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# THE BARGAINING SET $M_o$ FOR CONVEX GAMES

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The theory of the bargaining sets for cooperative n-person games with side payments, initiated by R. J. Aumann and M. Maschler in [1], has further been developed in many papers, some of them mentioned in the second edition of Owen's book [5]. The particular case of convex games has also been considered in [4]. In a previous paper ([3]), a combinatorial characterization of the payoff vectors belonging to the bargaining set $M_o$ has been obtained. In this paper, for convex games a new characterization will be obtained, precisely we shall show that a payoff vector $x$ belongs to $M_o - C(G)$, if and only if the maximum excess w.r.t. $x$ is reached for at least two coalitions (Theorem 2.9). In order to make the paper self contained, some definitions and previous results will be given in the first section, the case of convex games will be considered in the second section.

## 1. Definitions and previous results.

A game with coalition structures is a triplet $G_F = (I,v,F)$, where $I$ is a finite set, the set of players, $|I| = n$, then $v: P(I) \to R$ is the characteristic function, subject to $v(\phi) = 0$, and $F$ is the set of admissible payoff vectors. Any partition $\mathcal{S}$ of $I$ is a coalition structure. The set of admissible payoff vectors for $\mathcal{S}$ is $F_{\mathcal{S}} = \{x \mid x \in R^n,\ x(S) = v(S),\ \forall\ S \in \mathcal{S}\}$. The set $F$ of admissible payoff vectors for $G$ is the union of all $F_{\mathcal{S}}$, (see [1]).

For $x \in F$ and $S \in P(I)$ the excess is $e(x,S) = v(S) - x(S)$, where $x(S) = \sum_{i\in S} x_i$, if $S \neq \phi$, and $x(\phi) = 0$. The cores of $G$ is $C(G) = \{x \mid x \in F,$

$e(x,S) \leq 0,\ \forall\ S \in P(I)\}$, (see [2]).

Consider any $x \in F$ for some coalition structure $\mathcal{S}$. Denote a generic coalition structure by $\mathcal{T} = (P_1,\dots,P_\pi;\ N_1,\dots,N_\nu;\ 0_1,\dots,0_\omega)$, where $P_i$, $i=\overline{1,\pi}$, $N_j$, $j=\overline{1,\nu}$, $0_k$, $k=\overline{1,\omega}$, stand for the coalitions with positive, negative and zero excesses, respectively. Denote $P = \bigcup_1^\pi P_i$, $N = \bigcup_1^\nu N_j$, $0 = \bigcup_1^\omega 0_k$. If $x \notin C(G)$, then there coalition structures $\mathcal{T}$ with $P \neq \phi$.

<u>Definition 1.1</u>. Any coalition structure $\mathcal{T}$ with $P \neq \phi$ will be called a bargaining proposal w.r.t. $(x,\mathcal{S})$. A bargaining proposal with $N = \phi$ is a trivial bargaining proposal.

As in [3], we suppose that: (A) there is no trivial bargaining proposal w.r.t. $(x,\mathcal{S})$.

<u>Definition 1.2</u>. If $\mathcal{T}$ is a bargaining proposal, then any $y \in F_{\mathcal{T}}$ such that $y_h \geq x_h$, $\forall\ h \in P$ and $y_h > x_h$ for some $h \in P_i$, $\forall\ i = \overline{1,\pi}$ will be called a bargaining distribution of the gain provided by $\mathcal{T}$.

Note that for a given $\mathcal{T}$ the set $D(\mathcal{T})$ of bargaining distributions for $\mathcal{T}$ is nonempty and in general is an infinite set.

Let $\mathcal{T}$ be a bargaining proposal w.r.t. $(x,\mathcal{S})$ and $y \in D(\mathcal{T})$ a bargaining distribution. The double excess of any $S \in P(I)$, w.r.t. $(x,\mathcal{S})$ and $(y,\mathcal{T})$ is

(1.1) $$e(x,S;y,\mathcal{T}) + e(x,S) = [y(S \cap P) - x(S \cap P)].$$

Consider any other bargaining proposal $\mathcal{T}^* = (P_1^*,\dots,P_{\pi^*}^*;\ \dots)$ and assume

(B) $$P^* \cap P \neq \phi,\quad P^* = \bigcup_1^{\pi^*} P_{i^*}^*$$

We can suppose without loss of generality that there is an integer $r$, $1 \leq r \leq \pi^*$, such that

(1.2) $$\begin{aligned} &P_{i^*}^* \cap P \neq \phi, \quad i^* = \overline{1,r} \\ &P_{i^*}^* \cap P = \phi, \quad i^* = \overline{r+1,\pi^*} \quad \text{if } r < \pi^* . \end{aligned}$$

Definition 1.3. Any bargaining proposal $\mathcal{T}^*$ subject to (B) with

(1.3) $$e(x,P^*_{i^*};y,\mathcal{T}) \geq 0, \quad i^* = \overline{1,r}$$

for all $y \in D(\mathcal{T})$, will be called a bargaining counter proposal w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$.

Definition 1.4. If $\mathcal{T}^*$ is a bargaining proposal subject to (B) and $y \in D(\mathcal{T})$, then any $z \in F_{\mathcal{T}^*}$ such that

(1.4) $$z_h \geq y_h, \quad \forall\ h \in P^* \cap P$$
$$z_h \geq x_h, \quad \forall\ h \in P - P^* \cap P$$

will be called a bargaining counter distribution w.r.t. $y$.

These two concepts are related by the following result:

Theorem 1.5. ([3], Th.2.4): Consider a pair $(\mathcal{T},\mathcal{T}^*)$ of bargaining proposals subject to (B). Then, $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$, if and only if for every $y \in D(\mathcal{T})$ there exists a bargaining counter distribution $z \in F_{\mathcal{T}^*}$ w.r.t. $y$.

A combinatorial characterization of the bargaining counter proposals has previously been proved:

Theorem 1.6. ([3], Th.2.7): Consider a pair $(\mathcal{T},\mathcal{T}^*)$ of bargaining proposals subject to (B) and let $r$, $1 \leq r \leq \pi^*$, be the integer defined by (1.2). Then $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$, if and only if

(C) $$e(x,P^*_{i^*}) \geq \sum_{i^* \in I(P^*_{i^*})} e(x,P_i), \quad i^* = \overline{1,r}$$

where $I(P^*_{i^*}) = \{i \mid P_i \cap P^*_{i^*} \neq \phi\}$, $i^* = \overline{1,r}$.

Definition 1.7. The bargaining set $M_o$ is the set of all stable $x \in F$. Any $x \in F$ is stable, if either $x \in C(G)$, or for any bargaining proposal $\mathcal{T}$ w.r.t. $(x,\mathcal{S})$, there exists a bargaining counter proposal $\mathcal{T}^*$ w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$.

From this definition and Theorem 1.6 follows a combinatorial characterization of the payoff vectors belonging to $M_o - C(G)$:

Theorem 1.8. Any $x \in F$, $x \notin C(G)$, belongs to $M_o$, if and only if for each bargaining proposal $\mathcal{T} = (P_1,\dots,P_\pi; \dots)$ w.r.t. $(x,\mathcal{S})$, there exists another bargaining proposal $\mathcal{T}^* = (P^*,\dots,P^*_{\pi^*}; \dots)$ w.r.t. $(x,\mathcal{S})$, such that the pair $(\mathcal{T},\mathcal{T}^*)$ satisfies (B) and (C).

It has been shown in [3], (Th.3.3), that: Theorem 1.8 still holds if we consider in the statement only bargaining proposals $\mathcal{T}^*$ of the form $\mathcal{T}^* = (P^*,I-P^*)$. In the following, we shall show that we can also confine ourselves to bargaining proposals of the form $\mathcal{T} = (P,I-P)$. From this result we shall further derive a numerically better characterization of the payoff vectors belonging to $M_o-C(G)$.

## 2. Combinatorial results for convex games.

Definition 2.1. A game with coalition structures $G = (I,v,F)$ is called a convex game, if

(2.1) $$v(s) + v(T) \leq v(S \cup T) + v(S \cap T), \quad \forall\ S,\ T \in P(I),$$

i.e. the characteristic function is supermodular.

Lemma 2.2. In a convex game, for any $x \in F$ we have

(2.2) $$e(x,S) + e(x,T) \leq e(x,S \cap T) + e(x,S \cup T), \quad \forall\ S,\ T \in P(I),$$

i.e. the excess function is supermodular.

Proof. Follows from definition 2.1, taking into account the identity

(2.3) $$e(x,S) + e(x,T) - e(x,S \cap T) - e(x,S \cup T) = v(S) + v(T) - v(S \cap T)-v(S \cup T)$$

for all $x \in F$ and all $S,\ T \in P(I)$.

Lemma 2.3. If $G$ is convex and $\mathcal{T} = (P_1,\dots,P_\pi; \dots)$ is a bargaining proposal, then $\mathcal{T}' = (P,I - P)$ is also a bargaining proposal.

Proof. As $e(x,P_i) > o$, $i = \overline{1,\pi}$, the result follows from the inequality

(2.3) $$\sum_1^{\pi} e(x,P_i) \leq e(x,P)$$

derived from the convexity condition, because $P_i$, $i = \overline{1,\pi}$, are pairwise disjoint.

Note that $e(x,I - P) \leq o$, because there are no trivial bargaining proposals.

A bargaining proposal of the form $(P,I - P)$ will be called an one coalition bargaining proposal; the same words will be used for counter proposals.

Theorem 2.4. If $G$ is convex, then there exists a bargaining counter proposal for each bargaining proposal w.r.t. $(x,S)$, if and only if there exists a bargaining counter proposal for each one coalition bargaining proposal w.r.t. $(x,S)$.

Proof. Consider a bargaining proposal $T = (P_1,\ldots,P_\pi; \ldots)$, $\pi > 1$, w.r.t. $(x,S)$. Then, according to Lemma 2.3, $T' = (P,I - P)$ is also a bargaining proposal. Let $T^* = (P_1^*,\ldots,P_{\pi^*}^*; \ldots)$ be a bargaining counter proposal w.r.t. $(x,S)$ and $T'$. As $e(x,I - P) \leq o$, if $r$ is the number defined by (1.2), then from Theorem 1.6 we get

$$(2.4) \qquad e(x,P_{i^*}^*) \geq e(x,P), \quad i^* = \overline{1,r}.$$

From (2.3) and (2.4) we obtain

$$(2.5) \qquad e(x,P_{i^*}^*) \geq \sum_1^\pi e(x,P_i), \quad i^* = \overline{1,r}.$$

As the number $r$ is the same for $T$ and $T'$, because $T'$ is a nontrivial bargaining proposal, from (2.5) we get (C). Then, Theorem 1.6 shows that $T^*$ is also a bargaining counter proposal w.r.t. $(x,S)$ and $T$.

Theorem 2.5. If $G$ is convex, then $x \in M_o - C(G)$ if and only if for each coalition $P$ with $e(x,P) > o$, there exists a coalition $P^*$ with $P^* \cap P \neq \phi$ such that

$$(2.6) \qquad e(x,P^*) \geq e(x,P).$$

Proof. This is a corollary of Theorem 2.4, taking into account the remark which follows Theorem 1.8.

Obviously, from (2.6) an algorithm for checking whether $x \in M_o - C(G)$ in a convex game, can be derived. However, some computations may be saved if we use other combinatorial results that follow.

Let us denote by $V = P(I) - \{\phi\}$ and $v_S = S$, $S \in V$. We define the set of unordered pairs $E = \{(v_S, v_T) \mid v_S \in V,\ v_T \in V,\ S \cap T \neq \phi\}$. In this way an undirected graph $H = (V,E)$ has been attached to the game $G$. For a fixed $x \in F$ we define a weight function $w(v_S) = e(x,S)$, $S \in V$. Let us denote by $V^+$ the set of vertices with positive weights and $H^+$ the subgraph of $H$ induced by $V^+$. Theorem 2.5 above can be expressed in graph theoretical terms as follows:

Theorem 2.6. If $G$ is convex, then $x \in M_o - C(G)$ if and only if for each vertex $v_P \in V^+$ there exists an adjacent vertex $v_{P^*} \in V^+$ such that $w(v_{P^*}) \geq$ $\geq w(v_P)$.

Let us denote by $w_o > w_1 > \dots > w_t > o$, $t \geq o$, the weights of vertices in $V^+$ and

$$V_g^+ = \{v_S \mid v_S \in V^+,\ w(v_S) = w_g\},\quad g = \overline{o,t}. \tag{2.7}$$

Lemma 2.7. If $G$ is convex and $v_{S_o} \in V_o^+$, then any vertex $v_S \in V^+$ is adjacent to $v_{S_o}$.

Proof. If $v_S$ were not adjacent to $v_{S_o}$, then from Lemma 2.2 we get

$$w(v_{S_o}) + w(v_S) \leq w(v_{S_o \cup S}) \tag{2.8}$$

hence $v_{S_o \cup S} \in V^+$ and $w(v_{S_o \cup S}) > w_o$ which is impossible.

Lemma 2.8. If $G$ is convex, then $V_o^+$ is a clique of $H^+$ and any vertex $v_S \in V^+ - V_o^+$, if any, is adjacent to all vertices of $V_o^+$.

Proof. Follows from Lemma 2.7.

Note that if $V^+ - V_o^+ \neq \phi$, then $V_o^+$ is not a maximal clique, because any vertex $v_S \in V^+ - V_o^+$ can be added to $V_o^+$ to obtain a larger clique. However, two vertices of $V^+ - V_o^+$ might be adjacent or not.

Theorem 2.9. If $G$ is convex, then $x \in M_o - C(G)$ if and only if $|V_o^+| > 1$

Proof. If $|V_o^+| = 1$, i.e. $V_o^+ = \{v_{S_0}\}$, then either $|V^+| = 1$, or for any vertex $v_S \in V^+ - V_o^+$ we have $w(v_S) < w(v_{S_o}) = w_o$. In both cases by Theorem 2.6 we get $x \notin M_o - C(G)$. If $|V_o^+| > 1$, then by Lemma 2.8 there is an adjacent vertex $v_{S_o} \in V_o^+$ for each vertex $v_S \in V^+$ and this means that for these vertices $w(v_{S_o}) \geq w(v_S)$, hence by Theorem 2.6 we have $x \in M_0 - C(G)$.

This last result is a computationally simple criterion to check whether $x \in F$ belongs to $M_0 - C(G)$ in a convex game: compute all excesses for $x$ and determine whether the maximum excess is reached for at least two coalitions. If yes, then $x \in M_o - C(G)$, if not, then $x \notin M_o - C(G)$.

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