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ON THE PRIMARY REGULATION PROBLEM FOR LINEAR SYSTEMS IN HILBERT SPACES

R. Rabah, D. Ionescu

Abstract. The primary regulation problem (PRP) for linear systems with bounded operators on Hilbert spaces is examined. Some aspects concerning the invariant subspaces for bounded operators on Hilbert spaces are considered and used for the obtaining of the necessary and sufficient conditions of the primary regulation problem. A generalization of the stabilization problem is also given.

1. Introduction

In this paper we consider the primary problem for the regulation of the following linear system:

(1a) $\dot{x} = Ax + Bu$

(1b) $y = Cx$

where $x(t) \varepsilon \mathcal{X}$, $u(t) \varepsilon \mathcal{U}$, $y(t) \varepsilon \mathcal{Y}$; $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, A, B, C being linear and continuous: $A \varepsilon L(\mathcal{X})$, $B \varepsilon L(\mathcal{U},\mathcal{X})$, $C \varepsilon L(\mathcal{X},\mathcal{Y})$.

Generally speaking the regulation problem will involve the determination of the following system

(2a) $u(t) = F\, w(t) + K\, y(t)$

(2b) $\dot{w}(t) = L\, w(t) + G\, y(t)$

where $w(t) \varepsilon \mathcal{W}$ also a Hilbert space, and $F \varepsilon L(\mathcal{W}, \mathcal{U})$, $K \varepsilon L(\mathcal{Y}, \mathcal{U})$, $L \varepsilon L(\mathcal{W})$, $G \varepsilon L(\mathcal{Y}, \mathcal{W})$ are linear and continuous operators, in order to stabilize the output.

This problem was approached by numerous authors. In the case of the finite dimensional systems an exhaustive study is done by Wonham [11]. For infinite dimensional systems the problem of finding the compensator was the aim of the works of Jacobson [4], Schumacher [9], Lagemann [5], Curtain [2]. In these works, they considered the following aspect of the above mentioned problem; the operator A is unbounded and the stabilization of the system is made by a finite dimensional system, i.e., $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$ are finite dimensional spaces and F, K, L, G are matrices.

The case when $u(t) = F\, x(t)$ will lead to the stabilization problem. The cornerstone in the problem is to find an operator $F \varepsilon L(\mathcal{X}, \mathcal{U})$ carrying out a feedback such as the system $\dot{x} = (A + BF)x$ is stable in certain sense (exponentially, strongly, weakly). In case that this achievement is not possible, as well as in case when the requirements of the state stability is too strong, one try to stabilize only the output $y(t)$, which can be in particular a projection of the system space on a given subspace. Furthermore, the study of the output feedback stabilization will allow further construction for a dynamic compensator.

2. Definitions and Preliminaries

For easy reference, we shall state here some results on invariant subspaces:

Definition 1: The subspace $\mathcal{V}$ is a called (A,B) invariant if there is an operator $F \in L(\mathcal{X}, \mathcal{U})$ such that $\mathcal{V}$ is $(A + BF)$-invariant.

There are also some other invariance notions, which, for the finite dimensional spaces are equivalent, and different for infinite dimensional Hilbert spaces. (see Pandolfi [6] and Schmidt [8] for more details).

Lemma 1: Suppose that $\mathcal{V}$ is (A, B)-invariant. Then there is an operator $F \in L(\mathcal{X}, \mathcal{U})$ such that $\mathcal{V}$ is $\exp\big((A = BF)t\big)$-invariant for all $t \geqslant 0$.

This result is a trivial consequence of definition 2 and of the exponential properties.

Let us note $\mathcal{N}(A,B) = \bigcap_{t \geqslant 0} \ker B^*\exp(A^*t)$ and $\mathcal{L}(A,B) = \mathcal{N}^{\perp}(A,B)$. It follows immediately that $\mathcal{L}(A,B)$ is the minimal closed subspace containing ImB and being A-invariant. Similarly $\mathcal{N}(A,B)$ is the maximal closed subspace containing KerD* and being A*-invariant.

Lemma 2: The subspace $\mathcal{N}(A,B)$ is $(A^* + F^*B^*)$-invariant for all $F \in L(\mathcal{X}, \mathcal{U})$ and $\mathcal{L}(A,B)$ is (A + BF) invariant.

Proof: If $x \in \mathcal{N}(A,B)$, then $B^*A^{*i}x = 0$ for every $i \in N$. Thus $B^*A^{*i}(A^*+F^*B^*)n = B^*A^{*i+2}n = 0$ and $B^*x = 0$.

Let us consider the next subspace:

$$\mathcal{X}_w^-(A) = \left\{x \,\middle|\, \exp(At)x \xrightarrow{w} 0 \text{ when } + \to \infty\right\}$$

It can easily be proved that $\mathcal{X}_{w}^{-}$ is a closed subspace and that it is also A-invariant.

Definition 2: The system (1a) is called exponential stabilizable (respectively stabilizable, weak stabilizable) if there is a $F \varepsilon L(\mathcal{X},\mathcal{U})$ such that $\|e^{(A+BF)t}xu\| \leqslant Ne^{-\varepsilon t}\|x\|$. For some $N, \varepsilon > 0$, and for every $x \varepsilon \mathcal{X}$ (respectively $e^{At}x \xrightarrow{s} 0$, $e^{At}x \xrightarrow{w} 0$ for every $x \varepsilon \mathcal{X}$).

We will give hereafter a result on the stabilization and the central idea of the proof, which will be used further on for the primary regulation problem.

For the result formulation we will decompose the state space as follows: $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ where $\mathcal{X}_1 = \mathcal{L}(A,B)$ and $\mathcal{X}_2 = \mathcal{L}^{\perp}(A,B)$. Let P_i be the orthogonal projector on $\mathcal{X}_i$, A_{ij} the restriction of P_iAP_j to $\mathcal{X}_j$ and $P_iB = B_i$ with $i,j = 1,2$. Setting $x_i = P_ix$ we can write now the system (1a) as follows:

$$(3) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{22} & x_1 \\ 0 & A_{22} & x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

We used here the following $A_{21} = 0$ and $B_2 = 0$ because $\mathcal{X}_i$ is A-invariant and $\mathrm{Im}B \subset \mathcal{X}_i$.

Proposition 1. [7]

The system (1a) is exponentially stabilizable iff

i) the pair (A_{11}, B_1) is exponentially stabilizable.

ii) the operator A_{22} is exponentially stable.

Proof: Let us take $F \in L(\mathcal{X}, \mathcal{U})$. The decomposition stated above allows us to write F such as (F_1, F_2), where $F_1 \in L(\mathcal{X}_1, \mathcal{U})$ and $F_2 \in L(\mathcal{X}_2, \mathcal{U})$. The operator $(A + BF)$ can be written as:

$$A + BF = \begin{pmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 \\ 0 & A_{22} \end{pmatrix} \tag{4}$$

If F is an exponential stabilization operator of the system (A,B), the operators $(A_{11} + B_1F_1)$ and A_{22} will be exponentially stable. Thus (A_{11}, B_1) is exponentially stabilizable by F_1 and A_{22} is exponentially stable. Assume now that the conditions i) and ii) are satisfied and F_1 is the operator which is stabilizing (A_{11}, B_1). The operator $\begin{pmatrix} A_{11}+B_1F_1 & A_{12} \\ 0 & A_{22} \end{pmatrix}$ will be exponentially stable (cf. [7] and [9]). Thus the operator $(F_1, 0)$ stabilizes exponentially the system (A,B), See [10] for another approach.

Definition 3. We will say that the primary exponential regulation (respectively strong, weak) admits a solution if there is $F \in L(\mathcal{X}, \mathcal{U})$ such that $Ce^{(A+BF)t}x_0 \to 0$ exponentially (resp. strongly, weakly) when $t \to \infty$.

Remark. F is a solution of the exponential PRP iff $\|Ce^{(A+BF)t}x_0\| \leqslant Ne^{-\varepsilon t}\|x_0\|$ for certain N, ε positive and for all $x_0 \in X$.

Lemma 3. The weak PRP admits a solution F iff $\operatorname{Im} C^* \subset \overline{\mathcal{X}}_w(A^*+F^*B^*)$.

Proof: F is a solution of the weak PRP iff $< C\, e^{(A+BF)t} x,y > \to 0$ when $f \to \infty$ for every $(x,y)\ \varepsilon\ \mathcal{X} \times \mathcal{X}$. This is equivalent with: $< x, e^{(A^*+F^*B^*)t} C^*y > \to 0$ for $t \to \infty$ and thus equivalent with $\mathrm{Im}C^* \subset \mathcal{X}_w^-(A^*+F^*B^*)$.

Lemma 4: Let it be W a closed subspace, (A,B) maximal invariant and contained in KerC. In order that F_0 be a solution of the weak PRP. It is necessary that:

$$(5) \quad W^{\perp} \subset \mathcal{X}_w^-((A^*+F_0^*B^*)$$

By the maximality of W we understand that W contains all the (A,B) invariant subspaces.

Proof: If F_0 is a solution of the weak PRP, the Lemma 3 gives $\mathrm{Im}C^* \subset \mathcal{X}_w^-(A^*+F_0^*B^*)$. Then $\mathcal{M} = (\mathcal{X}_w^-(A^*+F_0^*B^*)^{\perp} \subset \mathrm{Ker}C$. But $\mathcal{M}$ is (A+BF)-invariant, i.e., it is (A,B)-invariant and consequently $\mathcal{M} \subset W$ and (5) follows.

Remark. It is obvious that the condition (5) is also necessary for the strong and exponential PRP.

Lemma 5: Let W be a closed subspace (A,B) invariant and contained in KerC and satisfying (5). Then F_0 is a solution of the weak PRP.

Proof: Because $W \subset \mathrm{Ker}C$, we have $\mathrm{Im}C^* \subset W^{\perp}$ and then $\mathrm{Im}C^* \subset \mathcal{X}_w^-(A^*+F_0^*B^*)$. The Lemma 3 is giving then the result.

Remark: If W is the maximal (A,B) invariant closed subspace contained in KerC, then the condition (5) is necessary and sufficient in order that F_0 be a solution of the weak PRP.

In the case of finite dimensional systems there is always such a subspace and the condition (5) can be written as $\mathcal{X}^+(A+BF_0) \subset W$ (cf[11], [3]).

3. Main Results

We will establish some extensions of the Proposition 1 concerning the stabilization.

Theorem 1. In order that (A,B) be strongly (respectively weakly) stabilizable it is necessary that the following conditions be satisfied:

i) (A_{11},B_1) is strongly stabilizable (resp. weakly)

ii) A_{11} is exponentially stable. (resp. weakly)

Moreover if one of the following two conditions is verified:

iii) (A_{11},B_1) is exponentially stabilizable.

iv) A_{11} is exponentially stable.

then the system (A,B) is strongly (resp. weakly) stabilizable.

Proof: Let F be the operator stabilizing (A,B). Using the decomposition (4) one obtains that for every initial condition $x_0=\mathrm{Col}(x_0^1;x_0^2)$ the state of the system $x(t) = \mathrm{Col}(x_1(t),x_2(t))$ will be given by the following relations:

$$x_1(t)=e^{(A_{11}+B_1F_1)t}x_0^1 + \int_0^t e^{(A_{11}+B_1F_1)t}(A_{12}+B_1F_2)x_2(\tau)d\tau$$

$$x_2(t) = e^{A_{22}t}x_0^2$$

We will have then $x_2(t) \to 0$ for $t \to \infty$ (resp. $x_2(t) \xrightarrow{w} 0$) which is implying ii). Furthermore putting $x_0^2 = 0$, we get $\exp((A_{11}+B_1F_1)t)x_0^1 \to 0$ (resp. weakly) for every $x_0^1 \in X_1$ which means that (A_{11},B_1) is stabilizable by F_1. In the second assertion, we will establish the result for the strong stabilization; the weak stabilization will follow by applying the same scheme in its demonstration.

Assume now that A_{22} is exponentially stable and that (A_{11},B_1) is strongly stabilizable by F_1. Let us put again $F = (F_1,0)$. We will have then $\|x_2(t)\| \leqslant N e^{-\alpha t}$ for positive α and N. The condition i) gives $\exp\big((A_{11}+B_1F_1)t\big)x_0^1 \to 0$ when $t\to\infty$ for every $x_0^1 \in X$. It remains to evaluate the term

$$\int_0^t \exp\big((A_{11} + B_1F_1)(t-\tau)\big)A_{12}x_2(\tau)d\tau.$$

Let us note $S_1(t) = \exp\big((A_{11} + B_1F_1)t\big)$. We will have then

$$\int_0^t S_1(t-\tau)A_{12}x_2(\tau)d\tau = S_1(t-T)\int_0^T S_1(T-\tau)A_{12}x_2(\tau)d\tau$$

$$+ \int_T^t S_1(t-\tau)A_{12}x_2(\tau)d\tau$$

Let's consider first the second term. It can easily be seen that $\|S_1(t-\tau)\| \leqslant M_1$, where M_1 is a constant because, $S_1(t-\tau)x^1 \to 0$ when $t \to \infty$. Thus $\|\int_T^\infty S_1(t-\tau)A_{12}x_2(\tau)d\tau\| \leqslant M_2\int_T^\infty \|x_2(\tau)\|d\tau < \frac{\varepsilon}{2}$ for an arbitrary ε and $T > T_0(\varepsilon)$, where M_2 is another constant.

Furthermore, the first term can be written as $S_1(t-T)x_T$ and for $t > T_1(\varepsilon)$ one obtains $\|S_1(t-\tau)x_T\| < \frac{\varepsilon}{2}$. Finally, for all $\varepsilon > 0$, one can find $T_1(\varepsilon)$ such that for $t > T_1(\varepsilon)$ we have $\|\int_0^t S_1(t-\tau)A_{12}x_2(\tau)d\tau\| < \varepsilon$. Thus the second term from the expression of $x_1(t)$ tends toward 0 when $t\to\infty$. The approach is analogous if we will suppose that (A_{11},B_1) is exponentially stabilizable and A_{22} strongly stable.

Remark. If $A_{12} = 0$ (X_1 and X_2 are A-invariant; this is the case when A is autoadjunct) then the conditions i) and ii) are necessary and sufficient for the stabilization (see [7],[1]).

In finite dimension spaces, the condition i) is automatically verified because the couple (A_{11},B_1) is controllable.

Theorem 2. Assume that there is W, a maximal (A,B) invariant closed subspace contained in KerC. Then a necessary condition for the existence of PRP solution is that:

$$(6) \quad \mathcal{W}^\perp \cap \mathcal{N}(A,B) \subset \mathcal{X}^-_w(A^*)$$

Proof: For the sake of the notation we will note $\mathcal{N}(A,B) \overset{\Delta}{=} \mathcal{N}$. Assume that F_0 is a PRP solution (exponential, strong or weak). Applying the Lemma 4 we have $\mathcal{W}^\perp \subset \mathcal{X}^-_w(A^*+F_0^*B^*)$. Thus $\mathcal{W}^\perp \cap \mathcal{N} \subset \mathcal{X}^-_w(A^*+F_0^*B^*)$. Lets set $\mathcal{X}_2 = \mathcal{W}^\perp \cap \mathcal{N}$ and $\mathcal{X}_1 = \mathcal{X}_2^\perp$. Let's prove that $\mathcal{X}_2$ is $(A^*+F^*B^*)$-invariant for every $F \in L(\mathcal{X},\mathcal{U})$.

Actually let be $F_w \in L(\mathcal{X},\mathcal{U})$ such that $\mathcal{W}$ is (A+BF)-invariant (Definition 1). Therefore, for $x \in \mathcal{X}_2$ we have $(A^*+F^*B^*)x = A^*x$ and because $\mathcal{N}$ is A^*-invariant, we will have $A^*x \in \mathcal{N}$. Furthermore, for all $y \in \mathcal{W}$ it will follow that:

$$<y,(A^*+F^*B^*)x> \; = \; <y,A^*x> = <y,(A^*+F^*B^*)x> = <(A+F_1B)y,x> = 0$$

because $(A+F_1B)y \in \mathcal{W}$. Thus $(A^*+F^*B^*)x \in \mathcal{W}^\perp$.

In particular, this result gives the invariance of $\mathcal{X}_2$ with respect of A^* and those of $\mathcal{X}_1$ with respect of A.

In this way the decomposition of the space $\mathcal{X}$ in $\mathcal{X}_1 + \mathcal{X}_2$ is achieved. This will lead to the decomposition of the operators A,B and F_0 as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{12} \end{pmatrix} \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad \text{and } F = (F_1, F_2)$$

and also

$$A + BF_0 = \begin{pmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 \\ 0 & A_{22} \end{pmatrix}$$

Let assume that $x \in \mathcal{W}^{\perp} \cap \mathcal{N}$. Then $x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ and because $\mathcal{X}_2 \subset \mathcal{X}_w^-(A^*+F_0^*B^*)$ one obtain that $x_2 \in \mathcal{X}_w^-(A_{22}^*)$. Thus $\mathcal{W}^{\perp} \cap \mathcal{N} \subset \mathcal{X}_w^-(A^*)$.

To enunciate the following result, we must first make a finer decomposition of the phase space (analogous of those of Wonham [11]).

Let $\mathcal{W}$ be a (A,B)-closed invariant contained in KerC and $F \in L(\mathcal{X}, \mathcal{U})$ the corresponding operator.

Let it be $\mathcal{L} = \mathcal{L}(A,B)$.

Let's take $\mathcal{X}_1 = \mathcal{L} \cap \mathcal{W}$, $\mathcal{X}_2 = \mathcal{L} \cap \mathcal{W}^{\perp}$, $\mathcal{X}_3 = \mathcal{N} \cap \mathcal{W}$ and $\mathcal{X}_4 = \mathcal{N} \cap \mathcal{W}^{\perp}$. Let P_i be the orthogonal projectors onto $\mathcal{X}_i$, $i = \overline{1,4}$, let's note A_{ij} the restriction of $P_i(A+BF)P_j$ at $\mathcal{X}_j$ $i,j = \overline{1,4}$. $\mathcal{L} = \mathcal{X}_1 \quad \mathcal{X}_2$ is (A,B)-invariant, which is implying that $A_{31} = 0$, $A_{32} = 0$, $A_{41} = 0$ and $A_{42} = 0$; $\mathcal{X}_1$ is (A+BF)-invariant and will imply that $A_{21} = 0$; $\mathcal{W}$ is (A+BF)-invariant which is now finally implying that $A_{23} = 0$ and $A_{43} = 0$

Thus the operator (A+BF) can be written as

$$A+BF = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}$$

In an analogous manner we can establish that the operators B and C can be written as:

$$B = \mathrm{Col}(B_1, B_2, 0, 0)$$

$$C = (0, C_2, 0, C_4)$$

Theorem 3: Let $\mathcal{W}$ be a closed (A,B) invariant subspace contained in KerC. The next conditions are sufficient for the existence of an exponential PRP solution:

i) (A_{22}, B_2) is exponentially stabilizable

ii) A_{44} is exponentially stable.

Proof: Let be F_2 the operator stabilizing the pair (A_{22}, B_2). Let's write $F_0 = F + (0, F_2, 0, 0)$. For an initial state $x_0 = \mathrm{col}(x_0^1, x_0^2, x_0^3, x_0^4)$ the output (1a) of the system will be given by:

$$(7) \quad y(t) = C_2 x_2(t) + C_4 x_4(t)$$

where

$$x_4(t) = \exp(A_{44} t) x_0^4$$

and

$$x_2(t) = \exp\big((A_{22} + B_2 F_2)t\big) x_0^2 + \int_0^t \exp\big((A_{22} + B_2 F_2)(t - \tau)\big) A_{44} x_4(\tau) d\tau$$

The conditions i) and ii) of the theorem assure the stabilization of the pair $(\tilde{A}, \tilde{B})$ (cf. Proposition 1) where:

$$\tilde{A} = \begin{pmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} B_2 \\ 0 \end{pmatrix}$$

Thus $\|x_2(t)\| \leqslant N e^{-\alpha T} \|x_0^2\|$, $\|x_4(t)\| \leqslant M e^{-\beta t} \|x_0^4\|$. The expression (7) of the output allows to write simply that $\|y(t)\| \leqslant K e^{-\gamma t} \|x_0\|$.

Theorem 4. Let W be such as in theorem 3. If the following conditions i) and ii) are satisfied

i) A_{22} is strongly (respectively weakly) stable

ii) (A_{22}, B_2) is strongly (respectively weakly) stabilizable

and in one of this two cases the stability (or the stabilizability) is exponential, then there is a strong (resp. weak) solution of PRP.

Proof: The proof is analogous to those of theorem 3; the difference is that we have to use theorem 1 instead of proposition 1.

4. Conclusion

The results established here concern the dynamical systems with bounded parameters. The case when the A operator is unbounded but an infinitesimal generator of a C_0 semigroup is of a particular interest because it is enclosing a large class of systems with distributed parameters.

This case will be the object of future investigations of the authors.

On the other hand, the diversity of the invariance approaches for the infinite dimensional systems and the difficulties in obtaining a complete description of these notions implies that the results obtained here, and also those concerning the case when A is unbounded, are less rich like for the finite dimensional case.

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