system* generated by the above family is defined as

$$\dot{\zeta}(t) = \vec{Z}_{\sigma(t)}(\zeta(t)). \tag{1.1}$$

In this article, we are interested in the case of time-periodic switched systems. Accordingly, from now on, by convention, we will assume that there are switched times

$$0 = \tau_0 < \tau_1 < \dots \tau_{m-1} < \tau_m = T,$$

for some $T > 0$ such that

$$\sigma(t) = j \quad \text{for } \ t \in [\tau_{j-1}, \tau_j[, \quad \forall\, j = 1, \dots, m \quad \text{with } \ m \geq 2$$

and $\sigma(t)$ is $T$-periodic. This means that at any time $t$, there is an *active subsystem*: namely the system $\dot{\zeta} = \vec{Z}_1(\zeta)$ on the time-interval $[0, \tau_1[$, jumping to the system $\dot{\zeta} = \vec{Z}_2(\zeta)$ on the time-interval $[\tau_1, \tau_2[$, and so on, with all the alternation between these systems repeating in a $T$-periodic fashion. For $m = 2$, the system switches between only two subsystems in a period. This is the minimal situation for a periodic switched system; although very simple, this case is already suitable to produce rich dynamics as we will illustrate with some examples in this article.

Professor Bacciotti achieved very important contributions in the development of the theory of switched systems (see [9, 10, 11, 12, 13, 14, 15]). In particular, he deeply investigated the problem of stability for these systems, which is of great significance for the applications to control theory (see [2, 22]), when the single active subsystems are linear, and with special emphasis to the case of periodic switching signals. In this context, it may be worth to observe that even the apparently simpler case of only two linear planar systems which are activated alternatively in a periodic manner, exhibits interesting phenomena [11].

In [13] it is claimed that: "In spite of their apparent simplicity, switched systems may exhibit a very complicated dynamical behavior". We fully agree with this point of view and, indeed, it is the aim of this paper to show how very simple time-periodic switched systems may exhibit a very complex, namely "chaotic", behavior.

In order to focus our analysis to some specific situations and also following our recent work [59], we will consider switched systems determined by subsystems that are time-reversible, which, in our applications, will consists of planar autonomous systems which present a mirror symmetry with respect to the $x$-axis or the $y$-axis. More in detail, we will confine ourselves to the case of Liénard or Rayleigh type subsystems. To support the above claim by Andrea Bacciotti in [13], we will show, by rather natural examples, that the presence of only two subsystems alternating in a periodic fashion may produce very complicated dynamics.

Before proceeding further, we need to introduce in a precise manner the concept of "chaos" that we are considering in the present work.

In spite of the huge and still growing literature involving the concept of chaos from the theoretical or the applied point of view, there are various competing technical definitions which have been considered by different authors, such as: transitivity and sensitive dependence on initial conditions [8, 35, 42, 83], horseshoe structure [71, 72], positivity of the maximal Lyapunov exponent [3, 62], positive topological entropy [1, 23, 21, 32], chaos according to Li-Yorke [47] (see also [19, 41]), or according to Devaney [31] (see also [16, 70]), or in the sense of Block and Coppel [20], just to mention a few well-known instances. A list of some different definitions and their comparison can be found in [7, 39, 52, 63]. Other aspects which are related to measure theoretic concepts, such as existence of invariant measures, ergodicity or mixing, although connected to the above definitions, will be not considered here (see, for instance, [82] for more information). Whatever definition of chaos one has in mind, it is a general agreement [31, 57, 72] to consider as a paradigmatic example the Bernoulli shift automorphism $\sigma : \Sigma_m \to \Sigma_m$ on the set $\Sigma_m := \{0, \dots, m-1\}^{\mathbb{Z}}$ of two-sided sequences on an alphabet $\{0, \dots, m-1\}$ of $m \geq 2$ symbols (with the discrete topology), defined as $\sigma(\mathbf{s}) = \mathbf{s}'$, where $\mathbf{s}' = (s_{i+1})_{i\in\mathbb{Z}}$ for $\mathbf{s} = (s_i)_{i\in\mathbb{Z}}$. The set $\Sigma_m$, with the product topology, is a compact metrizable space with distance $d(\mathbf{s}^1, \mathbf{s}^2) = \sum_{i\in\mathbb{Z}} \delta(s_i^1, s_i^2)/2^{|i|+1}$, where $\delta(a, b) = 0$ for $a = b$ and $\delta(a, b) = 1$ for $a \neq b$. The shift map $\sigma$ has all the properties usually attributed to the concept of chaos, like transitivity (a dense orbit), periodic points of arbitrarily large periods which constitute a dense set in $\Sigma_m$, uncountable many non-periodic points, sensitive dependence on initial conditions, positive topological entropy [1]. The celebrated Smale's horseshoe [71, 72, 73] provides a geometric framework according to which a diffeomorphism $\psi$ has a compact invariant set $\mathcal{I}$ such that $\psi|_\Lambda$ is *conjugate* to a Bernoulli shift (see also [57, Ch. III]). This means that (for some $m \geq 2$) there exists a continuous and bijective map $h : \mathcal{I} \to \Sigma_m$ such that the diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\ \psi\ } & \mathcal{I} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \Sigma_m & \xrightarrow[\ \sigma\ ]{} & \Sigma_m \end{array}$$

commutes and, therefore, $\psi|_{\mathcal{I}}$ has the same properties as $\sigma$ on $\Sigma_m$. In concrete examples coming from the applications, it is often difficult to prove the conjugation to a Bernoulli shift as in the case of the original Smale's horseshoe. This leads to a weaker concept of chaos, by assuming the continuous map $h$ (in the commutative diagram) to be surjective. In this case, we say that $\psi$ is semiconjugate to the Bernoulli shift or, following [24], that $\psi$ has a *horseshoe* factor. In this setting, the following definition of chaos will be applied.

**Definition 1.1.** Let $\psi : \mathcal{D}(\subset \mathrm{dom}(\psi) \subset \mathbb{R}^2) \to \psi(\mathcal{D}) \subset \mathbb{R}^2$ be a homeomorphism. We say that $\psi$ *induces chaotic dynamics on* $m$ *symbols in the set* $\mathcal{D}$ if there are $m \geq 2$ (nonempty) pairwise disjoint compact sets $H_0, \dots, H_{m-1} \subset \mathcal{D}$ such that, for each *itinerary* $(H_{s_i})_{i\in\mathbb{Z}}$ with $H_{s_i} \in \{H_0, \dots, H_{m-1}\}$ there exists a two-sided sequence of points $w_i \in H_{s_i}$ with $w_{i+1} = \psi(w_i)$, $\forall\, i \in \mathbb{Z}$ and, moreover, we can take the sequence $(w_i)_{i\in\mathbb{Z}}$ to be $k$-periodic if $(H_{s_i})_{i\in\mathbb{Z}}$ if $k$-periodic.

This is the so-called chaotic dynamics in the coin-tossing sense (see [39, 73]) as, for $m = 2$, it can be interpreted as follows: given a sequence of *heads* $\equiv 0$ and *tails* $\equiv 1$, like $\dots 000101 \dots$, there is at least a point $w$ which, under the action of the deterministic law $\psi$, at the $i$-th step, will be in $H_0$ or in $H_1$ according to the preassigned sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ of *heads* and *tails*. In other word, the map $\psi$, along with its iterates can reproduce any possible outcome of a coin-flipping experiment. This is the essence of deterministic chaos, according to Smale [73, p.42]: *Guessing whether heads or tails is the outcome of a coin toss is the paradigm of pure chance. On the other hand it is a deterministic process that governs the whole motion of a real coin, and hence the result, heads or tails, depends only on very subtle factors of the initiation of the toss. This is "sensitive dependence on initial conditions."* Definition 1.1 puts also a special emphasis on the periodic sequences, a fact that in other definitions of chaos, like [37, 38], is not always guaranteed. It has been proved in [55] that the presence of chaotic dynamics according to Definition 1.1 implies the existence of a compact invariant set $\mathcal{I} \subset H_0 \cup \dots \cup H_m$ for the map $\psi$ such that the set of periodic points of $\psi$ is dense in $\mathcal{I}$ and also $\psi|_{\mathcal{I}}$ is *semiconjugate* to the Bernoulli shift on $m$ symbols; moreover, the inverse image $h^{-1}(\mathbf{s})$ of any $k$-periodic sequence $\mathbf{s} \in \Sigma_m$ contains a $k$-periodic point of $\psi$ (this holds for every positive integer $k$). In this manner, our definition of chaotic dynamics turns out to be equivalent to other ones considered in the literature, like those in [25, 46, 84].

The plan of the paper is now the following. In Section 2 we recall some basic facts about reversible planar systems and propose a simple model of Rayleigh-Liénard type equations with symmetries, which appears to be new in the literature. Then, in Section 3 we prove the presence of chaotic-like dynamics (according to Definition 1.1) for a simple class of switched systems with symmetries. This is achieved as an application of our recent abstract result in [59]. Notice that Definition 1.1 guarantees the existence of periodic points of any order. Hence, when applied to the Poincaré map associated with the considered ODE system, we have not only a rigorous proof of the chaotic dynamics, but we also obtain the existence of infinitely many subharmonic solutions of any order. We will also show some numerical simulations giving evidence of our theoretical findings. All the numerical simulations in the article have been performed using Maple mathematical software [85].

## 2 Reversible planar systems and their periodic perturbations

*The presence and importance of time-reversal symmetry was recognized in the early days of dynamical systems by Birkhoff. He utilized it in his study of the restricted three-body problem in classical mechanics.* This quotation, borrowed from [45], is used as a starting point and a motivation for this section, where we wish to briefly introduce a class of differential systems which plays an important role, not only in the area of dynamical systems [4, 69], but also for its applications in different areas of classical and modern Physics: the *reversible systems.*

Let us consider a planar autonomous system of the form

$$\begin{cases} \dot{x} = X_0(x, y) \\ \dot{y} = Y_0(x, y) \end{cases} \tag{2.1}$$

having an equilibrium point at the origin, that is $X_0(0, 0) = Y_0(0, 0) = 0$. We assume that the vector field $\vec{Z}_0 := (X_0, Y_0) : \mathbb{R}^2 \to \mathbb{R}^2$ is locally Lipschitz continuous. This ensures the uniqueness of the solutions for the associated initial value problems, not only for system (2.1), but also for the associated periodically perturbed non-autonomous equation

$$\begin{cases} \dot{x} = X_0(x, y) \\ \dot{y} = Y_0(x, y) \end{cases} + e(t), \tag{2.2}$$

with $e(t) = (e_1(t), e_2(t))$ a $T$-periodic perturbation that we assume to be piecewise continuous. The same remarks also extend to other more general perturbation of (2.1), like

$$\begin{cases} \dot{x} = X(t, x, y) \\ \dot{y} = Y(t, x, y) \end{cases} \tag{2.3}$$

with $X, Y$ locally Lipschitz continuous in $(x, y)$ and continuous (or piecewise continuous) in $t$. Specific forms for $(X, Y)$ related to $(X_0, Y_0)$ will be discussed in the applications of Section 3.

Following [4, 5, 30] (see also [45, 69]), we say that the system (2.1) is time-reversible (or simply *reversible*), if there exists a transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ of the phase space, which is an involution (i.e., with $R^2 = Id$) that reverses the direction of time along the solution-operator (Poincaré map). More precisely, if we denote by $\phi_0(t, z)$ the solution of (2.1) with $(x(0), y(0)) = z$, the hypothesis of time-reversibility can be expressed by the relation

$$(R \circ \phi_0)(t, z) = \phi_0(-t, Rz), \tag{2.4}$$

as illustrated in Figure 1 where the involution is the reflection with respect to the $x$-axis.

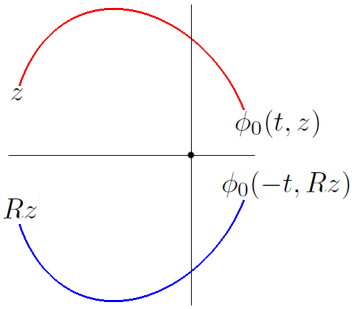



Figure 1: Example of time-reversibility in a planar system. The trajectories move around the origin in clockwise turns. The point $z$ in the upper half-plane (second quadrant) is shifted clockwise to the right (first quadrant) to a new point $\phi_0(t, z)$ after some $t > 0$. The symmetric $Rz$ of the point $z$, in the lower half-plane (third quadrant) moving back counterclockwise to the time $-t$ ends at the symmetric position of $\phi_0(t, z)$, (in the fourth quadrant) so that we have $\phi_0(-t, Rz) = R(\phi_0(t, z))$.

Reversible systems share some similar features with the Hamiltonian ones, although there are also some important differences. According to [65], "nonconservative reversible differential equations can display Hamiltonian-like behaviour near their symmetric cycles as well as behaviour typical of dissipative and expansive systems near their asymmetric cycles. This hybrid nature makes them interesting models in which to study within a single system the differences and crossover between conservative and dissipative behaviour". The surveys of Roberts and Quispel [65] and of Lamb and Roberts [45] explore these issues also from the historical development of this subject. Another important aspect related to the study of reversible planar systems is intimately linked to the theory of centers, dating back to the classical works of Poincaré (see [28, 29, 67] and the references therein).

In [57, Ch. V], J. Moser considered the case of linear involutions (in particular, reflections). As observed in [65], when the transformation $R$ is linear, the reversibility condition (2.4) can be equivalently expressed in terms of the vector field $\vec{Z}_0$ as

$$R \circ \vec{Z}_0 = -\vec{Z}_0 \circ R.$$

This, in turn, allows to find some simple symmetry conditions on the vector field $\vec{Z}_0$ when we are interested in planar systems where the orbits have a mirror symmetry with respect to horizontal or vertical axes. According to Roberto Conti [29, §12] we will say that system (2.2) is reversible with respect to a straight line $\ell$ through the origin if it is invariant with respect to reflection about $\ell$ and a reversion of time $t$.

Centers and symmetries with respect to a line were considered also in [81] in the case of $X_0(x,y) = y + q(x,y)$ and $Y_0(x,y) = -x - p(x,y)$ for $p$ and $q$ polynomials. For these systems general conditions were obtained by Conti in [29] (see also [67] and the references therein). This topic is discussed in the classical book of Sansone and Conti [68], as well.

Elementary calculations show that (2.2) is reversible with respect to the $y$-axis if and only if

$$X_0(-x,y) = X_0(x,y), \qquad -Y_0(-x,y) = Y_0(x,y) \tag{2.5}$$

and, analogously, the system is reversible with respect to the $x$-axis if and only if

$$-X_0(x,-y) = X_0(x,y), \qquad Y_0(x,-y) = Y_0(x,y). \tag{2.6}$$

When we apply these results to the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

written as the equivalent system in the Liénard plane

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x),$$

(for $F(x) := \int_0^x f(s)\,ds$) we obtain, according to (2.5), the mirror symmetry of the orbits with respect to the $y$-axis for $F$ even and $g$ odd. Periodic perturbations of these reversible Liénard systems were considered in [56] for $g(x) = x$ and $F(x) = ax^2$ and in [59] for more general choices of $F$ and $g$.

On the other hand, it is natural to exploit the symmetry with respect to the $x$-axis in the phase-plane given by (2.6) for the quadratic Liénard equation (cf. [40, 43, 66])

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{2.7}$$

or for the Rayleigh equation

$$\ddot{x} + f(\dot{x}) + g(x) = 0. \tag{2.8}$$

Equation (2.7) takes the form of system

$$\dot{x} = y, \qquad \dot{y} = -f(x)y^2 - g(x). \tag{2.9}$$

Hence, we have always symmetry with respect to the $x$-axis, independently on the conditions on $f$ and $g$. On the other hand, equation (2.8) takes the equivalent form

$$\dot{x} = y, \qquad \dot{y} = -f(y) - g(x). \tag{2.10}$$

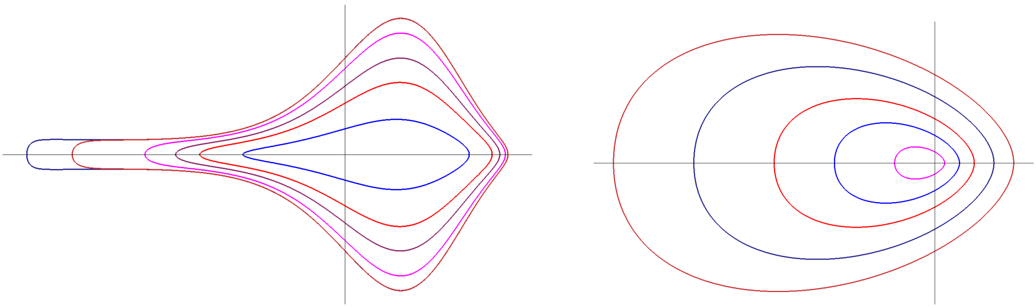


Figure 2: The left panel shows the phase-portrait of the quadratic Liénard system (2.9) for $f(x) = x^2 - x - 1$ and $g(x) = x^3 + 2x^2 + 2x$. Note that $f(x)$ and $g(x)$ have no particular symmetry. Nevertheless the trajectories present a mirror symmetry with respect to the $x$-axis, due to the presence of $y^2$ in (2.9). In this example the origin is a local center. The solutions of the initial problem $x(0) = x_0$, $y(0) = 0$ with $x_0 \gtrsim 1.79355$ blow-up very quickly, both in backward and in forward time. Also if we take $f(x) = x^2 - x + 1$, so that $f(x) > 0, \forall\, x$, we have a local center at the origin, but again the solutions of the initial problem $x(0) = x_0$, $y(0) = 0$ with $x_0 \gtrsim 0.36963$ blow-up very quickly, both in backward and in forward time.
The right panel shows a phase-portrait associated with one of the simplest examples of Rayleigh equation, namely $\ddot{x} + \lambda|\dot{x}| + Ax = 0$, with $\lambda > 0$, $A > 0$. This model was studied in [78] and reconsidered by Omari in [58] in the framework of the theory of upper and lower solutions (see also [36]). It is interesting to notice that the origin is a global center (as in figure) if and only if $4A > \lambda^2$. In this case, the center is isochronous with all the orbits having common period $T = \frac{4\pi}{\sqrt{4A-\lambda^2}}$. The present simulation is obtained for $\lambda = 1$ and $A = 3/2$.

Hence, we have symmetry with respect to the $x$-axis, provided that $f$ is even. Figure 2 shows two examples for (2.9) and (2.10) exhibiting trajectories which are symmetric with respect to the $x$-axis.

Motivated by the examples (2.9) and (2.10), we will focus our attention to a special class of reversible systems of the form

$$(S) \qquad \begin{cases} \dot{x} = h(y) \\ \dot{y} = -\lambda f(x,y) - g(x), \end{cases}$$

where, from now on, we assume that $\lambda > 0$ is a real parameter, $g, h : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions satisfying the following conditions

$$(g_{\text{sign}}) \qquad \exists\, x^0 : \quad g(x^0) = 0, \qquad g(x)(x - x^0) > 0,\ \forall x \neq x^0,$$

$$(h_{\text{sign}}) \qquad h(0) = 0, \qquad h \ \textit{strictly increasing}, \qquad h(-y) = -h(y),\ \forall\, y \in \mathbb{R},$$

$$(f_{\text{even}}) \qquad f(x,-y) = f(x,y), \quad f(x,0) = 0,\ \forall\, x \in \mathbb{R}.$$

As a consequence of the fact that $h$ is odd and $f$ is even with respect to $y$, we have that system $(S)$ is reversible with respect to the $x$-axis. Moreover, the above assumptions on $g, h, f$ imply that the point $Q := (x^0, 0)$ is the unique equilibrium point of system $(S)$ [1] In this context, it will be also useful to recall [68, Theorem 15, p.95], (from the book of Sansone-Conti) where conditions for an equilibrium point to be a center are obtained in case of (2.6).

The classification of singularities for reversible planar vector fields has been performed by Teixeira in [76] (see also [17] and the references therein). In [44], Labouriau and Sovrano proved the existence of chaotic dynamics (in a setting analogous to the one considered in our work [59]), for some switched systems which are periodic perturbations of reversible autonomous systems with a symmetry with respect to $y = 0$. Our aim now is to continue study of [44, 59] in this direction, with reference to the periodic perturbations of system $(S)$. We believe that this problem has its own interest, as it includes as special cases equations (2.9) and (2.10). Moreover, the choice of the function $h$ allows us to consider some $\phi$-Laplacian type Liénard and Rayleigh equations, about which we recall now some basic facts.

In [50], Manásevich and S. Sędziwy studied the problem of existence and uniqueness of limit cycles for the $p$-Laplacian Liénard equation

$$\frac{d}{dt}(\phi_p(\dot{x})) + \lambda f(x)\phi_p(\dot{x}) + g(x) = 0, \tag{2.11}$$

[1] Since we study systems with mirror symmetry with respect to the $x$-axis, there is no substantial difference from having the equilibrium point at the origin or at a point $Q_0$ on the $x$-axis.

where $\phi_p(s) = |s|^{p-2}s$ (with $\phi_p(0) = 0$) and $p > 1$. Equation (2.11) can be written as an equivalent planar system of the form

$$\dot{x} = \phi_q(y), \qquad \dot{y} = -\lambda f(x)y - g(x),$$

where $\phi_q(\xi) = \phi_p^{-1}(\xi)$, with $q > 1$ being the Hölder conjugate of $p > 1$.
As pointed out in [50], the same analysis can be extended to the $\phi$-Laplacian Liénard equation

$$\frac{d}{dt}(\phi(\dot{x})) + \lambda f(x)\phi(\dot{x}) + g(x) = 0, \tag{2.12}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism with $\phi(0) = 0$. In this case, equation (2.12) can be written as an equivalent planar system of the form

$$\dot{x} = h(y), \qquad \dot{y} = -\lambda f(x)y - g(x),$$

with $h := \phi^{-1}$. The same point of view has been also followed in [33, 74].

On the other hand, equations like

$$\frac{d}{dt}(\phi(\dot{x})) + \lambda f(x)\dot{x} + g(x) = 0, \tag{2.13}$$

where, as above, $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism with $\phi(0) = 0$, can be written as an equivalent planar system of the form

$$\dot{x} = h(y), \qquad \dot{y} = -\lambda f(x)h(y) - g(x),$$

with $h := \phi^{-1}$. This second point of view was followed in [53] and in [48].

An unifying point of view was adopted by Pérez-González, Torregrosa and Torres in [61] (see also [26, 27], for recent contributions in this direction), where a general equation (with also a nonlinear differential operator) of the form

$$\frac{d}{dt}(\phi(\dot{x})) + \lambda f_0(x)f_1(\dot{x}) + g(x) = 0, \tag{2.14}$$

was considered. Clearly, this model includes both the Liénard and Rayleigh equations, as well as (2.12) and (2.13). If we denote, as before, by $h$ the inverse of $\phi$, now we can consider the equivalent system

$$(S_{a,b}) \qquad \begin{cases} \dot{x} = h(y) \\ \dot{y} = -\lambda a(x)b(y) - g(x), \end{cases}$$

for $a(x) := f_0(x)$ and $b(y) := f_1(h(y))$. Note that system $(S_{a,b})$ fits in the class of equation $(S)$ for $f(x, y) := a(x)b(y)$. In this case, hypothesis $(f_{\text{even}})$ is satisfied

provided that $b$ is an even function with $b(0) = 0$.
As a final remark regarding these kind of equations, we observe that system $(S_{a,b})$ (as well as $(S)$) covers also the case of a scalar Liénard or Rayleigh equation with relativistic acceleration. Studies about the existence of limit cycles for these equations initiated in [61] and [53]. In this case, assuming by convention the speed of light in vacuum $c = 1$, the differential (relativistic acceleration) operator takes the form $\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right)$ and hence $\phi(s) = \frac{s}{\sqrt{1-s^2}}$ in equation (2.14). In this case $\phi : (-1,1) \to \mathbb{R}$ is an odd increasing homeomorphism and $h : \mathbb{R} \to (-1,1) \subset \mathbb{R}$ (the inverse of $\phi$), defined by

$$h(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}, \tag{2.15}$$

is an odd increasing map with $h(0) = 0$, so that hypothesis $(h_{\text{sign}})$ is satisfied.

The following result holds.

**Lemma 2.1.** *Assume* $(g_{\text{sign}})$, $(h_{\text{sign}})$ *and* $(f_{\text{even}})$. *Suppose that*

$$g_0 := \liminf_{s \to x^0} \frac{g(s)}{s - x^0} > 0 \tag{2.16}$$

*and*

$$h_0 := \liminf_{s \to 0} \frac{h(s)}{s} > 0 \tag{2.17}$$

*hold. Then there is* $\lambda^* > 0$ *such that, for each* $\lambda \in (0, \lambda^*)$ *the equilibrium point* $Q = (x^0, 0)$ *is a (local) center for system* $(S)$.

*Proof.* Without loss of generality, we deliver the proof in the case $x^0 = 0$, namely when the unique equilibrium point is the origin. The hypothesis for $f$ being locally Lipschitzian with $f(x,0) \equiv 0$ implies that there exist constants $f_0 > 0$ and $\delta_0 > 0$ such that

$$|f(x,y)| \le f_0|y|, \qquad \forall\, (x,y): \ |x| \le \delta_0, \ |y| \le \delta_0\,. \tag{2.18}$$

Let $\theta(t, z_0)$ be the angular coordinate associated with the solution of system $(S)$ with initial point $z_0 \neq 0 = (0,0)$. By the uniqueness of the solutions to the initial value problems associated to $(S)$ we know that $\theta$ is well defined on its maximal interval of existence. By the sign conditions on the vector field we know that the solutions move from left to right in the upper half-plane and from right to left in the lower half-plane. Moreover,

$$-\frac{d}{dt}\theta(t) = \frac{ydx - xdy}{x^2+y^2} = \frac{h(y)y + \lambda f(x,y)x + g(x)x}{x^2+y^2}\,,$$

for $(x, y) = (x(t), y(t)) = (x(t, z_0), y(t, z_0))$ the solution of $(S)$ with $(x(0), y(0)) = z_0 \neq 0$. For any $\varepsilon > 0$ and sufficiently small, let $\delta_\varepsilon \in (0, \delta_0)$ be such that $g(s)s \geq (g_0 - \varepsilon)s^2$ and $h(s)s \geq (h_0 - \varepsilon)s^2$ for all $|s| \leq \delta_\varepsilon$. Hence,

$$-\dot{\theta}(t) \geq \frac{(h_0 - \varepsilon)y(t)^2 - \lambda f_0 |x(t)||y(t)| + (g_0 - \varepsilon)x(t)^2}{x(t)^2 + y(t)^2},$$

as long as $|x(t)| \leq \delta_\varepsilon$ and $|y(t)| \leq \delta_\varepsilon$.
If we assume that

$$\lambda^2 f_0^2 < 4 g_0 h_0 \,, \tag{2.19}$$

then, for $\varepsilon > 0$ sufficiently small, the matrix $\mathscr{M}_\varepsilon := \begin{pmatrix} g_0 - \varepsilon & \lambda f_0/2 \\ \lambda f_0/2 & h_0 - \varepsilon \end{pmatrix}$ is positive definite. Hence, there exists $\eta = \eta_\varepsilon > 0$ such that $-\dot{\theta}(t) \geq \eta$. And therefore, $|\theta(t) - \theta(0)| \geq \eta t$ for all $t > 0$ such that $|x(t)| \leq \delta_\varepsilon$ and $|y(t)| \leq \delta_\varepsilon$ on $[0, t]$.

Suppose now $t > \tau^* := \pi/\eta$. This means that the solution starting at a point $(x^0, 0)$ with $x^0 < 0$ will meet again the $x$-axis at a first point $(x(\tau), 0)$ with $x(\tau) > 0$ for some time $\tau \in (0, \tau^*]$. By the mirror symmetry of the trajectories with respect to the $x$-axis, we conclude that the orbit starting at the point $P_0 = (0, x_0)$ with $x_0 < 0$ is closed. Of course, all the above argument is valid provided that the solution departing from $P_0$ lies in the neighborhood of the origin $[-\delta_\varepsilon, \delta_\varepsilon]^2$ during the time-interval $[0, \tau^*]$. But, since the trivial one $(x, y) \equiv (0, 0)$ is a solution of system $(S)$, an elementary continuity argument guarantees that our previous ansatz is correct provided that we choose $P_0$ sufficiently close to the origin.

Finally, it is trivial to observe that condition (2.19) is satisfied provided that

$$\lambda < \lambda^* := \frac{2\sqrt{g_0 h_0}}{f_0} \,. \tag{2.20}$$

We have thus proved that if (2.20) holds, then all the solutions of $(S)$ starting from a point $P_0 \neq 0$ in a neighborhood of the origin are closed and hence the origin is a (local) center.

If $x^0 \neq 0$, then we have only to translate the same argument, taking as the origin of the polar coordinate system the equilibrium point $Q = (x^0, 0)$ and using the formula $-\dot{\theta} = \frac{y dx - (x - x^0) dy}{(x - x^0)^2 + y^2}$ for the angular derivative. □

**Remark 2.2.** The assumption that the matrix $\mathscr{M}_\varepsilon$ is positive definite implies the condition that the origin be a focus or a center for the comparison linear system

$$\dot{x} = h_0 y, \quad \dot{y} = \pm \lambda f_0 y - g_0 x.$$

Hence, the conclusion of the proof in Lemma 2.1 can be also achieved by invoking a classical result [68, Theorem 15, p.95], from the book of Sansone-Conti.

We also observe that condition (2.19) is sharp. Indeed, if we apply it to the symmetric Rayleigh system

$$\ddot{x} + \lambda|\dot{x}| + Ax = 0, \tag{2.21}$$

we find that the origin is a center if $\lambda^2 < 4A$. The same condition was proved in [78] to be necessary and sufficient for the existence of a center to (2.21).

In general, we cannot give global information about the phase-portrait of system $(S)$ without imposing quite specific conditions on the functions $h, g, f$. Indeed, under the only assumptions of Lemma 2.1, we cannot prevent the existence of separatrices and regions of the plane like in [79, 80] where the trajectories become unbounded once they are entered. Moreover, without some specific growth assumptions of $f(x, y)$ with respect to the $y$-variable, the global continuability of the solutions is not guaranteed. However, as we will see in the two examples in Section 3, taking as $h$ the inverse of the nonlinearity associated to the relativistic acceleration, in some cases helps to have the solutions globally defined and also to provide natural conditions for the equilibrium to be a global center.

We suppose now that we have a switched system with two subsystems $(S_1)$ and $(S_2)$ where the functions in

$$(S_i) \qquad \begin{cases} \dot{x} = h_i(y) \\ \dot{y} = -\lambda_i f_i(x, y) - g_i(x), \end{cases}$$

satisfy the basic assumptions $(h_{\text{sign}})$, $(g_{\text{sign}})$ and $(f_{\text{even}})$, previously introduced. We also suppose that both the equilibrium points $Q_1 = (x_1^0, 0)$ and $Q_2 = (x_2^0, 0)$ are centers. Then we can construct two annular regions $\mathcal{A}_1$ and $\mathcal{A}_2$ around $Q_1$ and $Q_2$, respectively, which are filled by periodic orbits. Since the trajectories of the two systems have mirror symmetry with respect to the $x$-axis, we have that also $\mathcal{A}_1$ and $\mathcal{A}_2$ are symmetric with respect to the line $y = 0$. To be more specific, assume that $\mathcal{A}_i$ has the orbits $\Gamma_i^{\text{in}}$ and $\Gamma_i^{\text{out}}$ as inner and outer boundary, respectively. We denote by

$$\tau_i^{\text{in}} \quad \text{and} \quad \tau_i^{\text{out}}$$

the periods of the orbits $\Gamma_i^{\text{in}}$ and $\Gamma_i^{\text{out}}$, respectively.
Furthermore, we denote by $(c_i^-, 0)$ and $(c_i^+, 0)$ the two points of intersection of $\Gamma_i^{\text{in}}$ with the $x$-axis and by $(d_i^-, 0)$ and $(d_i^+, 0)$ the two points of intersection of $\Gamma_i^{\text{out}}$ with the $x$-axis, so that

$$d_i^- < c_i^- < x_i^0 < c_i^+ < d_i^+.$$

By construction, $\mathcal{A}_i \cap \{(x, y) : y = 0\} = \big([d_i^-, c_i^-] \times \{0\}\big) \cup \big([c_i^+, d_i^+] \times \{0\}\big)$.
Similarly, we denote by $(x_i^0, h_i^-)$ and $(x_i^0, h_i^+)$ the two points of intersection of $\Gamma_i^{\text{in}}$ with

the vertical line $x = x_i^0$ and by $(x_i^0, k_i^-)$ and $(x_i^0, k_i^+)$ the two points of intersection of $\Gamma_i^{\text{out}}$ with the line $x = x_i^0$, so that

$$k_i^- < h_i^- < 0 < h_i^+ < k_i^+, \quad \text{and, by symmetry, } h_i^- = -h_i^+, \ k_i^- = -k_i^+.$$

By construction, $\mathcal{A}_i \cap \{(x, y) : x = x_i^0, y \geq 0\} = \{x_i^0\} \times [h_i^-, k_i^-]$.

Any of the sets

$$[d_i^-, c_i^-] \times \{0\}, \quad [c_i^+, d_i^+] \times \{0\}, \quad \{x_i^0\} \times [h_i^-, k_i^-]$$

will be called a *ray* of the annulus $\mathcal{A}_i$. Moreover, we also denote by $\mathcal{I}(\mathcal{A}_i)$ the inner region of $\mathcal{A}_i$, namely the bounded open component of $\mathbb{R}^2 \setminus \Gamma_i^{\text{in}}$ and by $\mathcal{E}(\mathcal{A}_i)$ the outer region of $\mathcal{A}_i$, namely the unbounded open component of $\mathbb{R}^2 \setminus \Gamma_i^{\text{out}}$. Following [59], we say that the two annuli $\mathcal{A}_1$ and $\mathcal{A}_2$ are *topologically linked* if there exist two rays $\mathfrak{r}_1$ and $\mathfrak{r}_2$ of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, such that

$$\mathfrak{r}_1 \subset \mathcal{I}(\mathcal{A}_2) \quad \text{and} \quad \mathfrak{r}_2 \subset \mathcal{I}(\mathcal{A}_1).$$

As proved in [59, Proposition 3.1] if two annuli are topologically linked, there are also parts of $\mathcal{A}_1$ in the exterior of $\mathcal{A}_2$ and vice-versa. Possible examples of linked annuli in case of systems $(S_i)$ are described in Figure 3.

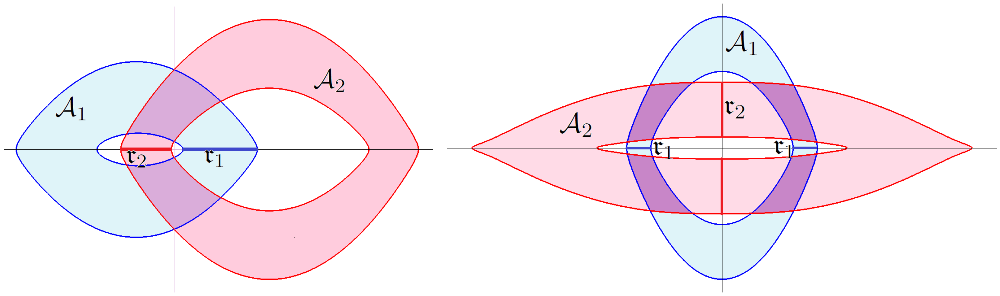



Figure 3: The left panel shows two linked annuli which are obtained by the switched subsystems $\dot{x} = h(y)$, $\dot{y} = -|h(y)| - 2x + \{e_1, e_2\}$ for $e_1 = -1$ (left annulus) and $e_2 = 6$ (right annulus). The right panel shows two linked annuli which are obtained by the switched subsystems $\dot{x} = h(y)$, $\dot{y} = -\frac{x^3}{1+x^2}(h(y))^2 - \{A_1, A_2\}x$ for $A_1 = 10$ (annulus elongated vertically) and $A_2 = 0.1$ (annulus elongated horizontally). In both the cases $h(y)$ is defined as in (2.15), so that the systems correspond to the equations with relativistic acceleration $\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + |\dot{x}| + Ax = e(t)$ (Rayleigh-type) and $\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + \frac{x^3}{1+x^2}(\dot{x})^2 + A(t)x = 0$ (quadratic Liénard-type), respectively.

In this setting, the following result holds for the $T$-periodic switched system which activates $(S_1)$ for an interval of time-length $T_1$ and $(S_2)$ for an interval of time-length $T_2$, with $T_1 + T_2 = T$. In other words, we have a planar system of the form

(1.1) having $(S_1)$ and $(S_2)$ as subsystems and such that $0 = \tau_0 < \tau_1 = T_1 < \tau_2 = T$, with $\tau_2 - \tau_1 = T_2$ .

**Theorem 2.3.** *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ be two annular regions filled by periodic orbits of the systems $(S_1)$ and $(S_2)$, respectively. Assume that:*

$(LC)$ $\quad$ *$\mathcal{A}_1$ and $\mathcal{A}_2$ are topologically linked* $\quad$ (Linked Condition)

$(TC)$ $\quad$ $\tau_1^{\mathrm{in}} \neq \tau_1^{\mathrm{out}}$ *and* $\tau_2^{\mathrm{in}} \neq \tau_2^{\mathrm{out}}$ $\quad$ (Twist Condition)

*Then, for every pair $(k,\ell)$ of positive integers with $m := k \times \ell \geq 2$, there are $T_1^* = T_1^*(k) > 0$ and $T_2^* = T_2^*(\ell) > 0$ such that for any $T_1 > T_1^*$ and $T_2 > T_2^*$ the Poincaré map associated with switched system* (1.1) *induces chaotic dynamics on $m$-symbols on a set $\mathcal{D} \subset \mathcal{A}_1 \cap \mathcal{A}_2$ .*

*Proof.* Our result is basically a version of [51, Theorem 4.1], with the improvement obtained in [59], where a more general notion of linked annuli was introduced. In [51] the result is stated for the planar Hamiltonian system

$$\dot{z} = J\nabla H(z) + e_{1,2}(t)$$

which is a periodic switched system having $\dot{z} = J\nabla H(z) + e_1$ and $\dot{z} = J\nabla H(z) + e_2$ as subsystems. Here $J$ denotes the symplectic matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable Hamiltonian function. In [51, Theorem 4.1] the $(LC)$ condition is slightly more restrictive that ours (but, as already observed, a proper generalization was obtained in [59]). In any case, the assumption in [51] is that there are two annular regions linked together and each one filled by the periodic orbits of one (respectively) of the autonomous subsystems. In our case, we have the same geometry obtained by means of two reversible systems as in [59] instead of two Hamiltonian systems. Since the theorems in [51, 59] are of topological nature and do not depend on a particular Hamiltonian structure, we have that the results in the above cited articles apply to our situation and hence our theorem follows. We also observe that essentially the same proof lead to [59, Theorem 1.1] where we considered another type of switched reversible systems with mirror symmetry with respect to the $y$-axis. Accordingly, we refer the reader to [59] for the missing details. □

**Remark 2.4.** The minimal lengths $T_1^*$ and $T_2^*$ of the intervals in which each of the subsystems $(S_1)$ and $(S_2)$ is active can be estimated from the gaps $\Delta_1 := |\tau_1^{\mathrm{out}} - \tau_1^{\mathrm{in}}|$ and $\Delta_2 := |\tau_2^{\mathrm{out}} - \tau_2^{\mathrm{in}}|$. Typically, the smaller are $\Delta_1$ and $\Delta_2$, the larger $T_1^*$ and $T_2^*$ are needed to be.

**Remark 2.5.** We have stated Theorem 2.3 for a switched system which has $(S_1)$ and $(S_2)$ as subsystems. This choice is made in view of the examples that we are going to present in Section 3. It is clear that *any* switched system having two subsystems satisfying $(LC)$ and $(TC)$ is suitable for the application of Theorem 2.3.

**Remark 2.6.** As observed in previous articles (see for instance [60] where a detailed discussion on this aspect was performed), the result expressed by Theorem 2.3 is stable under small perturbation of the coefficients in the $L^1$-norm on $[0,T]$.

## 3 Some examples

In this section we present two new examples of application of our method to periodically switched reversible systems. Previous examples have been recently obtained for some particular switching-type reversible systems in [44, 59]. Here we focus our attention to the case in which the second order differential operator is replaced by a relativistic acceleration, a research topic which has received a great deal of interest in the last decade [18, 53, 77]. To show the effectiveness of our method, we will present two examples, the first one for a Rayleigh-type equation and the second one for a Liénard-type equation with a quadratic term in the derivative (as considered in [66]).

As a first example, we consider the periodically forced equation

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + \lambda|\dot{x}| + Ax = e(t) \tag{3.1}$$

which is the analogous of the case studied in [78], but with a relativistic acceleration. Here we assume $\lambda > 0$ and $A > 0$ and $e : \mathbb{R} \to \mathbb{R}$ is a $T$-periodic function.

As a second example, we consider

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + \lambda\varphi(x)\dot{x}^2 + A(t)x = 0, \tag{3.2}$$

where $\lambda > 0$, $\varphi : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function and $A : \mathbb{R} \to \mathbb{R}$ is a $T$-periodic function with $A(t) > 0$ for all $t$.

For equation (3.1) we suppose that $e(t)$ is a stepwise function of the form

$$e(t) = \begin{cases} e_1, & \text{for } \ 0 \le t < T_1 \\ e_2, & \text{for } \ T_1 \le t < T_1 + T_2 = T, \end{cases} \tag{3.3}$$

with $e_1 \neq e_2$.

Analogously, for (3.2) we suppose that $A(t)$ is a stepwise function of the form

$$A(t) = \begin{cases} A_1, & \text{for } 0 \le t < T_1 \\ A_2, & \text{for } T_1 \le t < T_1 + T_2 = T, \end{cases} \tag{3.4}$$

with $A_1 \neq A_2$ and $A_1, A_2 \in \mathbb{R}_0^+ = (0, +\infty)$. In this manner, for both equations we enter in the setting of the periodic switched systems with two active subsystems.

Let us start now with the analysis of equation (3.1), with $e(t)$ as in (3.3). If we set

$$y := \frac{\dot{x}}{\sqrt{1-\dot{x}^2}}, \qquad \dot{x} = h(y) := \frac{y}{\sqrt{1+y^2}}, \tag{3.5}$$

we can write equation (3.1) as an equivalent switched system $(S_1)$-$(S_2)$, with

$$(S_i) \qquad \begin{cases} \dot{x} = h(y) \\ \dot{y} = -\lambda |h(y)| - Ax + e_i, \end{cases} \qquad i = 1, 2.$$

Each of these subsystems has a unique equilibrium point $Q_i = (x_i^0, 0)$ with $x_i^0 = \frac{e_i}{A}$. The change of variables $x \to x + x_i^0$, $y \to y$ moves the equilibrium point $Q_i$ to the origin. Therefore, we can study the system

$$\begin{cases} \dot{x} = h(y) \\ \dot{y} = -\lambda |h(y)| - Ax \end{cases} \tag{3.6}$$

as a universal model for both $(S_1)$ and $(S_2)$. The vector field in system (3.6) is globally Lipschitz. Thus the solutions are globally defined in time. It is immediate to check that we enter in the setting of Lemma 2.1 with $h_0 = 1$, $g_0 = A$. Moreover, $f_0 = 1$. Then we have that the origin is a local center of (3.6) provided that

$$\lambda < 2\sqrt{A} \tag{3.7}$$

(as in [78]). Actually a stronger result holds. Indeed we have the following.

**Proposition 3.1.** *Under condition* (3.7)*, the origin is a global center for system* (3.6)*.*

*Proof.* We argue as in the proof of Lemma 2.1. Passing to the polar coordinates we find that

$$-\dot{\theta} = \frac{h(y)y + \lambda |h(y)| x + Ax^2}{x^2 + y^2} \ge \frac{h(y)y - \lambda |h(y)||x| + Ax^2}{x^2 + y^2}. \tag{3.8}$$

We can consider the numerator in the above expression as a quadratic form with associated symmetric matrix given by $\mathscr{M}(x,y) := \begin{pmatrix} A & \lambda\frac{h(y)}{2y} \\ \lambda\frac{h(y)}{2y} & \frac{h(y)}{y} \end{pmatrix}$. Now, since

$$\det(\mathscr{M}(x,y)) = \frac{h(y)}{4y}\left(4A - \frac{h(y)}{y}\lambda^2\right) \geq \frac{h(y)}{4y}(4A-\lambda^2) \geq \delta_R > 0,$$

for each $y$ in a compact interval $[-R,R]$, arguing similarly as in the proof of Lemma 2.1, we can conclude that the trajectories starting from a point $P_0 = (x_0,0)$ with $x_0 < 0$ enter the upper half-plane and then hit again the $x$-axis at a point $(x_1,0)$ with $x_1 > 0$.

To make this argument a little more precise, observe also that the solution of (3.6) starting at the point $P_0$ enters the upper half-plane (as $-g(x_0) > 0$ for $x_0 < 0$) with $\dot{y} > 0$ until the solution achieves its maximum in the $y$-component when it crosses the part of the isocline $x = -\frac{\lambda}{A}|h(y)|$ at some point $(u_1,y_1)$ with $x_0 < u_1 < 0$ and $y_1 > 0$. Then, as long as the solution remains in the upper half-plane, we have $0 < y(t) \leq y_1$ and therefore $h(y(t))/y(t) = 1/\sqrt{1+y(t)^2} \geq 1/\sqrt{1+y_1^2} = c_1 > 0$. This justifies the above assertions on the quadratic form.

Symmetrically, the trajectories starting at a point $(x_0,0)$ with $x_0 > 0$ enter the lower half-plane and then hit again the $x$-axis at a point $(x_1,0)$ with $x_1 < 0$. This proves that, indeed, the origin is a global center for system (3.6) if (3.7) is satisfied. □

Now, the period of the small orbits is asymptotic to the period of the orbits of

$$\dot{x} = y, \quad \dot{y} = -\lambda|y| - Ax,$$

which is

$$\tau_0 := \frac{4\pi}{\sqrt{4A-\lambda^2}}.$$

On the other hand, it is easy to show that, by virtue of the fact that $h(y)$ is bounded, the period of the large orbits tends to infinity. A visual interpretation of this occurrence is given in Figure 4.

**Remark 3.2.** The result that we have obtained for (3.6) is sharp. Indeed, for $\lambda > 2\sqrt{A}$ we do not have a center anymore. However, due to fact that $h(y)/y \to 0$ for $y \to \pm\infty$, we have that the numerator in (3.8) is positive and bounded away from zero on larger orbits. This implies that we can produce in any case an annular region filled by periodic orbits (outside a bounded neighborhood of the origin). Figure 5 illustrates this situation.

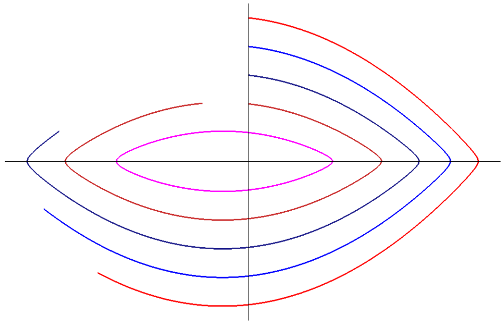


Figure 4: The simulation shows the solutions of (3.6) with $\lambda = 1$ and $A = 2$ (the case of a center, according to (3.7)) starting at the points $(0, y_0)$ with $y_0 = 5, 10, 15, 20, 25$ and a fixed time interval. The presence of a period gap between the smaller and larger orbits is evident.

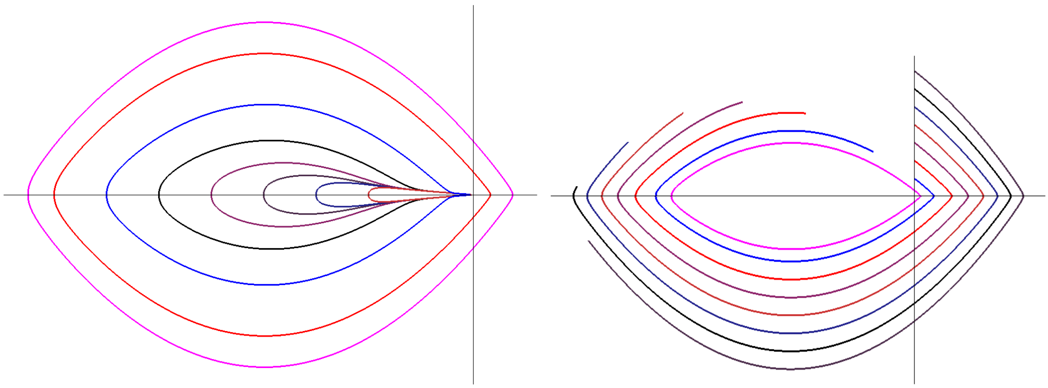


Figure 5: The simulation shows the solutions of (3.6) with $\lambda = 4$ and $A = 1$. The origin is no more a local center, due to the failure of condition (3.7) and a region at the left-hand side of the origin filled by homoclinic solutions appears. However, larger orbits are periodic and fill an open annulus around the origin (as shown in the left panel). Moreover, it is possible to construct periodic annular regions with a twist condition at the boundary as shown in the right panel where we consider periodic orbits starting at the points $(0, y_0)$ with $y_0 = 1, 3, 6, 9, 12, 15, 18, 21$ and a fixed time interval. The presence of a period gap between the smaller and larger orbits is evident. This shows that it will be possible to apply Theorem 2.3 to (3.1) also when condition (3.7) is not satisfied.

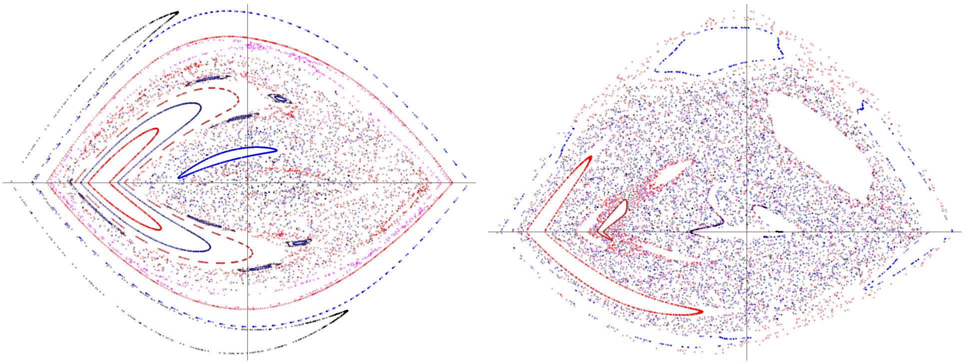
At this point, for $e_1 \neq e_2$, we can construct two linked annuli like in the left panel of Figure 3 and, using the gap between the periods of the orbits, we can apply Theorem 2.3 in order to prove rigorously the presence of complex dynamics associated with system (3.1), provided that the two switching times are sufficiently large. Furthermore, according to Remark 2.6, we can obtain the same result for a (possibly smooth) periodic forcing term $e(t)$ which is close to a stepwise function in the $L^1$-norm. Figure 6 shows the Poincaré section for the periodically forced planar system (equivalent to (3.1))

$$\begin{cases} \dot{x} = h(y) \\ \dot{y} = -\lambda|h(y)| - Ax + e(t), \end{cases} \tag{3.9}$$

for $\lambda = 1$, $A = 2$ (so that condition (3.7) is satisfied) and for $\lambda = 4$, $A = 1$ (so that condition (3.7) is not satisfied) and

$$e(t) = K \tanh(n \sin(\omega t + \alpha)). \tag{3.10}$$

The function $\tanh(nx)$ for $n$ an integer sufficiently large provides a good approximation of $\text{sign}(x)$ (see [34, 75]). Hence the function $e(t)$ defined in (3.10) is close to a stepwise periodic function in the $L^1$-norm.



(a) Poincaré section for the case $\lambda = 1$, $A = 2$ (cf. left panel of Figure 3).

(b) Poincaré section for the case $\lambda = 4$, $A = 1$ (cf. Figure 5).

Figure 6: The figures are obtained by considering 800 iterations of the Poincaré map for (3.9) and starting from different initial points. The resulting points are spread in a rectangle of size $[-16, 15] \times [-360, 370]$, starting from 15 initial points (left panel) and, respectively, in a rectangle of size $[-21, 18] \times [-140, 250]$, starting from 16 initial points (right panel). Accordingly, the aspect-ratio of both the figures is modified in order to put in better evidence the resulting structure. The simulation has been performed for $e(t)$ as in (3.10) with $K = 3$, $n = 6$, $\omega = 0.05$ and $\alpha = 0.5$. Thus the period of the forcing term $e(t)$ is $T = 40\pi \sim 125.66$ which is reasonably large. The distribution of the points in the Poincaré section shows the typical patterns of alternation of regions of stability/instability, including subharmonic solutions of large order and *island chains* according to [6] (compare also with [64, Figure 1] for a similar pattern emerging from a completely different equation). The presence of a rich and complex structure is clear in both the examples.

Coming to the second example, we study now equation (3.2), with $A(t) > 0$ as in (3.4) and consider the equivalent switched system $(S_1)$-$(S_2)$, with

$$(S_i) \qquad \begin{cases} \dot{x} = h(y) \\ \dot{y} = -\lambda\varphi(x)(h(y))^2 - A_i x \end{cases} \qquad i = 1, 2.$$

Clearly, each of the subsystems $(S_i)$ fits in the class of type $(S_{a,b})$ with $a(x) = \varphi(x)$ and $b(y) = (h(y))^2$. Then, for $(x, y)$ in a neighborhood of the origin such that $|\varphi(x)| \leq |\varphi(0)| + 1$ for $|x| \leq \delta_0$ and $y^2 \leq \mu_0|y|$ for $|y| \leq \delta_0 < 1$ we have

$$|f(x, y)| = |\varphi(x)|h(y)^2 \leq (|\varphi(0)| + 1)y^2 \leq (|\varphi(0)| + 1)\mu_0|y| = \kappa_0|y|,$$

for $\kappa_0 := (|\varphi(0)| + 1)\mu_0$. Hence, we can apply Lemma 2.1 with $h_0 = 1$, $g_0 = A_i > 0$ and (2.18) holds with $f_0 = \kappa_0$, and $\mu_0 > 0$ as small as we like provided that we take $\delta_0$ sufficiently small (depending on $\mu_0$). In particular, for any fixed $\lambda > 0$ we can tale $\mu_0$ and then $\kappa_0$ so that $\lambda^2\kappa_0^2 < 4\min\{A_1, A_2\}$, thus concluding that the origin is a local center for system $(S_i)$ (for $i = 1, 2$).

The question if the local center is a global one is much more delicate and, to simplify our analysis, we will suppose now that $\varphi$ is bounded. In this case, discharging a multiplicative factor on the parameter $\lambda$, without loss of generality, we can suppose that

$$(\varphi_{\text{bound}}) \qquad |\varphi(x)| \leq 1, \qquad \forall\, x \in \mathbb{R}.$$

Then, the following result holds, where $A$ stands for $A_1$ or $A_2$.

**Proposition 3.3.** *Assume* $(\varphi_{\text{bound}})$. *Then, for* $h$ *as in* (3.5), *the origin is a global center for the system*

$$\dot{x} = h(y), \qquad \dot{y} = -\lambda\varphi(x)(h(y))^2 - Ax, \tag{3.11}$$

*for every* $\lambda > 0$ *and* $A > 0$.

*Proof.* First of all, we observe that the solutions of system (3.11) are globally defined in time. This follows from standard results from the theory of ODEs, as $h(y)$ and $\varphi(x)$ are bounded and $g(x) = Ax$ is globally Lipschitz. Next, arguing like in Lemma 2.1 and Proposition 3.1 we pass to the polar coordinates and we obtain

$$-\dot{\theta} = \frac{h(y)y + \lambda\varphi(x)h(y)^2 x + Ax^2}{x^2 + y^2} \geq \frac{h(y)y - \lambda|h(y)||x| + Ax^2}{x^2 + y^2}. \tag{3.12}$$

In fact, the inequality follows from $(\varphi_{\text{bound}})$ and by the fact that $|h(y)| \leq 1$ and therefore $h(y)^2 \leq |h(y)|$.

Thus,from (3.12) we get the same lower bound for $-\dot{\theta}$ as in (3.8) and from now on we just can repeat the argument in the proof of Proposition 3.1 and conclude that there all the large orbits are closed.

Now we are in the following situation. We have a small orbit, say $\gamma_1$, passing through a point $P_1 = (x_1, 0)$ with $x_1 < 0$ such that for every $P_0 = (x_0, 0)$ with $x_1 \le x_0 < 0$ the trajectory of system (3.11) passing through $P_0$ is closed (in fact, we know that the origin is a local center). We also have a large orbit, say $\gamma_2$, passing through a point $P_2 = (x_2, 0)$ with $x_2 < x_1 < 0$ such that for every $P_0 = (x_0, 0)$ with $x_0 \le x_2 < 0$ the trajectory of system (3.11) passing through $P_0$ is closed (this has been just achieved in the first part of the proof). It remains to prove that also the orbits between $\gamma_1$ and $\gamma_2$ are closed.
To this aim, let us consider now a point $P_0 = (x_0, 0)$ with $x_2 < x_0 < x_1 < 0$ and let $\gamma^+(P_0)$ be the positive semi-orbit of system (3.11) passing through $P_0$. We claim that $\gamma^+(P_0)$ hits the positive $x$-axis at a point $P_0' = (u_0, 0)$ with $u_0 > 0$. Once this claim is proved, we have that the orbit path $\widehat{P_0 P_0'}$ of $\gamma^+(P_0)$, by reflection with respect to $x$-axis, generates a closed orbit of system (3.11). To prove our claim, suppose, by contradiction, that there is a point $P_0$ as above such that $\gamma^+(P_0)$ does not cross the positive $x$-axis. This implies that $\gamma^+(P_0)$ will remain in the upper half-plane between $\gamma_1$ and $\gamma_2$; call $\mathcal{D}$ this region and observe that $0 \notin \mathcal{D}$. The Poincaré-Bendixson theory implies that in the simply connected region $\mathcal{D}$ there should be an equilibrium point of (3.11), but this is impossible, as the only equilibrium point of the system is the origin.

Hence, we have proved that any positive semi-orbit departing from a point $P_0 = (x_0, 0)$ with $x_0 < 0$ is periodic. Symmetrically, the trajectories starting at a point $(x_0, 0)$ with $x_0 > 0$ enter the lower half-plane and then hit again the $x$-axis at a point $(x_1, 0)$ with $x_1 < 0$ and then are closed by the mirror symmetry with respect to the $x$-axis.

In this manner we have verified that the origin is a global center for system (3.11). □

Now, in order to enter in the setting of Theorem 2.3 we just need to link together two annular regions for systems $(S_1)$ and $(S_2)$ for $A_1 \neq A_2$, like in the right panel of Figure 3 and, using the gap between the periods of the orbits, we are able to prove rigorously the presence of complex dynamics associated with system (3.2), provided that the two switching times are sufficiently large. Furthermore, according to Remark 2.6, we can obtain the same result for a (possibly smooth) periodic positive weight function $A(t)$ which is close to a stepwise function like the one in (3.4) in the $L^1$-norm. Figure 9 shows the Poincaré section for the periodically forced planar system (equivalent to (3.2))

$$\begin{cases} \dot{x} = h(y) \\ \dot{y} = -\lambda\varphi(x)h(y)^2 - A(t)x, \end{cases} \tag{3.13}$$

where we have taken

$$\varphi(x) = \frac{x^2}{x^2+1} \tag{3.14}$$

and

$$A(t) = 2\big(A + K\tanh(n\sin(\omega t + \alpha))\big). \tag{3.15}$$

with the following parameters:

$$A = 10, \quad K = 9.9, \quad n = 6, \quad \omega = 1/10, \quad T = 20\pi \sim 62.832, \quad \alpha = 0. \tag{3.16}$$

Note that with this choice $A(t)$ is a smooth function close to a stepwise function which jumps between the values $A_1 = 39.8$ and $A_2 = 0.2$. Figure 7 gives evidence of the time-gap (twist condition) between smaller (faster) and larger orbits (slower), while Figure 8 show how linked annuli can be easily constructed for different values of the parameter $A > 0$.

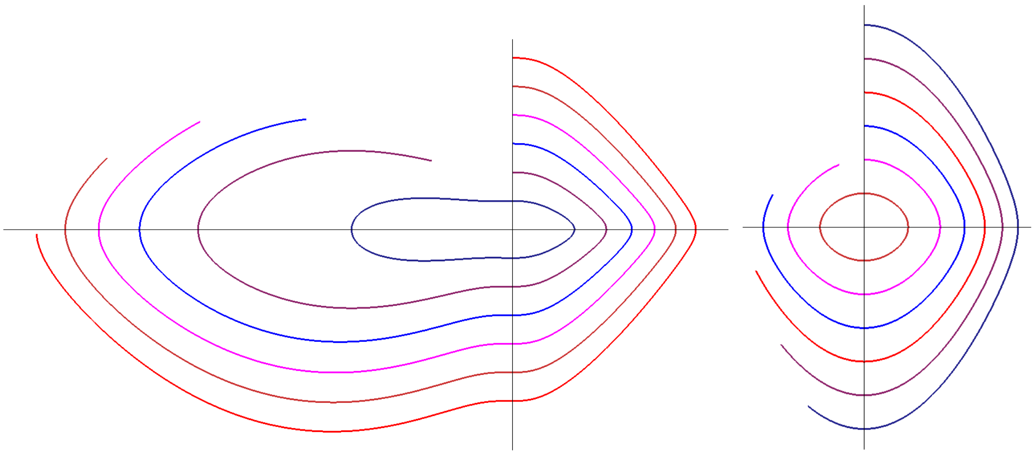


Figure 7: The simulation shows the solutions of (3.11) with $\lambda = 1$ and $A = 0.2$ (left panel) and those of the same system with $\lambda = 1$ and $A = 39.8$ (right panel). In both the cases the initial points are of the form $(0, y_0)$ with $y_0 = 1, 2, 3, 4, 5, 6$ and a fixed time interval. The presence of a period gap between the smaller and larger orbits is evident. It is interesting to observe also that the period of the orbits for $A = 0.2$ is notably larger than that for the case $A = 38.8$. In fact in the former case, the simulation is made for the time-interval $[0, 24]$, while in the latter the time interval is $[0, 1.3]$.

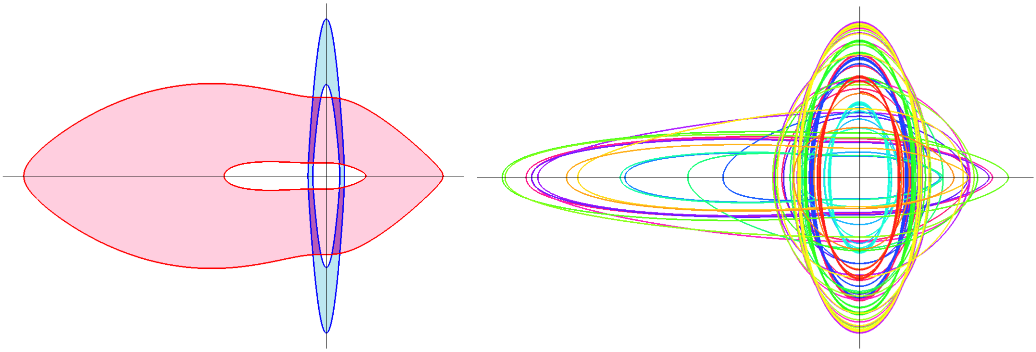


Figure 8: The left panel shows the example of two linked annuli obtained with $\lambda = 1$ and $\varphi(x)$ as in (3.14), for $A_1 = 38.8$ and $A_2 = 0.2$, respectively. The right panel shows a numerical simulation of the solution of system (3.13) for the time interval $[0, 400]$ and the initial point $P_0 = (0, 7)$. The function $A(t)$ is as in (3.15) with the choice of parameters as in (3.16). The dynamical behavior is that of a trajectory which roughly oscillates between the orbits of the (3.11) for $A_1 = 39.8$ in almost half of the period (with very rapid oscillations) and those of the same equation for $A = 0.2$ for almost the rest of the period (with very slow oscillations).

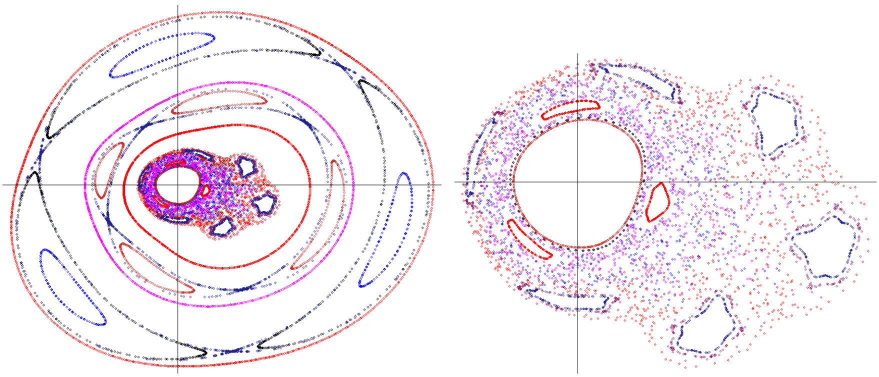


Figure 9: The figures are obtained by considering 800 iterations of the Poincaré map for (3.13) and starting from different initial points. The resulting points are spread in a rectangle of size $[-2.4, 3.7] \times [-17, 16]$, starting from 16 initial points Accordingly, the aspect-ratio of the figure is modified in order to put in better evidence the resulting structure. The simulation has been performed for $\varphi$ as in (3.14) and $\lambda = 1$. The weight function $A(t)$ is defined as in (3.15) with the parameters described in (3.16). The left panel shows the whole picture for all the initial points, while the right panel is a zoom of the same figure, by considering only the initial points which are near the origin. The emerging of a rich and complex structure is evident.

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