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# Boundary Regional Controllability of Semilinear Systems Involving Caputo Time Fractional Derivatives

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**Abstract**: We study boundary regional controllability problems for a class of semilinear fractional systems. Sufficient conditions for regional boundary controllability are proved by assuming that the associated linear system is approximately regionally boundary controllable. The main result is obtained by using fractional powers of an operator and the fixed point technique under the approximate controllability of the corresponding linear system in a suitable subregion of the space domain. An algorithm is also proposed and some numerical simulations performed to illustrate the effectiveness of the obtained theoretical results.

**Keywords**: Fractional derivatives and integrals, Nonlinear systems, Regional controllability, Semigroup operators, Fixed-point theorems.

**MSC2010**: 26A33, 93B05, 93C10

To Professor Andrea Bacciotti (in memoriam)

## 1 Introduction

Lately, fractional diffusion equations have garnered increasing attention and established themselves as invaluable tools for modeling various phenomena, particularly in the realms of physics, chemistry, engineering, medicine, and biology [19, 26, 27]. One of the most successful applications of fractional calculus manifest in diffusion models that elucidate the behavior of a diffusing particle, exhibiting mean square displacement rates either slower or faster than those observed in typical diffusion processes [1, 32, 37]. Indeed, anomalous diffusion is prevalent in numerous experiments, highlighting the superior capability of fractional equations to characterize intricate phenomena [6, 23]. Fractional systems can also be found in the field of electrochemistry, where they are employed to determine the concentration of analyzed electroactive species near the electrode surface with greater precision. This task is accomplished through the application of a fractional diffusion model [14, 29]. Owing to its extensive range of applications, the representation formula for the mild solutions of fractional sub-diffusion equations has undergone a comprehensive study: see, e.g., [8, 13, 22, 31, 41] and references therein.

It is widely recognized that the mathematical field of control theory encompasses various concepts, one of which is the notion of controllability [3–5]. The concept of controllability was initially introduced by Kalman in 1960 and involves guiding a system toward a desired state through the use of control techniques [16]. The issue of controllability, for both linear and nonlinear systems, for dynamic equations in both finite and infinite-dimensional spaces, have been extensively investigated [7, 12, 20, 21, 24, 40]. Nonetheless, in a significant number of practical applications, our primary concern lies in scenarios where the desired state of the problem at hand is confined to an internal or boundary subregion within the overall spatial domain. In such cases, the concept of regional controllability becomes crucial [17, 18, 38, 39].

An increasing number of researchers have directed their attention toward the study of regional controllability in linear time-fractional systems [9–11]. For nonlinear fractional systems within infinite-dimensional spaces, pioneering results have also been achieved [15, 25, 30, 34, 35].

The main objective of this work is to study the regional boundary controllability of the following class of distributed abstract fractional semilinear control systems involving a Caputo fractional order $\alpha \in (0, 1]$:

$$\begin{cases} {}^{C}D^{\alpha}_{0+} y(x,t) + Ay(x,t) = \mathcal{F}y(x,t) + \mathbf{B}u(t) & \text{in} \quad \Omega\times]0,T], \\ \dfrac{\partial y(\xi,t)}{\partial \nu_A} = 0 & \text{on} \quad \partial\Omega\times]0,T], \\ y(x,0) = y_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

with the space domain $\Omega$ being a subset of $\mathbb{R}^n$, $n \leq 3$. Here ${}^{C}D^{\alpha}_{0+}$ is the Caputo fractional derivative of order $\alpha$; $-A$ is the infinitesimal generator of an analytic semigroup $(\mathcal{T}(t))_{t\geq 0}$ on the Hilbert space $X := H^1(\Omega)$; $\mathcal{F}$ is a nonlinear operator; the control function $u$ takes values in $U = L^p([0,T], \mathcal{U})$, where $p \geq 2$ and $\mathcal{U}$ is a Banach space; and $\mathbf{B}$ is a linear control operator from $U$ into $L^p([0,T], X)$.

Semilinear systems, described by fractional equations, present an array of challenges and opportunities in the context of regional controllability. Our work is dedicated to establishing sufficient conditions for regional boundary controllability of the Caputo system (1.1), particularly when the operator $-A$ generates an analytic semigroup. This endeavor leverages the tools of fractional calculus and the Picard fixed-point theorem.

The paper is arranged as follows. Here, we have motivated and introduced problem (1.1) under investigation. In Section 2, we present some preliminaries that will be useful throughout the manuscript. Section 3 is devoted to study the boundary controllability on a subregion of the boundary of the evolution domain of the system: under some assumptions and by using the analytic approach, we prove results on the regional controllability by using the relation between internal and boundary controllability concepts (Theorems 3.1 and 3.7). We proceed with Section 4, presenting some numerical simulations that illustrate the effectiveness of the proposed methods. Finally, we end with Section 5 of conclusion, pointing also some directions of possible future research.

## 2 Preliminaries

In this section, we recall basic definitions of fractional operators, the notions of regional controllability, as well as some lemmas that will be useful afterwards.

**Definition 2.1** (See [19]). The Riemann–Liouville fractional integral of order $\alpha \in (0,1]$ with the lower limit 0, of a function $g$ at point $t > 0$, is defined by

$$\mathcal{I}^{\alpha}_{0+} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} g(\zeta) d\zeta, \quad \forall t > 0,$$

where $\Gamma$ is the gamma function.

**Definition 2.2** (See [19]). The Caputo fractional derivative of order $\alpha \in (0,1]$ with the lower limit 0, of a function $g$ at point $t > 0$, is given by

$$^{C}\mathrm{D}^{\alpha}_{0+} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{-\alpha} \frac{d}{d\zeta}(g(\zeta)) ds = \mathcal{I}^{\alpha}_{0+} g'(t). \tag{2.1}$$

Throughout the paper, we denote by $\rho(A)$ the resolvent set of $A$, assuming that $0 \in \rho(A)$. Then, for $0 < q < 1$, the fractional power of $A$ of order $q$ is well defined, being linear and closed on its domain $X_q = D(A^q)$. For details, see [28].

Next, we present the definition of mild solution of system (1.1).

**Definition 2.3** (See [8]). Let $t \in [0,T]$ and $u \in U$. The mild solution of (1.1), denoted by $y_u(\cdot)$, is a continuous function from $[0,T]$ to $X$ defined by the following expression:

$$\begin{aligned} y_u(t) = \mathcal{H}_{\alpha}(t) y_0 &+ \int_0^t (t-\zeta)^{\alpha-1} \mathcal{K}_{\alpha}(t-\zeta) \mathcal{F} y(\zeta, u) d\zeta \\ &+ \int_0^t (t-\zeta)^{\alpha-1} \mathcal{K}_{\alpha}(t-\zeta) \mathbf{B} u(\zeta) d\zeta, \end{aligned} \tag{2.2}$$

where

$$\mathcal{H}_\alpha(t) = \int_0^\infty \varphi_\alpha(\theta)\mathcal{T}(t^\alpha\theta)d\theta,$$

$$\mathcal{K}_\alpha(t) = \alpha \int_0^\infty \theta\varphi_\alpha(\theta)\mathcal{T}(t^\alpha\theta)d\theta,$$

and $\varphi_\alpha$ is a probability density function.

The above operators satisfy some well-known properties. The following lemma is useful for our purposes.

**Lemma 2.4** (See [33,36]). *For any $\alpha, q \in ]0,1]$, the following properties hold:*

*(i) The mapping $||\cdot||_{X^q} = ||A^q(\cdot)||_X$ defines a norm in $X^q$; $(X^q, ||\cdot||_{X^q})$ is a Banach space; and $\overline{X^q} = X$;*

*(ii) For every $t > 0$, there exists $C_q, M_{\alpha q} > 0$ such that*

$$||\mathcal{H}_\alpha(t)||_{\mathcal{L}(X,X^q)} \leq C_q t^{-\alpha q},$$

$$||\mathcal{K}_\alpha(t)||_{\mathcal{L}(X,X^q)} \leq \frac{M_{\alpha q}}{t^{\alpha q}};$$

*(iii) If $\mathcal{E}_\alpha(t) = t^{\alpha-1}\mathcal{K}_\alpha(t)$, then*

$$\mathcal{E}_\alpha \in L^1([0,T], \mathcal{L}(X, X^q)).$$

Throughout the text, $A_1 := ||\mathcal{E}_\alpha||_{L^1([0,T],\mathcal{L}(X,X^q))}$ with $\mathcal{E}_\alpha(t) = t^{\alpha-1}\mathcal{K}_\alpha(t)$.
Let $\omega$ be a subset of $\Omega$. The restriction operator in $\omega$ is defined as follows:

$$\begin{array}{rccc} \chi_\omega : & H^1(\Omega) & \longrightarrow & H^1(\omega) \\ & y & \longmapsto & y_{|\omega}. \end{array}$$

We now introduce the notion of regional controllability.

**Definition 2.5.** For any element $d_s$ of $H^1(\omega)$, if there exists a control $u \in U$ such that $\chi_\omega y_u(T) = d_s$, then we say that system (1.1) is exactly $\omega$-controllable (i.e., regionally exactly controllable in $\omega$). On the other hand, if for all $d_s$ in $H^1(\omega)$ and for all $\epsilon > 0$ there exists a control $u \in U$ such that $||\chi_\omega y_u(T) - d_s||_{H^1(\omega)} \leq \epsilon$, then we say that system (1.1) is approximately $\omega$-controllable (i.e., regionally approximately controllable in $\omega$).

Consider the trace operator $\gamma_0$ from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$, which is a continuous linear onto operator. Let us denote by $\gamma_0^*$ it's adjoint. For $\Gamma$ a subset of $\partial\Omega$, we define the restriction operator $\chi_\Gamma$ by

$$\begin{array}{rlcl} \chi_\Gamma: & H^{\frac{1}{2}}(\partial\Omega) & \longrightarrow & H^{\frac{1}{2}}(\Gamma) \\ & y & \longmapsto & y_{|\Gamma}, \end{array}$$

and we denote by $\chi_\Gamma^*$ it's adjoint.

Next, we give the definition of regional boundary controllability for system (1.1).

**Definition 2.6.** We say that system (1.1) is exactly (resp. approximately) boundary regionally controllable on $\Gamma$ ($\mathcal{B}$-controllable on $\Gamma$) if

$$\forall z_d \in H^{\frac{1}{2}}(\Gamma)\ \exists u \in U \quad \text{such that} \qquad \chi_\Gamma(\gamma_0 y_u(T)) = z_d$$

$$\left(\text{resp.} \qquad \forall z_d \in H^{\frac{1}{2}}(\Gamma)\ \exists \varepsilon > 0\ \exists u \in U \quad ||\chi_\Gamma(\gamma_0 y_u(T)) - z_d||_{H^{\frac{1}{2}}(\Gamma)} \leq \varepsilon\right).$$

## 3 Regional $\mathcal{B}$-Controllability

In this section, we study the possibility of finding a control function that steers system (1.1) to a desired state $z_d$ on $\Gamma$.

Let us define the operator $H_\Gamma^\alpha$ from $U$ into $H^{\frac{1}{2}}(\Gamma)$ by

$$H_\Gamma^\alpha u = \chi_\Gamma \gamma_0(\mathcal{E}_\alpha * \mathbf{B}u), \quad \forall u \in U,$$

where $*$ is the convolution operation.

In the next result, we suppose that the associate linear system to (1.1) (i.e., system (1.1) with $\mathcal{F} \equiv 0$) is approximately $\mathcal{B}$-controllable on $\Gamma$. Theorem 3.1 presents a direct result concerning the boundary controllability of system (1.1) on $\Gamma$.

**Theorem 3.1.** *If the subset $Im H_\Gamma^\alpha$ is closed and contains the element*

$$z_d - \chi_\Gamma \gamma_0 \mathcal{H}_\alpha(T) y_0 - \chi_\Gamma \gamma_0(\mathcal{E}_\alpha * \mathcal{F} y_{\tilde{u}})(T),$$

*then the exact $\mathcal{B}$-controllability on $\Gamma$ of system (1.1) at time $T$ is obtained by means of the control function*

$$\tilde{u}(t) = \mathrm{H}_\Gamma^{\alpha^\dagger}[z_d - \chi_\Gamma \gamma_0 \mathcal{H}_\alpha(t) y_0 - \chi_\Gamma \gamma_0(\mathcal{E}_\alpha * \mathcal{F} y_{\tilde{u}})(t)],$$

*where $H_\Gamma^{\alpha^\dagger}$ represents the pseudo inverse operator defined by $H_\Gamma^{\alpha^\dagger} := H_\Gamma^{\alpha^*}\left(H_\Gamma^\alpha H_\Gamma^{\alpha^*}\right)^{-1}$.*

*Proof.* The mild solution of (1.1) associated with the control $\tilde{u}$ is given by

$$y_{\tilde{u}}(t) = \mathcal{H}_\alpha(t)y_0 + (\mathcal{E}_\alpha * \mathcal{F}y_{\tilde{u}})(t) + (\mathcal{E}_\alpha * \mathbf{B}\tilde{u})(t), \tag{3.1}$$

which implies that

$$\chi_\Gamma \gamma_0 y_{\tilde{u}}(T) = \chi_\Gamma \gamma_0(\mathcal{H}_\alpha(T)y_0 + (\mathcal{E}_\alpha * \mathcal{F}y_{\tilde{u}})(T)) + H_\Gamma^\alpha \tilde{u}(T).$$

Therefore,

$$\begin{aligned}\chi_\Gamma \gamma_0 y_{\tilde{u}}(T) = {} & \chi_\Gamma \gamma_0(\mathcal{H}_\alpha(T)y_0 + (\mathcal{E}_\alpha * \mathcal{F}y_{\tilde{u}})(T)) \\ & + H_\Gamma^\alpha \mathrm{H}_\Gamma^{\alpha\dagger}[z_d - \chi_\Gamma \gamma_0(\mathcal{H}_\alpha(T)y_0 - (\mathcal{E}_\alpha * \mathcal{F}y_{\tilde{u}})(T))].\end{aligned}$$

Since $H_\Gamma^\alpha \mathrm{H}_\Gamma^{\alpha\dagger}$ is the orthogonal projection on $\mathrm{Im}H_\Gamma^\alpha$ and

$$[z_d - \chi_\Gamma \gamma_0(\mathcal{H}_\alpha(T)y_0 - (\mathcal{E}_\alpha * \mathcal{F}y_{\tilde{u}})(T))] \in \mathrm{Im}H_\Gamma^\alpha,$$

then $\chi_\Gamma \gamma_0 y_{\tilde{u}}(T) = z_d$. The proof is complete. □

Next, we prove the regional $\mathcal{B}$-controllability in the analytical setting by using the internal regional controllability result. More precisely, we formulate and establish some conditions for the regional $\mathcal{B}$-controllability on $\Gamma$ of system (1.1) with $y_0 = 0$. To do this, we first prove the connection between internal and boundary regional controllability, where we choose a suitable sub-region $\omega_c$. Secondly, we show that, under some assumptions, the internal regional controllability in $\omega_c$ is implied by approximate regional controllability of the corresponding linear system.

Let $\omega_c$ be a subregion of $\Omega$ such that $\Gamma \subseteq \partial\omega_c$. Proposition 3.2 presents a link between internal regional controllability and the boundary one.

**Proposition 3.2.** *The exact (resp. approximate) $\omega_c$-controllability of system* (1.1) *implies the exact (resp. approximate) regional $\mathcal{B}$-controllability on $\Gamma$.*

*Proof.* We define the operator $R$ from $H^{\frac{1}{2}}(\partial\omega_c)$ into $H^1(\omega_c)$ such that $\gamma_{0_{|\omega_c}} Rg = g$ for all $g \in H^{\frac{1}{2}}(\partial\omega_c)$. Let $z_d \in H^{\frac{1}{2}}(\Gamma)$. By using the trace theorem, there exists $R\overline{z}_d \in H^1(\omega_c)$ with a bounded support such that $\gamma_{0_{|\omega_c}} R\overline{z}_d = \overline{z}_d$. Then,

(i) If system (1.1) is exactly $\omega_c-$controllable, then, for any $d_s \in H^1(\omega_c)$, there exists $u \in U$ such that $\chi_{\omega_c} y_u(T) = d_s$. Since $R\overline{z}_d \in H^1(\omega_c)$, we get that

$$\exists u \in U \quad \text{such that} \quad \gamma_{0_{|\omega_c}}(\chi_\omega y_u(T)) = \overline{z}_d.$$

Thus, $\chi_{\Gamma_{|\partial\omega_c}} \gamma_{0_{|\omega_c}}(\chi_{\omega_c} y_u(T)) = z_d$ and $\chi_\Gamma \gamma_0(y_u(T)) = z_d$.

(ii) In the case where system (1.1) is approximately regionally controllable in $\omega_c$, for any $d_s \in H^1(\omega_c)$ and for all $\varepsilon > 0$, there exists a control $u \in U$ such that $||\chi_{\omega_c} y_u(T) - d_s||_{H^1(\omega_c)} \leq \varepsilon$. We have $R\overline{z}_d \in H^1(\omega_c)$ and

$$\forall \varepsilon > 0 \quad \exists u \in U \quad |||\chi_{\omega_c} y_u(T) - R\overline{z}_d||_{H^1(\omega_c)} \leq \varepsilon.$$

Moreover, by the continuity of $\gamma_0$ on $H^1(\omega_c)$, we have

$$||\gamma_{0_{|\omega_c}} \chi_{\omega_c} y_u(T) - \overline{z}_d||_{H^{\frac{1}{2}}(\partial\omega_c)} \leq \varepsilon.$$

Therefore,

$$||\chi_{\Gamma_{|\omega_c}} \gamma_{0_{|\omega_c}} \chi_{\omega_c} y_u(T) - z_d||_{H^{\frac{1}{2}}(\Gamma)} \leq \varepsilon,$$

and system (1.1) is approximate boundary regional controllable on $\Gamma$. □

To formulate and prove a result of regional controllability in a subregion $\omega_c$ such that $\Gamma \subseteq \partial\omega_c$, we make use of the following hypotheses:
$(H_1)$ The corresponding linear system of (1.1) is approximately $\omega_c-$controllable.
$(H_2)$ The nonlinear operator $\mathcal{F} : L^p([0,T], X^q) \longrightarrow L^p([0,T], X)$ satisfies

$$\begin{cases} \mathcal{F}(0) = 0, \\ ||\mathcal{F}z - \mathcal{F}y||_{L^p([0,T],X)} \leq F_N(||z||, ||y||)||z - y||_{L^p([0,T],X^q)}, \end{cases}$$

where $F_N : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is such that

$$\lim_{(\sigma_1,\sigma_2)\to(0,0)} F_N(\sigma_1, \sigma_2) = 0.$$

We introduce the following operators:

$$\begin{array}{rccc} \mathrm{H}^{\alpha}_{\omega_c} : & U & \longrightarrow & H^1(\omega_c) \\ & u & \longmapsto & \chi_{\omega_c}(\mathcal{E}_\alpha * \mathbf{B}u), \end{array}$$

and

$$\Psi(d_s, u) = \mathrm{H}^{\alpha^\dagger}_{\omega_c}(d_s - \chi_\omega(\mathcal{E}_\alpha * \mathcal{F}y_u)(T)) \quad d_s \in \mathrm{Im}(H^\alpha_{\omega_c}), u \in U,$$

where $H^{\alpha^\dagger}_{\omega_c}$ is the pseudo-inverse operator of $H^\alpha_{\omega_c}$.
The following lemmas are also needed.

**Lemma 3.3** (See [34]). *For any $d_s \in Im(H^\alpha_{\omega_c})$, the mapping*

$$||d_s||_{Im(H^\alpha_{\omega_c})} = ||H^{\alpha^\dagger}_{\omega_c} d_s||_U$$

*is a norm on $Im(H^\alpha_{\omega_c})$.*

**Lemma 3.4.** *If the control operator is bounded, then the control function $u$ satisfies the inequality*

$$||\mathcal{E}_\alpha * \mathbf{B}u||_{L^p([0,T],X^q)} \leq \mu ||u||_U, \qquad \mu > 0. \tag{3.2}$$

*Proof.* The result holds with $\mu = A_1 ||\mathbf{B}||_{\mathcal{L}(X,\mathcal{U})}$ by Young's inequality [2]. □

**Remark 3.5.** If $\mathbf{B}$ is unbounded, then we suppose that the inequality (3.2) holds.

**Lemma 3.6.** *Assume that hypothesis ($H_2$) holds. Then there exists $\kappa > 0$ and $m_\kappa > 0$ such that $f : u \longrightarrow y_u$ is a Lipschitz mapping from $\mathrm{B}(0, m_\kappa)$ to $\mathrm{B}(0,\kappa)$, where $\mathrm{B}(0,r)$ is the ball of center 0 and radius $r > 0$.*

*Proof.* First, we show that there exists $\kappa > 0$ such that

$$m_\kappa = \frac{\kappa}{\mu}(1 - A_1 \sup_{\theta \leq \kappa} F_N(\theta, 0)) > 0. \tag{3.3}$$

We have, by hypothesis ($H_2$), that

$$\lim_{(\sigma_1,\sigma_2)\rightarrow(0,0)} F_N(\sigma_1,\sigma_2) = 0.$$

Then $\exists\, \kappa > 0$ and $\exists\, \nu > 0$ such that

$$F_N(\sigma_1,\sigma_2) < \nu < \frac{1}{A_2 + A_1} \qquad \forall \sigma_1, \sigma_2 \leq \kappa, \tag{3.4}$$

where $A_2 > 0$. This gives

$$\sup_{\sigma_i \leq \kappa} F_N(\sigma_1,\sigma_2) \leq \nu < \frac{1}{A_2 + A_1}$$

with $A_1 \sup\limits_{\sigma_i \leq \kappa} F_N(\sigma_1,\sigma_2) < 1$. In particular, $m_\kappa > 0$. Let $u, v \in \mathrm{B}(0, m_\kappa)$. Then,

$$||y_u - y_v||_{L^p([0,T],X^q)} \leq ||\mathcal{E}_\alpha * (\mathcal{F}y_u - \mathcal{F}y_v)||_{L^p([0,T],X^q)} + ||\mathcal{E}_\alpha * \mathbf{B}(u - v)||_{L^p([0,T],X^q)}.$$

Using hypothesis ($H_2$) and Lemma 3.3, we obtain that

$$\begin{aligned} ||y_u - y_v||_{L^p([0,T],X^q)} \leq A_1 &\sup_{\sigma_i \leq \kappa} F_N(\sigma_1,\sigma_2) \\ &\times ||y_u - y_v||_{L^p([0,T],X^q)} + \mu ||u - v||_U. \end{aligned} \tag{3.5}$$

Then, $f$ is Lipschitz with the Lipschitz constant $\dfrac{\mu}{1 - A_1 \sup\limits_{\sigma_i \leq \kappa} F_N(\sigma_1,\sigma_2)}$. □

We prove that system (1.1) is $\omega_c$-controllable by studying the existence of a fixed point for operator $\Psi(d_s, \cdot)$.

**Theorem 3.7.** *If hypotheses* $(H_1)$ *and* $(H_2)$ *hold, together with*

$$||\chi_{\omega_c}\mathcal{E}_\alpha(\cdot)||_{\mathcal{L}(X,Im(H^\alpha_{\omega_c}))} = g_\alpha \in L^s([0,T]), \quad \frac{1}{p} + \frac{1}{s} = 1, \tag{3.6}$$

*then there exists* $\kappa > 0$, $m_\kappa > 0$, *and* $\rho_\kappa > 0$ *such that, for any element* $d_s$ *of* $\mathrm{B}(0,\rho_\kappa)$, *a subset of* $Im(H^\alpha_{\omega_c})$, *we can find a control* $\tilde{u}$ *in* $\mathrm{B}(0, m_\kappa)$ *steering system* (1.1) *from* $y_0$ *to* $d_s$ *at time* $T$ *in* $\omega_c$.

*Proof.* We show that there exists $\rho_\kappa > 0$ where, for all $d_s \in \mathrm{B}(0,\rho_\kappa)$, the operator $\Psi(d_s,\cdot)$ defined from $\mathrm{B}(0,m_\kappa)$ into $\mathrm{B}(0,m_\kappa)$ has a fixed point, which is then a control steering system (1.1) to $d_s$ in $\omega_c$ at time $T$. The proof uses the classical fixed point theorem of a contraction mapping.

(i) We show that $\Psi(d_s,\cdot)$ is a contraction mapping. Let us consider $d_s$ in $\mathrm{Im}(H^\alpha_{\omega_c})$ and $u, v$ in $\mathrm{B}(0,m_\kappa)$, where $m_\kappa$ is defined by (3.3). Then,

$$\begin{aligned} ||\Psi(d_s,u) - \Psi(d_s,v)||_U &= ||(\chi_{\omega_c}\mathcal{E}_\alpha * (\mathcal{F}y_u - \mathcal{F}y_v))(T)||_{\mathrm{Im}\,(H^\alpha_{\omega_c})} \\ &\le ||g_\alpha||_{L^s([0,T])}||\mathcal{F}y_u - \mathcal{F}y_v||_{L^p([0,T],X)} \\ &\le ||g_\alpha||_{L^s([0,T])} \sup_{(\sigma_i\le\kappa)} F_N(\sigma_1,\sigma_2) \times ||y_u - y_v||_{L^p([0,T],X^q)}. \end{aligned}$$

It is known from Lemma 3.6 that $u \longrightarrow y_u$ is a Lipschitz mapping from $\mathrm{B}(0,m_\kappa)$ into $\mathrm{B}(0,\kappa)$. Then, we obtain that

$$||\Psi(d_s,u) - \Psi(d_s,v)||_U \le \frac{\mu||g_\alpha||_{L^s([0,T])} \sup\limits_{(\sigma_i\le\kappa)} F_N(\sigma_1,\sigma_2)}{1 - A_1 \sup\limits_{(\sigma_i\le\kappa)} F_N(\sigma_1,\sigma_2)}||u-v||_U. \tag{3.7}$$

If we consider $\mu||g_\alpha||_{L^s([0,T])} := A_2$ in inequality (3.4), we have

$$(A_1 + A_2) \sup_{(\sigma_i\le\kappa)} F_N(\sigma_1,\sigma_2) < 1.$$

Consequently,

$$A_s := \frac{\mu||g_\alpha||_{L^s([0,T])} \sup\limits_{(\sigma_i\le\kappa)} F_N(\sigma_1,\sigma_2)}{1 - A_1 \sup\limits_{(\sigma_i\le\kappa)} F_N(\sigma_1,\sigma_2)} < 1,$$

and thus $\Psi(d_s,\cdot)$ is a strict contraction mapping.

(ii) We now show that $\Psi(d_s,\cdot)$ maps $\mathrm{B}(0,m_\kappa)$ into itself. Consider $u\in\mathrm{B}(0,m_\kappa)$. We have $y_u\in\mathrm{B}(0,\kappa)$ and

$$\begin{aligned}
||\Psi(d_s,u)||_U &= ||d_s-(\chi_{\omega_c}\mathcal{E}_\alpha*\mathcal{F}y_u)(T)||_{\mathrm{Im}\,(H^\alpha_{\omega_c})}\\
&\le ||d_s||_{\mathrm{Im}\,(H^\alpha_{\omega_c})}+||(\chi_{\omega_c}\mathcal{E}_\alpha*\mathcal{F}y_u)(T)||_{\mathrm{Im}\,()}\\
&\le ||d_s||_{\mathrm{Im}\,H^\alpha_{\omega_c}}+||g_\alpha||_{L^s([0,T])}\kappa\sup_{(\theta\le\kappa)}F_N(\theta,0).
\end{aligned}$$

Therefore, if

$$||d_s||_{\mathrm{Im}\,(H^\alpha_{\omega_c})}\le m_\kappa-||g_\alpha||_{L^s([0,T])}\kappa\sup_{(\theta\le\kappa)}F_N(\theta,0),$$

then $\Psi(d_s,u)\in\mathrm{B}(0,m_\kappa)$. Now, take

$$\rho_\kappa=\frac{\kappa}{\mu}(1-(A_1+A_2)\sup_{\theta\le\kappa}F_N(\theta,0)),$$

which is a positive constant. Then $\Psi(\mathrm{B}(0,\rho_\kappa),\mathrm{B}(0,m_\kappa))\subset\mathrm{B}(0,m_\kappa)$.

By the contraction mapping theorem, the existence of a fixed point of $\Psi(d_s,\cdot)$ is shown. The theorem is proved. □

By the contraction mapping theorem, we can also obtain the following result.

**Corollary 3.8.** *The sequence*

$$\begin{cases} u_0=0,\\ u_{n+1}=\mathrm{H}^{\alpha\dagger}_{\omega_c}(d_s-\chi_{\omega_c}(\mathcal{E}_\alpha*\mathcal{F}y_{u_n})(T)),\end{cases}\tag{3.8}$$

*converges to* $\tilde{u}$ *in* $\mathrm{B}(0,m_\kappa)$.

# 4 Numerical Approach and Examples

In this section, based on the theoretical results of Section 3, we present an algorithm that allows us to find, numerically, a control function that steers our system to a target state on $\Gamma$ at time $T$: see Algorithm 1.

We apply our Algorithm 1 in two examples of time fractional diffusion systems.

**Example 4.1.** Let $\Omega=]0,1[\times]0,1[$ and $T=3$. Consider the two-dimensional fractional system with diffusion described as follows:

$$\begin{cases}
{}^C D^{0.3}_{0^+}\theta(x,y,t)-\dfrac{\partial^2}{\partial x^2}\theta(x,y,t)-\dfrac{\partial^2}{\partial y^2}\theta(x,y,t)=\theta^2(x,y,t)+\chi_D u(t) & \text{in } \Omega\times]0,3]\\
\dfrac{\partial\theta}{\partial\nu_A}(\xi,\nu,t)=0 & \text{on } \partial\Omega\times]0,3]\\
\theta(x,y,0)=0 & \text{in } \Omega.
\end{cases}\tag{4.1}$$

**Algorithm 1**

**Initial Data:**

• $\alpha$, $T$,$\Gamma$, the desired state on $\Gamma$ $z_d$, and the location of the considered zonal or pointwise actuator;

• region $\omega_c$ where $\Gamma \subseteq \partial\omega_c$ and $d_s$ the extension of $z_d$ in $\omega_c$;

• error estimate $\varepsilon$.

Initial datum: $r_1 = d_s$.

**Repeat**

• Solve equation $u_n = H^{\alpha\dagger}_{\omega_c} r_n$.

• Solve the semilinear system controlled by $u_n$ and obtain $y_{u_n}(T)$.

• Compute $r_{n+1}$ by the formula

$$r_{n+1} = r_n + (d_s - \chi_{\omega_c} y_{u_n}(T)), \quad n \geq 2.$$

**Until**

$$||r_{n+1} - r_n||_{Im(\mathrm{H}^{\alpha}_{\omega_c})} \leq \varepsilon.$$

In this example, we consider a bounded control operator as a zonal actuator. We take $D = [0, 0.2] \times [0.2, 0.4]$ as the location of the actuator.

The desired state on $\Gamma = \{0\} \times [0, 0.1]$ is taken as $\theta_d(y) = 7y^3 - 13y^2 + 3$. Applying Algorithm 1 with $\varepsilon = 10^{-3}$, $\omega_c = [0, 0.3] \times [0, 0.1]$, and the extension of $z_d$ in $\omega_c$

$$d_s(x, y) = 10 \left( \frac{x^3}{46} - \frac{x^2}{62} + 0.1 \right) \left(7y^3 - 13y^2 + 3\right),$$

we obtain the results given in Figures 1 and 2. Figure 1 shows that the state in time $T$ is very close to $d_s$ in the subregion $\omega_c$. After the projection on $\Gamma$, Figure 2 shows that the target state is obtained with a reconstruction error $1.7 \times 10^{-4}$ on $\Gamma$ with cost $||\tilde{u}||^2_{L^2(0,T)} = 0.71$.

**Example 4.2.** Now we consider the same $\Omega$ as in Example 4.1, $T = 2$, and the following semilinear diffusion system with Caputo fractional derivative:

$$\begin{cases} {}^C D^{0.6}_{0^+} z(x,y,t) - \left( \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \right) z(x,y,t) \\ \qquad\qquad = z^2(x,y,t) + \delta_{\{b_1,b_2\}}(x,y) u(t) & \text{in } \Omega \times ]0,2]\,, \\ \dfrac{\partial z}{\partial \nu_A}(\xi, \nu, t) = 0 & \text{on } \partial\Omega \times ]0,2]\,, \\ z(x,y,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.2}$$

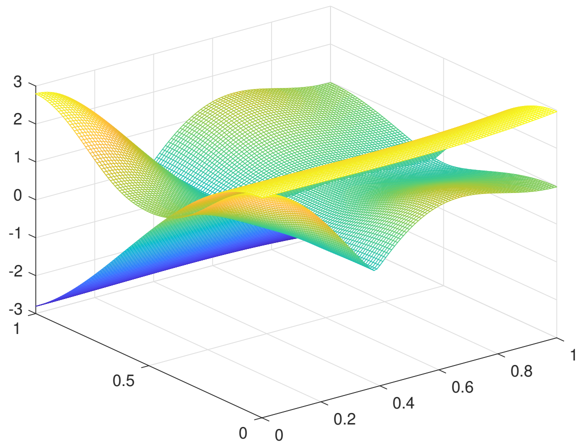


Figure 1: Reached state and the state $d_s$ in $\Omega$ for Example 4.1.

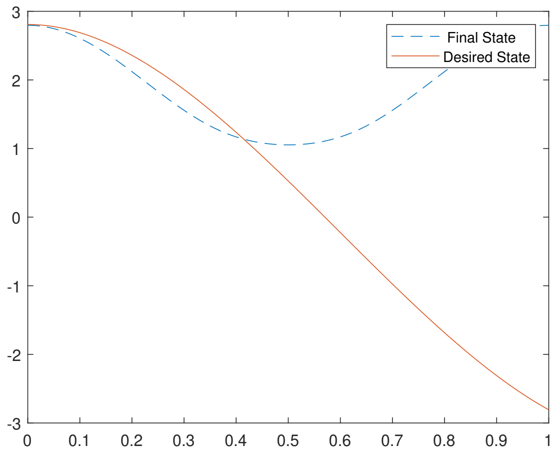


Figure 2: Desired state and reached one on $\Gamma$ for Example 4.1.

In this second example, the control operator $B$ is unbounded but it satisfies the condition (3.2). In this example, the desired state is $z_d(y) = 2\left(\frac{y^3}{42} - \frac{y^2}{1.3} + 0.1\right)$. Let us consider $\Gamma = \{0\} \times [0, 0.3]$, $\omega_c = [0, 0.2] \times [0, 0.3]$, $b_1 = 0.48$, $b_2 = 0.70$,

$$d_s(x, y) = 20\left(\frac{x^3}{4} - \frac{x^2}{65} + 0.1\right)\left(\frac{y^3}{42} - \frac{y^2}{1.3} + 0.1\right),$$

and $\varepsilon = 10^{-3}$. Using Algorithm 1 with our data, we obtain the results of Figures 3–5.

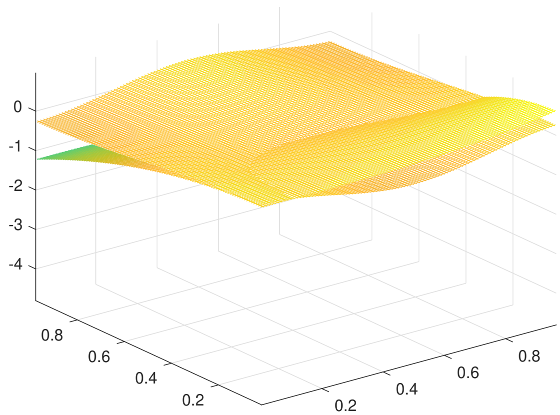


Figure 3: Reached state and the state $d_s$ in $\Omega$ for Example 4.2.

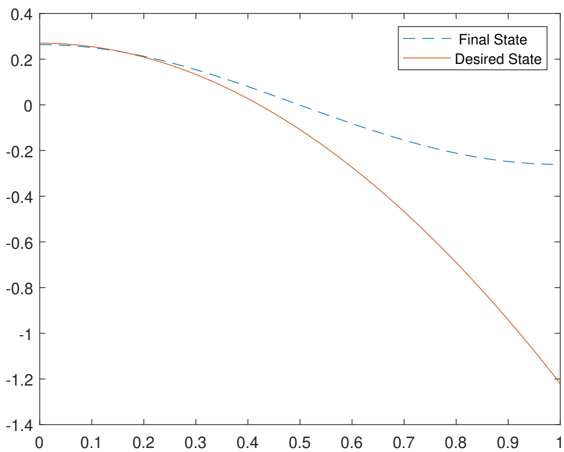


Figure 4: Desired state and reached one on $\Gamma$ for Example 4.2.

In Figure 3, we see that the final state at time $T$ is very close to the extension of the target state $d_s$ in $\omega_c$. From Figure 4, we can see that the reconstruction error is smaller than $10^{-3}$. Figure 5 presents the evolution of the control function $\tilde{u}(t)$ with

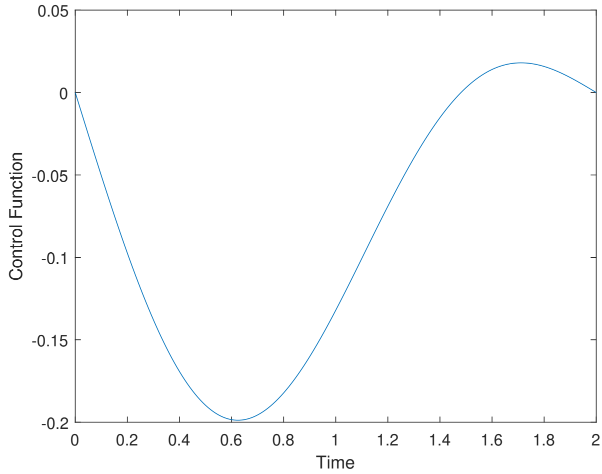


Figure 5: Evolution of the control function of Example 4.2.

the cost $||\tilde{u}||^2_{L^2(0,T)} = 2 \times 10^{-4}$.

# 5 Conclusion

We have proved sufficient conditions for the boundary regional controllability of semilinear Caputo fractional systems. More precisely, the boundary regional controllability of a class of semilinear Caputo systems is implied by the controllability in a suitable sub-region of the evolution domain. Further, the regional controllability problem is transformed into a fixed point problem of an appropriate nonlinear operator. Under some conditions in the nonlinear part of the system, we have guaranteed the existence of a fixed point of this operator and established the regional controllability of the considered system. Finally, we have presented two successful numerical simulations to illustrate our theoretical study.

For future work, it would be interesting to investigate the validity of the obtained results for other different fractional systems, e.g., Hadamard or Caputo–Fabrizio.

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