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SOME FIXED POINT THEOREMS IN UNIFORM SPACES

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Abstract. Some results on common fixed points for pairs of commuting mappings defined on a sequentially complete Hausdorff uniform space are proved. Convergence theorems for sequences of fixed points are also established.

1. Introduction. One of the numerous generalizations of the Banach contraction principle is that of Acharya [1], according to which a selfmapping S on a sequentially complete (Hausdorff) uniform space $(X,\mathcal{U})$ has a unique fixed point if it satisfies the following condition: for each $x,y \in X$

(*) $$(x,y) \in V \text{ implies } (Sx,Sy) \in k\,V,$$

where $0 < k < 1$, and V is an arbitrary member of a base generated by an associated family of pseudometrics of the uniformity $\mathcal{U}$.

A number of papers have appeared which establish fixed point theorems for mappings satisfying more general conditions than the condition (*).

The aim of this paper is to prove two common fixed point theorems of Acharya's type for pairs of commuting mappings. Also two convergence theorems for sequences of fixed points are established. Special cases of the presented results appear in [2], [5], [6], and [10].

2. Preliminaries. Throughout this paper $(X,\mathcal{U})$ is assumed to be a sequentially complete Hausdorff uniform space.

A sequence $\{x_n\}$ in X is convergent to a point x in X if for each $U \in \mathcal{U}$, there exists a natural number N such that $(x_n,x) \in U$ for each $n \geq N$.

Let $\mathcal{P}$ be a family of pseudometrics on X generating the uniformity $\mathcal{U}$. Define

$$\mathcal{C} = \{\bigcap_{i=1}^{n} V_{(p_i, r_i)}: \ p_i \in \mathcal{P},\ r_i > 0,\ n = 1,2,\ldots\},$$

where $V_{(p,r)} = \{(x,y): \ p(x,y) < r\}$. For each $V \in \mathcal{C}$ and $\alpha > 0$ define $\alpha V = \bigcap_{i=1}^{n} V_{(p_i, \alpha r_i)}$. The following properties will be used in the sequel (see [1]):

(i) $\alpha(\beta V) = (\alpha\beta)V$,

(ii) $\alpha V \circ \beta V \subset (\alpha+\beta)V$,

(iii) $\alpha V \subset \beta V$ if $\alpha < \beta$,

(iv) $(x,y) \in \alpha V_{(p,r_1)} \circ \beta V_{(p,r_2)}$ implies $p(x,y) < \alpha r_1 + \beta r_2$.

Recall also that for each $V \in \mathcal{C}$ there exists a pseudometric p (called the Minkowski pseudometric of V) such that $V = V_{(p,1)}$.

3. Fixed point theorems.

Theorem 1. Let T be a continuous selfmapping of X. Then T has a fixed point in X if and only if there exists a mapping S of X into $T(X)$, which commutes with T, and such that for any $V_i \in \mathcal{C}$ $(i=1,2,\ldots,5)$ and $x,y \in X$,

$$(Tx,Sx) \in V_1,\ (Ty,Sy) \in V_2,\ (Tx,Sy) \in V_3,\ (Ty,Sx) \in V_4,\ (Tx,Ty) \in V_5$$

implies

(A) $$(Sx,Sy) \in \alpha_1(x,y)V_1 \circ \ldots \circ \alpha_5(x,y)V_5,$$

where α_i $(i=1,2,\ldots,5)$ are nonnegative functions satisfying

(B) $$\sup_{x,y\in X}\{\alpha_1(x,y) + \ldots + \alpha_5(x,y)\} = k < 1.$$

Indeed, S and T then have a unique common fixed point.

Proof. Assume that u is a fixed point of T. Then the mapping S defined by $Sx = u$ $(x \in X)$ satisfies the required properties (each V_i contains the diagonal of $X \times X$).

To show the sufficiency, let $V \in G$ and let p be its Minkowski pseudo-metric. For any $x,y \in X$, let us take $p(Tx,Sx) = r_1$, $p(Ty,Sy) = r_2$, $p(Tx,Sy) = r_3$, $p(Ty,Sx) = r_4$ and $p(Tx,Ty) = r_5$. Let $\varepsilon > 0$. Then, by (i) - (iv), $(Tx,Sx) \in (r_1+\varepsilon)V$, $(Ty,Sy) \in (r_2+\varepsilon)V$, $(Tx,Sy) \in (r_3+\varepsilon)V$, $(Ty,Sx) \in (r_4+\varepsilon)V$ and $(Tx,Ty) \in (r_5+\varepsilon)V$. Thus by (A) we get (we shall write α_i instead of $\alpha_i(x,y)$)

$$(Sx,Sy) \in \alpha_1(r_1+\varepsilon)V \circ \ldots \circ \alpha_5(r_5+\varepsilon)V,$$

which implies

$$\begin{aligned} p(Sx,Sy) &< \alpha_1(r_1+\varepsilon) + \ldots + \alpha_5(r_5+\varepsilon) \\ &= \alpha_1\, p(Tx,Sx) + \alpha_2\, p(Ty,Sy) + \alpha_3\, p(Tx,Sy) + \alpha_4\, p(Ty,Sx) \\ &\qquad + \alpha_5\, p(Tx,Ty) + (\alpha_1 + \ldots + \alpha_5)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

(C) $$p(Sx,Sy) \leq \alpha_1\, p(Tx,Sx) + \alpha_2\, p(Ty,Sy) + \alpha_3\, p(Tx,Sy) + \alpha_4\, p(Ty,Sx) + \alpha_5\, p(Tx,Ty).$$

The condition (C) is equivalent to the following (cf. [9], p. 266):

(D) $$p(Sx,Sy) \leq k \max\{p(Tx,Ty),p(Tx,Sx),p(Ty,Sy),p(Tx,Sy),p(Ty,Sx)\}.$$

Now, the same procedure as in [3,Theorem 2.1] (or in [8,Theorem]) can be used to obtain a point u in X such that $p(Tu,Su) = 0$. Since V is arbitrary and X is Hausdorff, we get $Tu = Su$. Further, from (D) it follows that $p(SSu,Su) \leq k\, p(SSu,Su)$, hence $p(SSu,Su) = 0$, which implies (by the same argument as previously) $SSu = Su$. Since S and T commute, we obtain $TSu = STu = SSu = Su$, that is Su is a common fixed point of S and T.

Clearly, Su is the unique common fixed point of S and T.

Remark 1. (1) Theorem 1 improves Theorem 3.1 of [5] in which $\alpha_3 = \alpha_4$ is assumed.

(2) With $T = id_X$, Theorem 1 reduces to Theorem 1 of [10].

Theorem 2. Let $\{S_n\}$ and $\{T_n\}$ be two sequences of selfmappings of X such that for each n the mappings S_n and T_n satisfy the hypotheses of Theorem 1 (for the same functions α_i). If $\{S_n\}$ and $\{T_n\}$ converge to S and T, then S and T have a unique common fixed point u and the sequence $\{u_n\}$, where u_n is the unique common fixed point of S_n and T_n, converges to the point u.

Proof. From Theorem 1 it follows, that each pair S_n, T_n has a unique common fixed point u_n. Let $V \in G$ and let p be the Minkowski pseudometric of V. Let $x,y \in X$. Substituting S_n and T_n for S and T, resp., in the condition (D) and taking $n \to \infty$, we obtain that S and T satisfy (D). Further, since $S_n(X) \subset T_n(X)$ for each n, we have $S(X) \subset T(X)$, too. Thus, by Theorem 1, S and T have a unique common fixed point u.

Now, we show the $\{u_n\}$ converges to u. We have

$$\begin{aligned} p(u_n,u) &\leq p(S_n u_n, S_n u) + p(S_n u, Su) \\ &\leq p(S_n u, Su) + k \max\{p(u_n, T_n u), p(T_n u, S_n u_{\_}), p(u_n, S_n u)\} \\ &\leq p(S_n u, Su) + k \max\{p(u_n,u) + p(Tu, T_n u), p(T_n u, S_n u), \\ &\qquad p(u_n,u) + p(Su, S_n u)\}. \end{aligned}$$

There exists an N such that for each $n \geq N$

$$\max\{p(S_n u, Su), p(Tu, T_n u), p(T_n u, S_n u)\} < \frac{1-k}{1+k}.$$

Thus we get

$$p(u_n,u) < \frac{1-k}{1+k} + k(p(u_n,u) + \frac{1-k}{1+k}),$$

hence

$$p(u_n,u) < 1,$$

that is $(u_n,u) \in V_{(p,1)} = V$. Since V is arbitrary, it follows that $\{u_n\}$ converges to u.

Remark 2. (1) Theorem 2 with $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$ appears in [5,Theorem 4.2].

(2) With $T_n = id_X$, we get Theorem 2 of [10].

Theorem 3. Let $\{S_n\}$ and $\{T_n\}$ be two sequences of selfmappings of X such that each pair S_n, T_n has a common fixed point u_n. If $\{S_n\}$ and $\{T_n\}$ converge uniformly to S and T, resp., S and T satisfy the hypotheses of Theorem 1, then the sequence $\{u_n\}$ converges to the common fixed point of S and T.

Proof. Let u be the unique common fixed point of S and T (by Theorem 1). Let $V = V_{(p,1)} \in G$. Since S and T are the uniform limits of $\{S_n\}$ and $\{T_n\}$, resp., there exists an N such that $(T_nu_n,Tu_n) \in V$ and $(S_nu_n,Su_n) \in V$ for each $n \geq N$. We have

(E) $$p(u_n,u) \leq p(S_nu_n,Su_n) + p(Su_n,Su),$$

and by (C),

$$\begin{aligned} p(Su_n,Su) &\leq k \max\{p(Tu_n,Tu),p(Tu_n,Su_n),p(Tu,Su),\\ &\qquad p(Tu_n,Su),p(Tu,Su_n)\}\\ &= k\{\max p(Tu_n,u),p(Tu_n,Su_n)\}. \end{aligned}$$

Suppose $p(Su_n,Su) \leq k\, p(Tu_n,u) \leq k\,(p(Tu_n,T_nu_n) + p(u_n,u))$. Then, by (E), we have

$$p(u_n,u) \leq \frac{1}{1-k} p(S_nu_n,Su_n) + \frac{k}{1-k} p(Tu_n,T_nu_n).$$

On the other hand, if $p(Su_n, Su) \quad k\, p(Tu_n, Su_n) \quad k\,(p(Tu_n, T_n u_n) + p(S_n u_n, Su_n))$, we get by (E)

$$p(u_n, u) \leq (1 + k) p(S_n u_n, Su_n) + k\, p(Tu_n, T_n u_n).$$

In either case,

$$p(u_n, u) \leq \frac{1}{1 - k} p(S_n u_n, Su_n) + \frac{k}{1 - k} p(Tu_n, T_n u_n),$$

and, since $V = V_{(p,1)}$, we get $p(u_n, u) < \frac{1 + k}{1 - k}$. Thus $(u_n, u) \in (\frac{1 + k}{1 - k})V$ for each $n \geq N$. As V is arbitrary, $\{u_n\}$ converges to u.

<u>Remark 3</u>. (1) With $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$, Theorem 3 appears in [5, Theorem 4.4].

(2) With $T_n = id_X$, Theorem 3 reduces to Theorem 3 of [10].

<u>Theorem 4</u>. Let S and T be two continuous selfmappings of X. Then S and T have a common fixed point in X if and only if there exist mappings A and B of X into $S(X) \cap T(X)$ such that $AS = SA$, $BT = TB$ and such that for any $V_i \in \mathcal{G}$ $(i=1,2,\ldots,5)$ and $x, y \in X$,

$$(Sx, Ax) \in V_1,\ (Ty, By) \in V_2,\ (Sx, By) \in V_3,\ (Ty, Ax) \in V_4,\ (Sx, Ty) \in V_5$$

implies

$$\text{(F)} \qquad (Ax, By) \in \alpha_1(x,y)V_1 \circ \ldots \circ \alpha_5(x,y)V_5,$$

where α_i $(i=1,2,\ldots,5)$ are nonnegative functions satisfying condition (B), and $\alpha_3 = \alpha_4$.

Indeed, A, B, S, and T then have a unique common fixed point.

<u>Proof</u>. If S and T have a common fixed point u, then the mappings A and B defined by $Ax = Bx = u$ $(x \in X)$ have the required properties.

Now, let $V = V_{(p,1)} \in \mathcal{G}$ and $x, y \in X$. The same argument as that of Theorem 1 yields

$$p(Ax,By) \leq \alpha_1\, p(Sx,Ax) + \alpha_2\, p(Ty,By) + \alpha_3\, p(Sx,By) + \alpha_4\, p(Ty,Ax) + \alpha_5\, p(Sx,Ty),$$

which is equivalent to the following condition

(G) $$p(Ax,By) \leq k \max\{p(Sx,Ty), p(Sx,Ax), p(Ty,By), \tfrac{1}{2}[p(Sx,By + p(Ty,Ax)]\}.$$

Now, Theorem 2 of [7] is applicable to obtain a point $u \in X$ such that $p(Su,Tu) = 0$, $p(Au,Tu) = 0$, $p(Bu,Su) = 0$. This implies (by Hausdorffness) $Su = Tu = Au = Bu$. By (G) we have then $p(Au,BAu) \leq k\, p(Au,BAu)$, hence $p(Au,BAu) = 0$, and it follows as above that $BAu = Au$. Thus, Au is a common fixed point of A and B. Further, $SAu = ASu = AAu = Au$, and, similarly, $TAu = Au$. Therefore Au is a common (unique by (G)) fixed point of A, B, S and T.

Remark 4. (1) The assumption $\alpha_3 = \alpha_4$ is necessary in Theorem 4. A suitable counterexample appears in [4] (for $A \neq B$ and $S = T = id_X$). There is an open question whether there exists a common fixed point if $\alpha_3 \neq \alpha_4$, $S \neq T$ and $A = B$.

(2) In order to derive a convergence theorem analogous to Theorem 2 [Theorem 3], we need only that either $S_n \to S$ and $A_n \to A$ (pointwise) or $T_n \to T$ and $B_n \to B$ (uniformly)].

(3) Taking $A = B$ and $\alpha_3 = \alpha_4 = 0$, we get Theorem of [6]. Theorem 4 shows also that the hypothesis of continuity of A in [6, Theorem] is superfluous.

(4) Taking $A = B$ and $S = T$, we get Theorem 3.1 of [5].

(5) Taking $S = T = id_X$, we get Theorem 4 of [10].

4. Concluding remark. The results presented in this paper illustrate how one can derive a uniform space version of a contractive type theorem concerning a

complete metric space. Having defined a condition in terms of elements of a uniformity, we modify it to the one in terms of pseudometrics. In most cases only the pseudometric axioms are required to deriving a point such that the (pseudometric) distance from it to its image is equal to zero. This, as a result of Hausdorffness, gives a fixed point. So, new results in this spirit require only an examination whether there the full strength of metric axioms is used in proving the corresponding metric space theorem.

References

[1] Acharya, S. P. "Some results on fixed points in uniform spaces," Yokohama Math J. 22 (1974), 105-116.

[2] Acharya, S. P. "Convergence of a sequence of fixed points in a uniform space," Mat. Vesnik 13 (1976), 131-141.

[3] Das, K.M. and Naik, K.V. "Common fixed point theorems for commuting maps on a metric space," Proc. Amer. Math. Soc. 77 (1979), 369-373.

[4] Fisher, B. "On a conjecture on common fixed points," Math. Sem. Notes 6 (1978), 153-156.

[5] Khan, M.S. "Commuting mappings and fixed points in uniform spaces," Bull. Acad. Polon.:Math. 29 (1981), 499-507.

[6] Khan, M.S., and Fisher, B. "Common fixed points in uniform spaces," Math. Sem. Notes 10 (1982), 373-377.

[7] Kubiak, T. "Common fixed points of pairwise commuting mappings, Math. Nachr. 119 (1984) (to appear).

[8] Ranganathan, S. "A fixed point theorem for commuting mappings," Math. Sem. Notes 6 (1978), 351-357.

[9] Rhoades, B.E. "A comparison of various definitions of contractive mappings," Trans. Amer. Math. Soc. 226 (1977), 257-290.

[10] Rhoades, B.E. "Fixed point theorems in a uniform space," Publ. Inst. Math. 25 (1979), 153-156.