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REPRESENTATIONS OF DUAL ALGEBRAS AND OPERATORS OF CLASS C_0

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Dedicated to the memory of Professors Mendel Haimovici, Gheorghe Mihoc, Grigore C. Moisil, and Tiberiu Popoviciu

1. Introduction

Since S. Brown's seminal paper [1], the study of the weak*-continuous representations of dual algebras has led to significant progress in the study of Hilbert space operators (cf. e.g., [2]). The aim of this note is to emphasize some general features of the theory of these representations, and to connect them with the theory of contractions of class C_0. The plan of the paper is as follows. In Section 2 we introduce the concept of an elementary representation, and we prove a general realization theorem generalizing results from [3]. In Section 3 we discuss the representations of quotient algebras of H^∞, and their relation with the class C_0. We hope that this modest last part will inspire further research in this area.

2. Remarks on dual algebras

Let $\mathcal{A}$ be a complex Banach algebra such that there exists a subspace $\mathcal{A}_*$ of the Banach space dual $\mathcal{A}^*$ of $\mathcal{A}$ with the property that the unit ball of $\mathcal{A}$ is $\sigma(\mathcal{A}, \mathcal{A}_*)$-compact. Then $\mathcal{A}$ endowed with the weak topology $\sigma(\mathcal{A}, \mathcal{A}_*)$ is called a dual algebra, and that topology is called the w*-topology on $\mathcal{A}$. The duality between $\mathcal{A}$ and $\mathcal{A}_*$ will be denoted by $\langle u,x\rangle = x(u)$ for u in $\mathcal{A}$ and x in $\mathcal{A}_*$. We note that there are algebras $\mathcal{A}$ that have several distinct dual algebra structures, e.g., ℓ^1. However, several important algebras in operator theory have a unique dual algebra structure. This was proved for von Neuman algebras by Sakai [4], and for H^∞ by Ando [5]. In particular, the algebra $\mathcal{L}(\mathcal{H})$ of all bounded, linear operators on a

Hilbert space $\mathcal{H}$ has a unique w*-topology, also called the ultraweak topology. More precisely, $\mathcal{L}(\mathcal{H})_*$ can be identified with the Banach space $C_1(\mathcal{H})$ of all trace-class operators on $\mathcal{H}$, and the duality is given by $\langle T,A\rangle = tr(AT)$ for A in $C_1(\mathcal{H})$ and T in $\mathcal{L}(\mathcal{H})$. We will use below the obvious fact that for ξ, η in $\mathcal{H}$, the equality $\phi(T) = (T\xi|\eta)$ defines a w*-continuous functional on $\mathcal{L}(\mathcal{H})$.

A continuous representation $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of a dual algebra $\mathcal{A}$ is said to be w*-continuous if it is continuous from the w*-topology of $\mathcal{A}$ to the w*-topology of $\mathcal{L}(\mathcal{H})$.

2.1. DEFINITION. Let $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a w*-continuous representation of the dual algebra $\mathcal{A}$.

(i) Φ is said to be **elementary** if, given x in $\mathcal{A}_*$, we can find ξ and η in $\mathcal{H}$ satisfying the relation $\langle u,x\rangle = (\Phi(u)\xi|\eta)$ for u in $\mathcal{A}$. (Here $(\cdot|\cdot)$ denotes the scalar product on $\mathcal{H}$.)

(ii) Let n be a cardinal number, $n \leq \aleph_0$. Φ is said to be n-**elementary** if, given an array $(x_{ij})_{0\leq i,j<n} \subset \mathcal{A}_*$, we can find vectors ξ_i, η_i in $\mathcal{H}$, $0\leq i<n$, satisfying the relations $\langle u,x_{ij}\rangle = (\Phi(u)\xi_i|\eta_j)$ for a in $\mathcal{A}$ and $0\leq i,j<n$.

(iii) Φ is said to be **completely elementary** if it is elementary for every finite value of n.

It is obvious that Φ *is 1-elementary if and only if* Φ is elementary. It is also easy to see that for finite n, Φ is n-elementary if and only if the mapping $\Phi\otimes I : \mathcal{A}\otimes M_n \to \mathcal{L}(\mathcal{H})\otimes M_n$ is elementary, where M_n denotes the algebra of $n\times n$ complex matrices.

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a w*-closed subalgebra. Then $\mathcal{A}$ becomes a dual algebra when endowed with the relative w*-topology. This will be the only dual algebra structure that we consider on $\mathcal{A}$.

2.2. DEFINITION. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a w*-closed algebra. Then $\mathcal{A}$ is said to be **elementary** (resp., n-**elementary, completely elemetary**) if the inclusion map of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$ is elementary (resp., n-elementary, completely elementary).

The concepts introduced in Definition 2.2 were called properties $(\mathbb{A}_n)$ in [2]. However $(\mathbb{A}_1)$ appeared earlier, under the name (D_σ), in [6]. The terminology adopted here was proposed by Azoff [7].

2.3. LEMMA. Let Φ be a w*-continuous representation of the dual algebra $\mathcal{A}$.

(i) If Φ is elementary then Φ is one-to-one, bounded below, $\Phi(\mathcal{A})$ is w*-closed, and Φ is a w*-homeomorphism between $\mathcal{A}$ and $\Phi(\mathcal{A})$.

(ii) If Φ is n-elementary (resp., completely elementary) then $\Phi(\mathcal{A})$ is n-elementary (resp., completely elementary).

PROOF: Let $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, and let $\phi : C_1(\mathcal{H}) \to \mathcal{A}_*$ be the w*-dual of Φ. if Φ is elementary it follows that ϕ is onto (in fact every element in $\mathcal{A}_*$ is of the form $\phi(T)$ with a rank-one operator T). We have a factorization $\phi = \psi p$, where p: $C_1(\mathcal{H}) \to C_1(\mathcal{H})/\ker(\phi)$ denotes the canonical projection and ψ is an isomorphism. Accordingly, we have $\Phi = p^*\psi^*$, where p* is the inclusion $(\ker(\phi))^{\perp} \subset \mathcal{L}(\mathcal{H})$, and ψ^* is a norm and w*-isomorphism of $\mathcal{A}$ onto $\Phi(\mathcal{A})$. Now (i) follows at once, and (ii) is also an easy consequence of these considerations.

There are two geometric results from [2] that extend to this general context. These results were the starting point for the dilation theory developed in [2] and [3], and we hope they will prove useful even in the general setting. Therefore we include sketches of their proofs.

2.4. DEFINITION. Let $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $\Psi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ be two w*-continuous representations of the dual algebra $\mathcal{A}$. We say that Ψ can be realized in Φ if there exists a closed, densely defined, injective, linear map X: $\mathcal{D}(X)(\subset \mathcal{K}) \to \mathcal{H}$ such that $P\Phi(u)X \subset X\Psi(u)$, $u \in \mathcal{A}$, where P denotes the orthogonal projection of $\mathcal{H}$ onto the range of X.

2.5. LEMMA. Let $\mathcal{A}, \mathcal{H}, \mathcal{K}, \Phi, \Psi$, X, and P be as in Definition 2.4. Then the space $\mathcal{M} = P\mathcal{H}$ is semi-invariant for Φ, i.e., $\mathcal{M} = \mathcal{U} \ominus \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are invariant for Φ and $\mathcal{V} \subset \mathcal{U}$.

PROOF: By Lemma 0 of [8] it suffices to show that $P\Phi(uv) | \mathcal{M} = (P\Phi(u) | \mathcal{M})(P\Phi(v) | \mathcal{M})$ for all u,v in $\mathcal{A}$. Now, for η in $\mathcal{D}(X)$ we have

$$P\Phi(uv)X\eta = X\Psi(uv)\eta = X\Psi(u)\Psi(v)\eta = P\Phi(u)X\Psi(v)\eta = P\Phi(u)P\Phi(v)X\eta,$$

and the lemma follows because $\mathcal{M} = (X\mathcal{D}(X))^{-}$.

In the sequel it will always be assumed that the Hilbert spaces considered are separable. This will allow us to avoid nonessential technical difficulties.

2.6. THEOREM. Let Φ and Ψ be two w*-continuous representations of the dual algebra $\mathcal{A}$. If Φ is $\aleph_0$-elementary then Ψ can be realized in Φ.

PROOF: Let $\{\xi_i : i \in I\}$ (resp., $\{\eta_j : j \in J\}$) be a cyclic set of minimum cardinality for $\Psi(\mathcal{A})$ (resp., $\Psi(\mathcal{A})^*$). It is clear that I and J are at most countable. Define elements x_{ij} in $\mathcal{A}_*$ by $\langle u, x_{ij}\rangle = (\Psi(u)\xi_i | \eta_j)$ for $i \in I$, $j \in J$ and $u \in \mathcal{A}$. Note that x_{ij} exists because Ψ was assumed to be w*-continuous. Since Φ is $\aleph_0$-elementary, we can find vectors ξ'_i and η'_j such that $\langle u, x_{ij}\rangle = (\Phi(u)\xi'_i | \eta'_j)$ for $i \in I$, $j \in J$, and $u \in \mathcal{A}$. Denote by $\mathcal{U}$ (resp., $\mathcal{W}$) the closed linear manifold generated by $\{\Phi(u)\xi'_i : u \in \mathcal{A},\ i \in I\}$ (resp., $\{\Phi(u)^*\eta'_j : u \in \mathcal{A},\ j \in J\}$. Set then $\mathcal{V} = \mathcal{U} \cap \mathcal{W}^{\perp}$ and $\mathcal{M} = \mathcal{U} \ominus \mathcal{V}$. Finally, denote by P the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Let $\mathcal{L}$ be the closed linear span of the set $\{\Psi(u)\xi_i \oplus P\,\Phi(u)\xi'_i : u \in \mathcal{A},\ i \in I\}$ in $\mathcal{K} \oplus \mathcal{M}$. It can be shown that $\mathcal{L}$ is the graph of an operator X: $\mathcal{D}(X) \to \mathcal{M}$ satisfying the requirements of Definition 2.4. This is done by making use of the relations $(\Phi(u)\xi'_i | \eta'_j) = (\Psi(u)\xi_i | \eta_j)$, $u \in \mathcal{A}$, $i \in I$, $j \in J$. We only show that $\mathcal{L}$ is the graph of an injective operator or, equivalently, for $\nu \oplus \mu \in \mathcal{L}$, we have $\mu = 0$ if and only if $\nu = 0$. Indeed, if $\nu \oplus \mu \in \mathcal{L}$, there exist finitely supported families $\{u_i^{(n)} : i \in I\} \subset \mathcal{A}$ such that $\nu = \lim_{n\to\infty} \sum_{i \in I} \Psi(u_i^{(n)})\xi_i$ and $\mu = \lim_{n\to\infty} \sum_{i\in I} P\,\Phi(u_i^{(n)})\xi'_i$. Since the set $\{\Psi(v)^*\eta_j : v \in \mathcal{A},\ j \in J\}$ is total, we have $\nu = 0$ if and only if

$$0 = \lim_{n\to\infty} \sum_{i\in I} (\Psi(u_i^{(n)})\xi_i | \Psi(v)^*\eta_j) = \lim_{n\to\infty} \sum_{i\in I} (\Psi(vu_i^{(n)})\xi_i | \eta_j), \quad v \in \mathcal{A},\ j \in J.$$

Using the relations mentioned above, this happens if and only if

$$0 = \lim_{n\to\infty} \sum_{i\in I} (\Phi(vu_i^{(n)})\xi'_i | \eta'_j) = \lim_{n\to\infty} \sum_{i\in I} (\Phi(u_i^{(n)})\xi'_i | \Phi(v)^*\eta'_j)$$

$$= \lim_{n\to\infty} \sum_{i\in I} (P\,\Phi(u_i^{(n)})\xi'_i | \Phi(v)^*\eta'_j) = (\mu | \Phi(v)^*\eta'_j), \quad v \in \mathcal{A},\ j \in J,$$

i.e., if and only if $\mu \in \mathcal{W}^{\perp}$. Since $\mu \in \mathcal{M}$, this can happen if and only if $\mu = 0$.

In the previous proof we only used the fact that Φ is n-elementary, where $n = \max(\mathrm{card}(I), \mathrm{card}(J))$. Therefore we also have the following result.

2.7. THEOREM. Let Φ and Ψ be two w*-continuous representations of the dual algebra $\mathcal{A}$. If Φ is elementary, and both $\Psi(\mathcal{A})$ and $\Psi(\mathcal{A})^*$ have cyclic vector, then Ψ can be realized in Φ.

Let us also note that, if $\mathcal{A}$ is $\aleph_0$-elementary, we have more freedom in choosing the vectors ξ_i in the proof of Theorem 2.6. Since each ξ_i belongs to the domain of X, we can always assume that $\mathcal{D}(X)$ contains any a priori given countable subset of $\mathcal{K}$.

We conclude this section with several open problems.

2.8. PROBLEM. Let Φ be a completely elementary representation of a dual algebra. Is Φ necessarily $\aleph_0$-elementary?

We mention that Problem 2.8 has an affirmative answer for contractive w*-continuous representations of H^∞ (cf. [2] and [9]).

2.9. PROBLEM. Given Φ and Ψ as in theorem 2.6, under what conditions can X be chosen to be bounded, bounded below, or an isometry.

If $\mathcal{A} = H^\infty$, there are many examples in which X can be chosen to be an isometry (cf. [2] and [3]), but even in that case the complete answer to Problem 2.9 is not known yet.

3. Representations of quotients of H^∞

The questions discussed in Section 2 were extensively studied for isometric w*-continuous representations of H^∞. We propose here the study from this point of view of w*-representations of an algebra of the form H^∞/mH^∞, where $m \in H^\infty$ is an inner function, i.e., $|m(e^{it})| = 1$ for almost every $t \in [0,2\pi)$. This quotient algebra can be identified with the dual of $\bar{m}H^1_0/H^1_0$ under the duality $\langle u+mH^\infty, x+H^1_0\rangle = \langle u,x\rangle$ for $u \in H^\infty$ and $x \in \bar{m}H^1_0$. This is the dual algebra structure which we will consider in the sequel.

Note that H^∞/mH^∞ is different in several respects from H^∞. Unlike H^∞, H^∞/mH^∞ is not a sup-norm algebra unless it is one-dimensional. In addition, H^∞/mH^∞ is not semisimple. On the other side, while the representation theory of H^∞ is related with the theory of arbitrary absolutely continuous contractions on Hilbert space, the corresponding theory for the quotients of H^∞ is related with the special class C_0 of contractions. This latter class is much better understood, and therefore we hope that a fairly complete characterization of the elementary contractive representations of H^∞/mH^∞ can be achieved. In this section we provide some positive evidence in this direction.

We recall now some basic facts from [10] about contractions of class C_0. An operator T is of class C_0 if it is a completely nonunitary contraction and the ideal $J = \{u: u \in H^\infty, u(T) = 0\}$ contains nonzero functions. The ideal J is then a principal ideal in H^∞, and as such is generated by an inner function m called the minimal function of T. *Given an inner function m, there are operators of class* C_0 with minimal function m. The archetypal operator with this property is the so-called Jordan block S(m) defined as follows: $S(m) = PS|\mathcal{H}(m)$, where S denotes the unilateral shift on H^2, and P denotes the othogonal projection of H^2 onto the subspace $\mathcal{H}(m) = H^2 \ominus mH^2$.

There is a one-to-one correspondence between contractive w*-continuous representations of H^∞/mH^∞, and contractions T of class C_0 such that $m(T) = 0$. More precisely, given T with $m(T) = 0$, the formula $\Phi_T(u+mH^\infty) = u(T)$, $u \in H^\infty$, defines a contractive, w*-continuous representation of H^∞/mH^∞. Moreover, every such *representation* Φ *coincides with* Φ_T, *where* $T = \Phi(\chi + mH^\infty)$, *and* $\chi(\lambda) = \lambda$ for λ in the unit disc.

We will denote by $\mathcal{A}_T$ the w*-closed algebra generated by an operator T. Thus, for an operator T of class C_0 with minimal function m, we can ask whether the representation Φ_T or the algebra $\mathcal{A}_T$ has one of the properties in Definitions 2.1. and 2.2. These properties for Φ_T and $\mathcal{A}_T$ are related by the following easy consequence of Lemma 2.3.

3.1. PROPOSITION. Let T be an operator of class C_0 with minimal function m, and *let n be a cardinal number. If* Φ_T *is n-elementary then* $\mathcal{A}_T$ is n-elementary, and

(3.2) $\mathcal{A}_T = \{u(T): u \in H^\infty\}$.

PROOF: It suffices to apply Lemma 2.3 and to remark that $\mathcal{A}_T$ is the w*-closure of the range of Φ_T.

It is worthwhile to mention that relation (3.2) does not apply to all operators of class C_0. To describe the general situation we recall the definition of the meromorphic functional calculus (cf. [10]). Let T be a completely nonunitary contraction on $\mathcal{H}$, $u,v \in H^\infty$, and $X \in \mathcal{L}(\mathcal{H})$. Suppose that v(T) is a quasiaffinity, i.e., *it is one-to-one and has dense range.* Then we have $X = (u/v)(T)$ if and only if

$v(T)X = u(T)$. Let us also recall from [10] that, if T is a contraction of class C_0 with minimal function m, v(T) is a quasiaffinity if and only if v and m are relatively prime.

3.3. PROPOSITION. For an operator T of class C_0 we have $\mathcal{A}_T = \{T\}''$ and, moreover, $\mathcal{A}_T$ consists of all bounded operators X of the form (u/v)(T), where v and the minimal function of T are relatively prime.

PROOF: Denote by $\mathcal{W}_T$ the weakly closed algebra generated by T. It was proved in [11] that $\mathcal{W}_T = \{T\}''$ consists of all meromorphic functions of T. It remains to be proved that $\mathcal{A}_T = \mathcal{W}_T$. For each operator X denote by X° the orthogonal sum of infinitely many copies of X. Then T° is also of class C_0, and clearly $\mathcal{W}_{T°} = \mathcal{A}_{T°} = \{X°: X \in \mathcal{A}_T\}$ because $X \to X°$ is a w*-homeomorphism, and the w*- and weak topologies coincide on $\{X°: X \in \mathcal{L}(\mathcal{H})\}$. On the other side, $\mathcal{W}_{T°} = \{T°\}'' = \{X°: X \in \{T\}''\} = \{X°: X \in \mathcal{W}_T\}$, and the equality $\mathcal{A}_T = \mathcal{W}_T$ follows at once.

The following result was proved by Sarason in [12] (cf. also [13] for another proof).

3.4. PROPOSITION. Let m be an inner function, and T = S(m). Then Φ_T (and hence $\mathcal{A}_T$) is elementary.

3.5. PROPOSITION. Let T_1 and T_2 be two operators such that $\mathcal{A}_{T_1}$ and $\mathcal{A}_{T_2}$ are n-elementary. Then $\mathcal{A}_{T_1 \oplus T_2}$ is also n-elementary.

PROOF: Suppose that $T_j \in \mathcal{L}(\mathcal{H}_j)$, and let ϕ be a w*-continuous functional on $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Since $\mathcal{A}_{T_j}$ is elementary, we can find x_j and y_j in $\mathcal{H}_j$ such that $(p(T_1)x_1 | y_1) = \phi(p(T_1) \oplus 0)$ and $(p(T_2)x_2 | y_2) = \phi(0 \oplus p(T_2))$ for all polynomials p. Then, with $x = x_1 \oplus x_2$ and $y = y_1 \oplus y_2$, we have $(p(T_1 \oplus T_2)x | y) = \phi(p(T_1 \oplus T_2))$. This shows that $\mathcal{A}_{T_1 \oplus T_2}$ is elementary. The proof of n-elementarity is left to the interested reader.

3.6. PROPOSITION. Let $T = S(m') \oplus S(m'')$ where m' and m" are two relatively prime inner function, and set $m = m'm''$. Then $\mathcal{A}_T$ is elementary, but Φ_T is elementary if and only if $m'H^\infty + m''H^\infty = H^\infty$.

PROOF: The fact that $\mathcal{A}_T$ is elementary follows immediately from Propositions 3.4 and 3.5. Assume that Φ_T is elementary. Then by Proposition 3.1 there is u in H^∞ such that $I \oplus 0 = u(S(m')) \oplus u(S(m''))$, i.e., $u(S(m')) = I$ and $u(S(m'')) = 0$. Equivalently,

$u-1 \in m'H^{\infty}$ and $u \in m''H^{\infty}$ so that $1 = u-(u-1) \in m'H^{\infty} + m''H^{\infty}$. Conversely, assume that $m'H^{\infty} + m''H^{\infty} = H^{\infty}$. Then we can prove that S(m) and T are similar, and hence Φ_T is elementary by Proposition 3.4. To show that S(m) and T are similar we choose g' and g'' in H^{∞} such that $g'm'+g''m'' = 1$, and set $P = (g''m'')(S(m))$. Since $g''m''-(g''m'')^2 = g''m''(1-g''m'') = g'g''m'm''$, we deduce that P is a projection, and hence S(m) is similar to $(S(m)|\text{ran } P) \oplus (S(m)|\ker P)$. It is an easy exercise to check that $S(m)|\ker P$ and $S(m)|\text{ran } P$ are unitariliy equivalent to S(m'') and S(m'), respectively. The proposition is proved.

In the previous proof we used the fact from [20] that $I \oplus 0$ belongs to the algebra $\mathcal{A}_{S(m')\oplus S(m'')}$ if m' and m'' are relatively prime.

We see from the preceding result that $\mathcal{A}_T$ can be elementary even when Φ_T is not elementary. It is interesting to note that there do exist operators T of class C_0 with a cyclic vector, such that m_T is a Blaschke product and $\mathcal{A}_T$ is not elementary. This was proved by Cassier in [14].

For the next result we need to recall a classification result from [15]. Recall firs that the operators T and T' are said to be quasisimilar if there exist quasi-affinities X and Y such that TX = XT' and T'Y = YT. A Jordan operator is an operator of the form $\bigoplus_{j=0}^{\infty} S(m_j)$, where m_j are inner and m_{j+1} divides m_j for all j. For every operator T of class C_0 (on a separable Hilbert space) there exists a unique Jordan operator $\bigoplus_{j=0}^{\infty} S(m_j)$ that is quasisimilar with T. The function m_0 is the minimal function of T.

3.7. THEOREM. Let T be an operator of class C_0 with Jordan model $\bigoplus_{j=0}^{\infty} S(m_j)$. If $\mathcal{A}_T$ is n-elementary then $m_j = m_0$ for all $j < n$.

PROOF: Assume that T acts on $\mathcal{H}$. By results in [16], there exists a cyclic subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $T|\mathcal{H}_0$ has minimal function m_0. Denote by $\mathcal{K}$ the direct sum of n copies of $\mathcal{H}_0$, and let $\Psi: \mathcal{A}_T \to \mathcal{L}(\mathcal{K})$ be defined by $\Psi(A) = (A|\mathcal{H}_0) \oplus (A|\mathcal{H}_0) \oplus \dots$ (n summands). Then $\Psi(\mathcal{A}_T)$ has a cyclic set consisting of n vectors and, by results in [16], $\Psi(\mathcal{A}_T)^*$ also has a cyclic set consisting of n vectors. Let Φ denote the inclusion map of $\mathcal{A}_T$ into $\mathcal{L}(\mathcal{H})$. By the remark following the proof of Theorem 2.6, Ψ can be realized in Φ. If X is the realization operator, we have $(PT|\mathcal{M})X = XT'$, where $T' = (T|\mathcal{H}_0) \oplus (T|\mathcal{H}_0) \oplus \dots$ (n summands), and $\mathcal{M} = (X\mathcal{K})^{-}$. Observe at this point that the Jordan model of T' is $S(m_0) \oplus S(m_0) \oplus \dots$ (n summands). Denote by T'' the restriction of $T' \oplus (PT|\mathcal{M})$ to the graph of X. Then T'' is of class C_0 and it is a quasiaffine transform (cf. [10]) of both $PT|\mathcal{M}$ and T' (the quasiaffinities being just the projections onto the two components). By results of [17] it follows that T', T", and $PT|\mathcal{M}$ all have Jordan model $S(m_0) \oplus S(m_0) \oplus \ldots$ (n summands). Since the Jordan model of $PT|\mathcal{M}$ is a compression of the Jordan model of T, the conclusion of the theorem follows at once.

3.8. COROLLARY. Let T be an operator of class C_0. If $\mathcal{A}_T$ is 2-elementary then T is reflexive.

PROOF: This follows immediately from [18] since $m_0/m_1 = 1$.

Let us note that Theorem 3.7. and Corollary 3.8 extend results from [19] concerning the case of algebraic operators. The paper [19] also contains other interesting results about algebraic operators.

Problems 2.8 and 2.9 remain open even in the particular context described in this section.

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