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EXTERIOR CONCURRENT VECTOR FIELDS ON A CONFORMAL COSYMPLECTIC MANIFOLD ENDOWED WITH A SASAKIAN STRUCTURE

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In the paper [1] we have defined and studied *conformal cosymplectic* manifolds $\widetilde{M}(U,\widetilde{\Omega},\widetilde{\eta},\xi,\widetilde{f},\widetilde{g})$, endowed with a *pseudo-Sasakian* structure. Mutatis mutandis, one may consider conformal cosymplectic (abr. c.c.) manifolds $\widetilde{M}(\Phi,\widetilde{\Omega},\widetilde{\eta},\xi,\widetilde{f},\widetilde{g})$ endowed with a *Sasakian structure*. In this case, the metric tensor $\widetilde{g}$ has a Riemannian structure, and the (1,1) - tensor field Φ of $\widetilde{M}$ defines an *almost contact Riemannian metric structure*, i.e. $\Phi^2 = -I + \xi\otimes\widetilde{\eta}$ [2].

The purpose of the present paper is to study the behaviour of an *exterior concurrent* vector field $X \in \mathcal{X}\widetilde{M}$ on a c.c. manifold $\widetilde{M}(\Phi,\widetilde{\Omega},\widetilde{\eta},\xi,\widetilde{f},\widetilde{g})$ endowed with a Sasakian structure. In general, let $(\widetilde{M},\widetilde{g})$ be an oriented connected Riemannian or pseudo-Riemannian C^∞- manifold. Let $\widetilde{\nabla}$ (resp. $d\widetilde{p}$) be the covariant derivative operator defined by $\widetilde{g}$ (resp. the line element of $\widetilde{M}$). Supposing that $\widetilde{\nabla}$ is torsion free and that $\widetilde{M}$ is not flat, we have given in [3], [4] the following

Definition. Any vector field $\widetilde{X} \in \mathcal{X}\widetilde{M}$ which satisfies $\widetilde{\nabla}^2\widetilde{X} = \widetilde{u}\wedge d\widetilde{p}$, for some $u \in \Lambda^1\widetilde{M}$ is called an *exterior concurrent* (abr. e.c.) vector field on $\widetilde{M}$.

We shall prove in this paper that the structure vector field ξ of $\widetilde{M}(\Phi,\widetilde{\Omega},\widetilde{\eta},\xi,\widetilde{f},\widetilde{g})$ is always e.c., and if $\widetilde{X}$ is any e.c. vector field on $\widetilde{M}$, then the Ricci curvature with respect to $\widetilde{X}$ is $\mathrm{Ric}(\widetilde{X}) = -2m(c + \widetilde{f}^2)$; $c = \mathrm{const.}$

In case $\widetilde{M}$ is compact, the constant c is necessarily negative, and any e.c. vector field $\widetilde{X}$ of $\widetilde{M}$ satisfies the integral formula

$$\int_{\tilde{M}}((2m-1)r_o^2 \langle \tilde{X},\tilde{X} \rangle \cos^2\mu + \frac{1}{2}\| L_{\tilde{X}}\tilde{g}\|^2 - (\mathrm{div}X)^2) = 0 ,$$

where $c = - r_o^2$, and $\tilde{f} = r_o^2 \sin\tilde{\mu}$.

Furthermore , any CR-submanifold of $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,\tilde{f},\tilde{g})$ is foliated by *vertical submanifolds* $M^\perp$, and if any vector field $X^\perp$ on $M^\perp$ is e.c. , then the *normal connection* of M is constant .

Let $\tilde{M}(\xi,\tilde{\eta},\tilde{\Omega},\tilde{f},\tilde{g})$ be an oriented connected (2m+1) - dimensional Riemannian C^∞ - manifold , and let $j:T\tilde{M} \to T^*\tilde{M}$ ($T\tilde{M}$ is the tangent bundle of $\tilde{M}$) be the canonical isomorphism defined by the metric tensor $\tilde{g}$. The structure tensor fields are defined as follows : ξ is Reeb's vector field ; $\tilde{\eta}$ is the dual form of ξ , i.e. $\tilde{\eta} = j(\xi)$; $\tilde{\Omega}$ is a 2-form of rank 2m , and $\xi = \mathrm{Ker}(\tilde{\Omega})$; $\tilde{f}$ is a nowhere vanishing scalar field .

If one has

$$\xi = a\,\mathrm{grad}\tilde{f} \ , \ a = \text{const} \neq 0 \tag{1}$$

$$d\tilde{\Omega} = 2\tilde{f}\ \tilde{\eta}\wedge\tilde{\Omega} \ , \tag{2}$$

then according to the general definition in [1] , we say that the treble $(\tilde{\Omega},\tilde{\eta},\tilde{f})$ defines a *conformal cosymplectic structure* (abr. c.c.-structure) .

By (1) one has $d\tilde{\eta} = 0$, and taking the exterior derivative in (2) one sees that

$$d\tilde{f} = c\tilde{\eta} \ , \ c = \text{const} \neq 0 \ . \tag{3}$$

Denote as usual by $\tilde{\nabla}$ the *covariant derivative operator* defined by $\tilde{g}$, which in the case under discussion has a Riemannian structure , and by $d\tilde{p}$ the *line element* of $\tilde{M}$ ($d\tilde{p}$ is a canonical vector 1 - form on $\tilde{M}$) .

Let now Φ be the (1,1) - tensor field of an *almost contact metric structure* [2], [3] , and let $\Gamma T\tilde{M} = \mathcal{X}\tilde{M}$ be the set of sections of $T\tilde{M}$. If for any vector fields $\tilde{U}$, $\tilde{V} \in \mathcal{X}\tilde{M}$ one has the following system of relationships

$$\Phi^2 = -I + \xi\otimes\tilde{\eta} \ , \ \tilde{g}(\Phi\tilde{U},\Phi\tilde{V}) = \tilde{g}(\tilde{U},\tilde{V}) - \tilde{\eta}(\tilde{U})\tilde{\eta}(\tilde{V}) \ , \ \Phi\xi = 0$$

$$\tilde{g}(\tilde{U},\xi) = \tilde{\eta}(\tilde{U}) \ , \quad \tilde{\Omega}(\tilde{U},\tilde{V}) = 2\langle \Phi U,V\rangle \ , \quad \tilde{\nabla}_U\xi = \tilde{f}\tilde{U} - \tilde{f}\tilde{\eta}(\tilde{U})\xi \tag{4}$$

$$\text{imply} \quad \tilde{\nabla}\xi = -\tilde{f}\Phi^2\tilde{dp} \ ,$$

we say that the e.c. manifold under consideration is endowed with a Sasakian structure (one may also call $\tilde{M}$, abusing language , a *conformal cosymplectic quasi-Sasakian manifold* [5]) .

It is worth noticing that (4) leads to the formula

$$\Phi\tilde{\nabla}\tilde{U} = \tilde{\nabla}\Phi\tilde{U} + \tilde{\eta}(\tilde{U})\Phi\tilde{dp} + \tilde{f}j(\Phi\tilde{U}) \ O\ \xi, \text{ for } \tilde{U}\in \mathcal{X}\tilde{M} \ , \tag{5}$$

which is coherent with the last equation in (4) .

Let us set , according to [6], $A^q(\tilde{M},T\tilde{M}) = \Gamma\mathrm{Hom}(\Lambda^q TM,TM)$ and notice that the elements of $A^q(\tilde{M},T\tilde{M})$ are differential q-forms on $\tilde{M}$, with values in $T\tilde{M}$. Let $d^{\tilde{\nabla}}: A^q(\tilde{M},T\tilde{M}) \to A^{q+1}(\tilde{M},T\tilde{M})$ be the *exterior covariant derivative operator* with respect to $\tilde{\nabla}$. Next , set $R\in\Gamma\mathrm{End}\Lambda(T\tilde{M})$ for the *curvature operator* on $\tilde{M}$, and let $\tilde{X}\in \mathcal{X}\tilde{M}$ be a globally defined vector field on $\tilde{M}$. Then the *second covariant differential* of X is (see [6]) expressed by

$$\tilde{\nabla}^2\tilde{X} = d^{\tilde{\nabla}}(\tilde{\nabla}\tilde{X}) = \tilde{R}(\tilde{U},\tilde{V})\tilde{X} \in A^2(\tilde{M}, T\tilde{M}) \ . \tag{6}$$

According to our definition [3], [4], if $\tilde{X}$ satisfies

$$\tilde{\nabla}^2 X = \tilde{u}\wedge\tilde{dp} \quad \text{for some} \quad \tilde{u}\in\Lambda^1\tilde{M} \ , \tag{7}$$

then $\tilde{X}$ is called an *exterior concurrent vector field* (abr. e.c.) on $\tilde{M}$. In consequence of this definition , $\tilde{u}$ is called the *concurrence form* , and by (6) $\tilde{M}$ is necessarily not flat .

Taking the second covariant derivative of the last equation (4) , one finds on behalf of (3)

$$\tilde{\nabla}^2\xi = (c+\tilde{f}^2)\tilde{\eta}\wedge\tilde{dp} \ . \tag{8}$$

Comparison of (8) and (7) shows that the structure vector field of $\widetilde{M}(\Phi,\widetilde{\Omega},\widetilde{\eta},\xi,\widetilde{f},\widetilde{g})$ is always e.c. In [4] we have proved the same property for the structure vector field of a Sasakian or a pseudo-Sasakian manifold .

Let now $\mathcal{O}(\widetilde{M})$ (resp. $\mathcal{O} = \mathrm{vect}\{e_A \mid A=0,1,\dots,2m\} \in \mathcal{O}(\widetilde{M})$) be the bundle of orthonormal frames of $\widetilde{M}$ (resp. an element of $\mathcal{O}(\widetilde{M})$) . If $\mathcal{O}^* = \mathrm{covect}\{\widetilde{\omega}_A\}$ is the coframe of $\mathcal{O}$, then one has

$$d\widetilde{p} = \widetilde{\omega}^A \otimes e_A . \tag{9}$$

Further , if $\widetilde{\theta}^A_B = \widetilde{\gamma}^A_{BC}\widetilde{\omega}^C \in \Lambda^1\widetilde{M}$ (resp. $\widetilde{\Theta}^A_B \in \Lambda^2\widetilde{M}$) are the local *connection forms* in $\mathcal{O}(\widetilde{M})$ (resp. the *curvature forms*) , then since $\widetilde{\nabla}$ is torsion free (E.CARTAN) the structure equations may be written in indexless form as

$$\widetilde{\nabla}e = \widetilde{\theta} \otimes e , \tag{10}$$

$$d\widetilde{\omega} = -\widetilde{\theta} \wedge \widetilde{\omega} , \tag{11}$$

$$d\widetilde{\theta} = -\widetilde{\theta} \wedge \widetilde{\theta} + \widetilde{\Theta} \tag{12}$$

But according to (4) $\mathcal{O} = \mathrm{vect}\{e_A\}$ is a Φ - basis [2] , and one easily finds by formula (10)

$$\widetilde{\theta}^a_b = \widetilde{\theta}^{a^*}_{b^*},\quad \widetilde{\theta}^{a^*}_b = \widetilde{\theta}^{b^*}_a ,\quad \widetilde{\theta}^a_o = \widetilde{f}\widetilde{\omega}^a,\quad \widetilde{\theta}^{a^*}_o = \widetilde{f}\widetilde{\omega}^{a^*} , \tag{13}$$

where $a = 1,2,\dots m$, $a^* = a + m$. By (11) , (13) and (8) one has

$$\begin{aligned} \widetilde{\Theta}^{a^*}_b - \widetilde{f}^2\widetilde{\omega}^b \wedge \widetilde{\omega}^{a^*} &= \Theta^{b^*}_a - \widetilde{f}^2\widetilde{\omega}^a \wedge \widetilde{\omega}^{b^*} ,\\ \widetilde{\Theta}^a_b - \widetilde{f}^2\,\widetilde{\omega}^b \wedge \widetilde{\omega}^a &= \widetilde{\Theta}^{a^*}_{b^*} - \widetilde{f}^2\widetilde{\omega}^{b^*} \wedge \widetilde{\omega}^{a^*} , \end{aligned} \tag{14}$$

and also

$$\widetilde{\Theta}^\alpha_o = (c + \widetilde{f}^2)\widetilde{\eta} \wedge \widetilde{\omega}^\alpha \quad ; \quad \alpha \in \{ a,a^* \} . \tag{15}$$

It is worth noticing that (15) allows to check equation (8) . Consider now the vector field $\widetilde{X} = \widetilde{X}^A e_A \in \mathcal{X}\widetilde{M}$. With respect to $\mathcal{O} = \mathrm{vect}\{e_A\}$, $\widetilde{\nabla}^2\widetilde{X}$ can be expressed as known by

$$\widetilde{\nabla}^2\widetilde{X} = \widetilde{\Theta}^A_B\widetilde{X}^B \otimes e_A \in A^2(\widetilde{M},T\widetilde{M}) . \tag{16}$$

Expressing now the fact that $\tilde{X}$ satisfies (7) , and taking into account (9) , one obtains

$$\tilde{u} = (c + f^2)\tilde{\omega} \ , \tag{17}$$

where we have set

$$\tilde{\omega} = j(\tilde{X}) = \sum_A \tilde{X}^A \tilde{\omega}^A \ . \tag{18}$$

Therefore , by (7) and (6) one may write

$$\tilde{\Theta}^A_B \tilde{X}^B = (c + \tilde{f}^2)\tilde{\omega} \wedge \tilde{\omega}^A \ . \tag{19}$$

Let now $\tilde{R}$ be the *Ricci tensor field* of $\tilde{\nabla}$. As known , $\tilde{R}$ is globally defined on $\tilde{M}$ by

$$\tilde{R}(\tilde{U},\tilde{V}) = \mathrm{tr}\ \tilde{R}(\cdot,\tilde{V})\tilde{V} \ , \text{ for any } \tilde{U},\tilde{V} \in \mathfrak{X}\tilde{M} \ . \tag{20}$$

From (6) , (19) and (20) , one obtains after some calculations

$$\tilde{R}(\tilde{X},\tilde{X}) = -\ 2m\ (c + \tilde{f}^{\,2})\ \langle \tilde{X},\tilde{X} \rangle \ . \tag{21}$$

Therefore , denoting by $\widetilde{Ric}(\tilde{X})$ the *Ricci curvature* with respect to $\tilde{X}$, the above equation becomes

$$\widetilde{Ric}(\tilde{X}) = -\ 2m\ (c + \tilde{f}^{\,2}) \ . \tag{22}$$

On the other hand , let $\mathrm{tr}\tilde{\nabla}^2\tilde{X} \in \mathfrak{X}\tilde{M}$ be the *contraction* of $\tilde{\nabla}^2\tilde{X}$ with respect to $\tilde{g}$ (i.e. $(\mathrm{tr}\ \tilde{\nabla}^2\tilde{X})_{\tilde{p}} = \sum \tilde{\nabla}^2(e_A,e_A)$) . From (7) , (17) and (18) we get

$$\mathrm{tr}\tilde{\nabla}^2\tilde{X} = (c + \tilde{f}^2)\tilde{X} \ . \tag{23}$$

Since by hypotheses $\tilde{X}$ is not parallel , then if $\tilde{M}$ is compact , it follows according to BOCHNER [6] , LICHNEROWICZ [7] , that the symmetric bilinear form $\tilde{B}(\tilde{X},\cdot) = \langle \mathrm{tr}\tilde{\nabla}^2\tilde{X} \ , \ \cdot \ \rangle$ must necessarily be negative definite . Therefore , if $\tilde{M}$ is compact , equation (23) shows that one must have

$$\tilde{c} + \tilde{f}^{\,2} = -\ \tilde{\lambda}^2 \ , \quad \tilde{\lambda} \in C^\infty\tilde{M} \ . \tag{24}$$

Hence , setting $c = -\ r_o^2$, we can write

$$\tilde{f} = r_o \sin\tilde{\mu} \ , \quad \tilde{\lambda} = r_o \cos\tilde{\mu} \ , \ \tilde{\mu} \in C^\infty\tilde{M} \ . \tag{25}$$

The following integral formula is obtained from (23) and (25) , if we take into account [6], and assume $\tilde{M}$ to be compact :

$$\int_{\tilde{M}} \left((2m-1)r_o^2 \langle \tilde{X},\tilde{X}\rangle \cos^2\tilde{\mu} + \frac{1}{2}\| L_{\tilde{X}}\tilde{g}\|^2 - (\operatorname{div}\tilde{X})^2\right) = 0 , \tag{26}$$

where L designates the Lie derivative. But since ξ is also e.c., one derives from (4)

$$\operatorname{div}\xi = \sum \langle \tilde{\nabla}_{e_A}\xi , e_A\rangle = 2m\tilde{f} , \tag{27}$$

and

$$\| L_\xi \tilde{g}\|^2(\tilde{p}) = \sum(\langle \tilde{\nabla}_{e_A}\xi, e_B\rangle + \langle e_A, \tilde{\nabla}_{e_B}\xi\rangle)^2 = 16m^2\tilde{f}^2 . \tag{28}$$

Hence, according to (25), (27) and (28), equation (26) becomes

$$\int_{\tilde{M}} r_o^2 \left((2m-1)\cos^2\tilde{\mu} + 4m^2\sin^2\tilde{\mu}\right) = 0 .$$

Now, in view of what we discuss in the last part of this paper, we shall write the expression of $\tilde{\nabla}^2\Phi\tilde{X}$, if $\tilde{X}$ is e.c. Making use of (4), (14) and (16), one finds

$$\tilde{\nabla}^2\Phi\tilde{X} = c[j(\tilde{X}) - \eta(\tilde{X})\tilde{\eta}]\wedge\Phi d\tilde{p} - \tilde{f}^2 j(\Phi\tilde{X})\wedge\Phi^2 d\tilde{p} . \tag{29}$$

THEOREM. Let $(\Phi,\tilde{\Omega},\tilde{\eta},\xi,\tilde{f},\tilde{g})$ be a conformal cosymplectic manifold endowed with a Sasakian structure. Regarding the concept of exterior concurrent vector field, one has the following properties : (i) the structure vector field ξ of $\tilde{M}$ is always e.c. ; (ii) if $\tilde{X}$ is any e.c. vector field on $\tilde{M}$, then the Ricci curvature with respect to $\tilde{X}$ is $\mathcal{Ric}(\tilde{X}) = -2m(c + \tilde{f}^2)$, where f is the structure scalar field of $\tilde{M}$; (iii) if $\tilde{M}$ is compact, then necessarily $\tilde{f} = r_o\sin\tilde{\mu}$ (r_o = const.), and any e.c. vector field $\tilde{X}$ satisfies (26).

In the last section of this paper we shall consider the following immersion $\chi : M \to \tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,\tilde{f},\tilde{g})$, where M is an ℓ-codimensional ($\ell < m$) submanifold of $\tilde{M}$, orthogonal to ξ and defined by means of the completely integrable Pfaff system

$$\omega^{r^*} = 0 \; ; \; r^*, s^* = 2m - \ell + 1 , \ldots, 2m . \tag{30}$$

Let us denote by $T_p(M)$ (resp. $T_p^{\perp}(M)$) the tangent space (resp. the normal space) at any $p \in M$ (we supress $\sim$ for the induced by χ elements) . Consider on $T_p(M)$ the two differentiable complementary distributions

$$D : p \to D_p = \mathrm{vect}\{ e_i \,,\, e_{i^*} \,,\, \xi \mid i= 1,..,m-\ell,\ i^*=i+m \} ,$$
$$D^{+} : p \to D_p^{+} = \mathrm{vect}\{ e_r \mid r=m-\ell+1 \,,\, \dots ,m \} . \tag{31}$$

One has clearly $\Phi D_p \subset T_p(M)$ and $\Phi D_p^{\perp} \subset T_p^{\perp}(M)$ for each $p \in M$. Hence , according to BEJANCU [8] and YANO and KON [2] M may be considered as a ξ - horizontal CR submanifold of $\tilde{M}$ [9] . Accordingly , D_p (resp. $D_p^{\perp}$) is the horizonatl (resp. the vertical) distribution on M . But by (4) , and with respect to the cobasis $\{\tilde{\omega}^A\}$, the structure 2 - form $\tilde{\Omega}$ is expressed by

$$\tilde{\Omega} = \sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a^*} . \tag{32}$$

From (30) one has on M

$$\Omega = \tilde{\Omega}/M = \sum \omega^i \wedge \omega^{i^*} . \tag{33}$$

Therefore , the *simple unit form* ϕ , which corresponds to the distribution D_p , is given (up to the sign) by

$$\phi = (\Omega \overset{m-\ell}{\wedge} \eta)/(m-\ell)! . \tag{34}$$

From (34) , on behalf of (2) and (3) one easily finds $d\phi = 0$.

Therefore , the ideal $I(D^{+}) = \{ \phi \in \Lambda M \,;\, \phi$ annihilates $D^{\perp}$, $dI \subset I \}$ is a *differentiable ideal* . Thus , we conclude that the vertical distribution $D^{\perp}$ is always *involutive* . One refinds in this way , in the case under discussion , a basic property of CR submanifolds (see also ROŞCA [10] , [11]) .

Let us denote now by M^{+} a leave of dimension ℓ of $D_p^{\perp}$, and suppose that $X^{+} \in \chi M^{+}$ is an exterior concurrent vector field on M^{+} . Then $\Phi X^{\perp} \in T_p^{\perp}(M^{\perp})$ is normal to M^{+} , and setting dp^{+} for the line element of M^{+} one derives from (29) :

$$\nabla^2 \Phi X^{+} = c j(X^{+}) \wedge \Phi dp \tag{35}$$

(in the above formula Φdp is a vector form in the normal bundle of $M^{\perp}$) .

Let $R^{N}(U^{\perp},V^{\perp})Z$ ($U^{\perp}$, $V^{\perp} \in T_p(M^{\perp})$; $Z \in T_p^{\perp}(M^{\perp})$) be the curvature tensor in the normal bundle of $M^{\perp}$, and assume that any vector field $X^{\perp}$ on $M^{\perp}$ is e.c. We derive then from (35) :

$$\Theta^{r^*}_{s^*} = - c\, \omega^{r} \wedge \omega^{s} , \qquad (36)$$

where $\Theta^{r^*}_{s^*}$ are the curvature forms associated with R^{N} . In view of equations (36) and referring to a known definition , we shall say that the normal connection ∇^{N} of $M^{\perp}$ is constant .

THEOREM . Any CR submanifold M of a $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,\tilde{f},\tilde{g})$ manifold is foliated by vertical submanifolds $M^{\perp}$, *and if any vector field* $X^{\perp}$ *on* $M^{\perp}$ is e.c. , then the normal connection of $M^{\perp}$ is constant .

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