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To the Memory of my Professors
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ON CRITICAL POINTS OF NONDIFFERENTIABLE FUNCTIONS

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Let H be a real separable Hilbert space with the inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Assume two operators, in general nonlinear, $A, B : H \to H$ and a proper convex lower semicontinuous (l.s.c.) function $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ are given. We look for eigensolutions, i.e. eigenvalue - eigenfunction pairs $(\lambda, u) \in \mathbb{R} \times H$, $0 \neq u \in D(\Phi)$, for the variational inequality of the second order

$$(1) \qquad (Au - \lambda Bu, v - u) + \Phi(v) - \Phi(u) \geq 0 \quad \forall\, v \in D(\Phi),$$

where the constraints are determined by $D(\Phi) = \{v \in H \mid \Phi(v) < \infty\}$, the effective domain of Φ. According to Von Karman's nonlinear model of plates, the buckling of a thin elastic plate, subjected to unilateral conditions, can be stated in terms of eigenvalues of the above type [3].

The study of problem (1) is a branch of the eigenvalue problem theory for a pair (A, B) of nonlinear operators, i.e. $Au = \lambda Bu$ with a normalization condition on the eigenfunction u. A comprehensive investigation of the latter problem is performed when both A and B are potential operators. Thus, to establish the existence of infinitely many distinct eigensolutions for the equation

$$(2) \qquad F'(u) - \lambda G'(u) = 0 \quad \text{with} \quad G(u) = r,$$

where $F, G : H \to \mathbb{R}$ are Fréchet - differentiable and $r > 0$ is a real number, we may

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employ the Ljusternik - Schnirelman theory, which generalizes Courant's max - min principle to nonlinear operators, (see e.g. [10], Chap.44).

The main difficulty for the existence of eigensolutions for the inequality (1) is the fact that Φ is not necessarily differentiable. To overcome this drawback, its previous investigators J.Naumann - H.U.Wenk [7], R.C.Riddell [8], R.S.Kubrusly-J.T.Oden [5] considered the Fréchet - differentiable Moreau's function

$$\Phi_\varepsilon(v) = \min_{w \in H} \{ \frac{1}{2\varepsilon} \| v - w \|^2 + \Phi(w) \},$$

depending on $\varepsilon > 0$ and applied the general theory to the penalized problem

$$F'(u_\varepsilon) + \Phi'_\varepsilon(u_\varepsilon) = \lambda_\varepsilon G'(u_\varepsilon).$$

After that, using the subgradient inequality and pseudo - monotonicity of F', they obtained an eigensolution of the variational inequality (1) by letting $\varepsilon \to 0$. The passage to the limit requires the boundedness of the eigenvalue sequence $\{\lambda_\varepsilon\}$ as ε approaches zero. The estimates are carried out under various technical conditions.

On the other hand, when Φ is the indicator function of a closed convex cone K, M.Kučera - J.Nečas - J.Souček [6] studied the eigensolutions $(\lambda, u) \in \mathbb{R} \times K$ of the variational inequality of the first kind

$$(3) \qquad (F'(u), v - u) - \lambda (G'(u), v - u) \geq 0 \quad \forall v \in K,$$

with $G(u) = r$. This problem is motivated by the bending problem for a beam that is supported at its ends, and has several one-side supports in between. By means of a penalty function, (3) can be approximated by an eigenvalue problem for a variational equation, on which an adapted Ljusternik - Schnirelman approach can be applied.

In the present note, we introduce the critical points of subdifferential functions restricted to a regular manifold. This definition is related to the Lagrange multipliers. We prove the existence of eigensolutions for an inclusion, equivalent to the variational nonlinear eigenvalue inequality (1).

1. CRITICAL POINTS OF LIPSCHITZIAN FUNCTIONS.

Let $f : H \to \mathbb{R}$ be a locally Lipschitzian function. This means that to any point $x \in D(f)$ there correspond a neighborhood U and a constant $K \geq 0$ such that

$$|f(y) - f(z)| \leq K \|y - z\|$$

for all $y, z \in U$. We define the *generalized directional derivative* of f at x in the direction w as

$$f^{\circ}(x,w) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+tw) - f(y)}{t} \;,$$

where of course w is a vector in H and t is a positive scalar. The *generalized subdifferential* of f at x is the set

$$\partial f(x) = \{ h \in H \mid f^{\circ}(x,w) \geq (h,w) \text{ for all } w \in H \}.$$

These definitions require no convexity assumptions on f. It follows that $\partial(-f) = \partial f$ and $0 \in \partial f(x)$ assures a local minimum or maximum at x. Moreover, $f'(x) \in \partial f(x)$ if f admits further a Gâteaux or Fréchet derivative $f'(x)$.

Let us denote by $G : H \to \mathbb{R}$ a Fréchet differentiable function. For any $r \in \mathbb{R}$, consider the level surface

$$\delta G_r = \{ v \in H \mid G(v) = r \}$$

and suppose that δG_r is a regular manifold, i.e. $G'(v) \neq 0$ for each $v \in \delta G_r$.

Denote also by

$$T_w = T_w(\delta G_r) = \{ z \in H \mid (G'(w,z) = 0 \}$$

and

$$N_w = N_w(\delta G_r) = \{ y \in H \mid (y, z) \leq 0 \quad \forall\, z \in T_w \}$$

the tangent space and normal cone to δG_r at w, respectively.

We say that $u \in \delta G_r$ is a *critical point* of the locally Lipschitzian function $J : H \to \mathbb{R}$ restricted to δG_r if there is a subgradient $h \in J(u)$ such that $(h,z) = 0$ for all $z \in T_u(\partial G_r)$. We assume implicitly that $\partial J(u) \cap N_u(\delta G_r) \neq \emptyset$.

In this framework, our main tool will be the following variant of

THE LAGRANGE MULTIPLIER RULE. The constrained critical points of the locally Lipschitzian function J to δG_r are eigenfunctions of the subdifferential inclusion

$$0 \in \partial J(u) + \lambda G'(u). \tag{4}$$

Proof. To each $x \in D(G')$ let be assigned $w \in H$ such that $(G'(x), w) = 1$. Then every $y \in H$ can be uniquely represented as

$$y = (G'(x), y) w + z \quad \text{with} \quad z \in T_x(\delta G_r).$$

We choose $x = u$ where u is a critical point of J with respect to δG_r and so there is $h \in \partial J(u)$ such that $(h, z) = 0$ for all $z \in T_u(\delta G_r)$. Then

$$(h, y) = (G'(u), y)(h, w) + (h, z) = (h, w)(G'(u), y) \quad \forall\, y \in H.$$

Hence, $h + \lambda G'(u) = 0$ with $\lambda = -(h, w)$ and (4) follows.

We consider now the minimum problem with a side condition

$$\min_{u \in \delta G_r} J(u) = J(u_r), \tag{5}$$

for any real number r. If the problem (5) has a solution, then, in view of the generalized Lagrange multiplier rule, due to F.H.Clarke ([2],p.228), there exist real numbers $\mu \geq 0$ and ν, not all zero, such that

$$0 \in \mu\, \partial J(u_r) + \nu\, G'(u_r). \tag{6}$$

Since δG_r is regular, $\mu \neq 0$ and we set $-\frac{\nu}{\mu} G'(u_r) = g$. We derive that $g \in \partial J(u_r)$ and $(g, z) = 0$ for $z \in T_{u_r}(\delta G_r)$. Hence u_r is a critical point of $J_{|\delta G_r}$.

In particular, any solution of (6) with μ and ν, not both zero, is a critical point of J restricted to the regular manifold δG_r.

We say that a function $f : H \to \mathbb{R}$ is *coercive* if $f(x) \to \infty$ as $\|x\| \to \infty$, and an operator from H into itself is *weakly continuous* if it maps weakly convergent sequences into strongly convergent sequences. Finally, by " $\rightharpoonup$ " we denote the weak convergence.

As a pivot in the construction of the eigensolutions of (4), we may take

PROPOSITION. Suppose the following assumptions hold:

j) $J : H \to \mathbb{R}$ is locally Lipschtzian, weakly l.s.c., and coercive;

jj) $G : H \to \mathbb{R}$ is Fréchet differentiable, weakly continuous, and $G(0) = 0$;

jjj) $G'(u) = 0$, or $0 \in \partial J(u)$, implies $u = 0$.

Then, for each $r \in \mathbb{R}$ such that $D(J) \cap \delta G_r \neq \emptyset$, there exists a nontrivial eigensolution (λ_r, u_r) of (4).

Proof. By *jjj*), there is a $u \in H$ such that $G(u) = r \neq 0$. The level set δG_r is weakly sequentially closed, because $G(u_n) = r$ for $u_n \rightharpoonup u$ implies $G(u) = r$, due to *jj*). According to the Weierstrass theorem, J possesses a minimum $u_r \in \delta G_r$. Since $G(0) = 0$ and $G(u_r) = r$ we infer that $u_r \neq 0$ and $G'(u_r) \neq 0$. By the Lagrange multiplier rule, the equation (4) has an eigensolution (λ_r, u_r). Finally, $\lambda_r \neq 0$, by *jjj*).

2. A NONLINEAR VARIATIONAL EIGENVALUE INCLUSION

Recall that the subdifferential of the convex function $f : H \to \mathbb{R}$ at x is the set of all elements $h \in H$ satisfying the subgradient inequality

$$f(y) \geq f(x) + (h, y - x) \quad \text{for all } y \in H,$$

or, in other words,

$$\partial f(x) = \{ h \in H \mid f(\cdot) - (h, \cdot) \text{ has } x \text{ as a global minimum point} \}.$$

On the other hand, since any proper convex l.s.c. function in a Hilbert space is subdifferentiable on a dense subset of its domain (see e.g. [1],p.192), the nonlinear variational eigenvalue inequality (1) is 'almost' equivalent to the eigenvalue inclusion

$$F'(u) - \lambda G'(u) + \partial\Phi(u) \ni 0, \tag{7}$$

whenever F' and G' are Fréchet potentials.

We establish the existence of eigensolutions of (7) in

THEOREM. Assume the following hypotheses are fulfilled:

i) $\Phi: H \to [0,+\infty]$ is a convex, locally Lipschitzian function with $\Phi(0) = 0$;

ii) $F: H \to \mathbb{R} \cup \{+\infty\}$ is convex, l.s.c., Fréchet differentiable and coercive;

iii) $G: H \to \mathbb{R}$ is Frêchet differentiable, weakly continuous, and $G(0) = 0$;

iv) $G'(u) = 0$, or $0 \in F'(u) + \partial\Phi(u)$, implies $u = 0$.

Then, for each $r \in \mathbb{R}$ such that $D(\Phi + F) \cap \partial G_r \neq \emptyset$, there exists a nontrivial eigensolution (λ_r, u_r) of the eigenvalue inclusion (7).

Proof. The assumptions assure that the sum $\phi + F$ is weakly l.s.c. and coercive. Using the above proposition, the conclusion follows easily.

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