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ON THE UNIQUENESS OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS*

A.R. Aftabizadeh

Abstract. The existence and uniqueness of solutions of third order non-linear boundary value problems is discussed. Two different methods have been used: (i) Contraction mapping principle and (ii) Successive approximations. It is found that the results by method (ii) are better than those given by (i) and obtained in [6].

1. Introduction.

It is known that if $f(x,y,y')$ is continuous and satisfies Lipschitz condition

$$|f(x,y,y') - f(x,z,z')| \leq K|y - z| + L|y' - z'|$$

on $[a,b] \times R^2$, then the boundary value problem

$$(1.1) \qquad \begin{aligned} y'' &= f(x,y,y') \\ y(a) = A, &\quad y(b) = B \end{aligned}$$

has a unique solution, provided $(b-a)$ is sufficiently small. In general, if the function $f(x,y,y')$ satisfies the Lipschitz condition over a compact region, then the boundary value problem (1.1) may not have a unique solution. See example 3.2, page 45 of [7].

In [1-6] several existence and uniqueness results have been obtained for the third order boundary value problems when the function $f(x,y,y',y'')$ satisfies different types of conditions. In this paper we discuss the existence and

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uniqueness of solutions of

$$y''' = f(x,y,y',y'') \tag{1.2}$$

with the following types of boundary conditions

$$y(a) = y'(a) = y'(b) = 0 \tag{1.3}$$

or

$$y(a) = y'(a) = y''(b) = 0. \tag{1.4}$$

We use the following transformation: Let

$$y'(x) = u(x). \tag{1.5}$$

Since $y(a) = 0$, one has

$$y(x) = \int_a^x u(t)dt. \tag{1.6}$$

Using (1.5) and (1.6) in (1.2) we have

$$u'' = f(x, \int_a^x u(t)dt, u(x), u'(x)) \tag{1.7}$$

with

$$u(a) = u(b) = 0 \tag{1.8}$$

or

$$u(a) = u'(b) = 0 \tag{1.9}$$

The existence and uniqueness of solutions of second order integro boundary value problem (1.7), (1.8) or (1.7), (1.9) assures the existence and uniqueness of solutions of the boundary value problem (1.2), (1.3) or (1.2), (1.4), respectively. So, when we state any theorem concerning equation (1.2), the proof will be given for second order integro-differential equation (1.7). We believe this method, gives us better results for such boundary conditions.

In section 2, we shall use Generalized Contraction Mapping Principle (Theorem 2.1) to discuss existence and uniqueness results for problem (1.2), (1.3) and (1.2), (1.4). In section 3, by using the successive approximations method given in [6], we have obtained unique solutions for the problems discussed in section 2. It is noted that the results obtained in section 3, indicate that the unique solution exists over a length of the (b-a) greater than that obtained in section 2, and also greater than that obtained in [6].

2. Application of Generalized Contraction Mapping Theorem.

We need the following theorem:

Theorem 2.1. Let T map a ball $B = \{w: \| w - y_0 \| \leq \mu\}$ of a complete normed linear space S into S. If there is an $\alpha \in (0,1)$ such that for all u, $v \in B$,

$$\| Tu - Tv \| \leq \alpha \| u - v \|$$

and

$$\| Ty_0 - y_0 \| \leq \mu(1 - \alpha), \tag{2.1}$$

then T has a unique fixed point y in B.

Theorem 2.2. Let $f(x,y,y',y'')$ be continuous and satisfies Lipschitz condition

$$|f(x,y,y',y'') - f(x,z,z',z'')| \leq K|y-z| + L|y'-z'| + M|y''-z''| \tag{2.2}$$

on

$$D = \{(x,y,u,v): \ a \leq x \leq b, \ |y| \leq (b-a)N, \ |u| \leq N, \ |v| \leq 4N/(b-a)\}$$

Let

$$m = \max_{a\leq x\leq b} |f(x,0,0,0)| \quad \text{and} \quad Q = \max_{D} |f(x,y,u,v)|.$$

Then if

$$K(b-a)^3/12 + L(b-a)^2/8 + M(b-a)/2 = \alpha < 1 \tag{2.3}$$

and either

$$m(b-a)^2 \leq 8N(1 - \alpha) \tag{2.4}$$

or

$$Q(b-a)^2 \le 8N, \tag{2.5}$$

boundary value problem (1.2), (1.3) has a unique solution.

Proof. Let the space S consist of continuously differentiable functions on $[a,b]$ with the norm

$$\| u \| = \max\{ \max_{a\le x\le b} |u(x)|, ((b-a)/4) \max_{a\le x\le b} |u'(x)| \}.$$

We show that problem (1.7), (1.8) has a unique solution. Define a mapping $T\colon S \to S$ by

$$(Tu)(x) = \int_a^b G(x,t)f(t,\int_a^t u(s)ds,u(t),u'(t))dt \tag{2.6}$$

where $G(x,t)$ is the Green's function for the boundary value problem

$$u'' = 0, \quad u(a) = u(b) = 0.$$

Let $u(x) = 0$ and B be the ball $\{w \in S\colon \| w-u_0 \| \le N$. Then if $u(x)$, $v(x) \in B$, we have, by (2.2) and (2.6),

$$\begin{aligned} |(Tu)(x) - (Tv)(x)| &\le \int_a^b |G(x,t)|((K(t-a)+L)\max|u-v| + M\max|u'-v'|)dt \\ &\le (K(b-a)^3\sqrt{3}/27 + 1(b-a)^2/8)\max|u-v| + (M(b-a)^2/8)\max|u'-v'| \end{aligned}$$

or

$$|(Tu)(x) - (Tv)(x)| \le (K(b-a)^3\sqrt{3}/27 + L(b-a)^2/8 + M(b-a)/2)\| u-v \|.$$

Similarly

$$(1/4)(b-a)|(Tu)'(x) - (Tv)'(x)| \le (K(b-a)^3/12 + L(b-a)^2/8 + M(b-a)/2)\| u-v \|,$$

from which it follows that

$$\| Tu - Tv\| \le \alpha\| u - v\| .$$

To apply Theorem 2.1, we need to show that (2.1) holds. If (2.4) holds, then since

$$|(Tu_0)(x)| \le \int_a^b |G(x,t)||f(t,0,0,0)|dt \le m(b-a)^2/8$$

and

$$R_2 = \{u(x): \quad |u(x)-u_0(x)| \le Q(x-a)(b-x)/2(1-\alpha)\}$$

$$R_3 = \{v(x): \quad |v(x)-u'_0(x)| \le Q((x-a)^2 + (b-x)^2)/2(1-\alpha)(b-a)\}$$

where $u_0(x)$ is defined by (3.1), $y_0(x) = \int_a^x u_0(t)dt$, and

$$Q = \max_{a\le x\le b} |f(x,\int_a^x u_0(t)dt, u_0(x), u'_0(x))|$$

and

$$K(b-a)^3/20 + L(b-a)^2 5/48 + M(b-a)/3 = \alpha < 1. \tag{3.2}$$

Then boundary value problem (1.2), (1.3) has a unique solution.

Proof. We show that boundary value problem (1.7), (1.8) has a unique solution. From (3.1) we have

$$|u_1(x)-u_0(x)| \le \int_a^b |G(x,t)||f(t,\int_a^t u_0(s)ds, u_0(t), u'_0(t))|dt < \frac{Q}{2}(x-a)(b-x) \tag{3.3}$$

and

$$|u'_1(x)-u'_0(x)| \le \int_a^b |G_x(x,t)||f(t,\int_a^t u_0(s)ds, u_0(t), u'_0(t))|dt < \frac{Q}{2(b-a)}[(x-a)^2+(b-x)^2]. \tag{3.4}$$

Hence, $u_1(x) \in D$. Now using Lipschitz condition, we have

$$|u_2(x)-u_1(x)| \le \int_a^b |G(x,t)|[K(t-a)+L)|u_1(t)-u_0(t)| + M|u'_1(t)-u'_0(t)|]dt \tag{3.5}$$

and

$$|u'_2(x)-u'_1(x)| \le \int_a^b |G_x(x,t)|[K(t-a)+L)|u_1(t)-u_0(t)| + M|u'_1(t)-u'_0(t)|]dt. \tag{3.6}$$

We need the following estimates:

$$\int_a^b |G(x,t)||u_1(t)-u_0(t)|(t-a) \le \frac{Q}{2}(b-x)(x-a)[\frac{1}{20}(b-a)^3] \tag{3.7}$$

(3.8) $$\int_a^b |G(x,t)||u_1(t)-u_0(t)|dt \leq \frac{1}{2} Q(b-x)(x-a)[\frac{5}{48}(b-a)^2]$$

(3.9) $$\int_a^b |G(x,t)||u_1'(t)-u_0'(t)|dt \leq \frac{1}{2} Q(b-x)(x-a)[\frac{1}{3}(b-a)]$$

(3.10) $$\int_a^b |G_x(x,t)||u_1(t)-u_0(t)|(t-a)dt \leq \frac{1}{2(b-a)} Q[(x-a)^2+(b-x)^2][\frac{1}{20}(b-a)^3]$$

(3.11) $$\int_a^b |G_x(x,t)||u_1(t)-u_0(t)|dt \leq \frac{1}{2(b-a)} Q[(x-a)^2+(b-x)^2][\frac{5}{48}(b-a)^2]$$

(3.12) $$\int_a^b |G_x(x,t)||u_1'(t)-u_0'(t)|dt \leq \frac{1}{2(b-a)} Q[(x-a)^2+(b-x)^2][\frac{1}{3}(b-a)].$$

Using the above inequalities we obtain

(3.13) $$|u_2(x)-u_1(x)| \leq \frac{\alpha}{2} Q(b-x)(x-a)$$

(3.14) $$|u_2'(x)-u_1'(x)| \leq \frac{\alpha}{2(b-a)} Q[(x-a)^2 + (b-x)^2].$$

Continuing this way, one has

(3.15) $$|u_{n+1}(x)-u_n(x)| \leq \frac{\alpha^n}{2} Q(b-x)(x-a)$$

(3.16) $$|u_{n+1}'(x)-u_n'(x)| \leq \frac{\alpha^n}{2(b-a)} Q[(x-a)^2 + (b-x)^2].$$

Then from these inequalities we successively have

(3.17) $$\begin{aligned} |u_{n+1}(x)-u_0(x)| &\leq |u_{n+1}(x)-u_n(x)| + \dots + |u_1(x)-u_0(x)| \\ &\leq (\alpha^n+\alpha^{n-1} + \dots + \alpha+1)\frac{1}{2} Q(b-x)(x-a) \\ &\leq \frac{1}{2(1-\alpha)} Q(b-x)(x-a). \end{aligned}$$

Similarly

(3.18) $$|u_{n+1}'(x)-u_0'(x)| \leq Q((b-x)^2 + (x-a)^2)/2(1-\alpha)(b-a)$$

By the definition of D, this means that $u_{n+1}(x) \in D$. Also since $\alpha < 1$, estimates (3.15) and (3.16) ensure that the sequence $\{u_n(x)\}$ converges to a limit, say $u(x)$. Hence, we have proved that the boundary value problem (1.7), (1.8) has at least one solution in D.

$$|(Tu_0)'(x)| \leq \int_a^b |G_x(x,t)||f(t,0,0,0)|dt \leq m(b-a)/2,$$

we have

$$\| Tu_0 - u_0 \| \leq N(1 - \alpha),$$

which is (2.1).

Next, suppose (2.5) holds. For any $u \in B$, we have

$$|u(x)| \leq N \quad \text{and} \quad |u'(x)| \leq 4N/(b-a),$$

hence by the hypothesis,

$$|f(x, \int_a^x u(t)dt, u(x), u'(x))| \leq Q.$$

It follows that

$$|(Tu)(x)| \leq \int_a^b |G(x,t)||f(t, \int_a^t u(s)ds, u(t), u'(t))|dt \leq Q(b-a)^2/8 \leq N$$

and

$$|(Tu)'(x)| \leq \int_a^b |G_x(x,t)||f(t, \int_a^t u(s)ds, u(t), u'(t))|dt \leq Q(b-a)/2,$$

hence

$$\| Tu \| \leq N.$$

Thus, if either (2.4) or (2.5) holds, Theorem 2.1 applies and the proof is complete.

Theorem 2.3. Suppose all conditions of Theorem 2.2 hold true on

$$D = \{(x,y,u,v): \; a \leq x \leq b, \; |y| \leq (b-a)N, \; |u| \leq N, \; |v| \leq 3\sqrt{3}\, N/(b-a)\}$$

and (2.3) is replaced by

(2.7) $$K(b-a)^3\sqrt{3}/27 + L(b-a)^2/8 + M(b-a)3\sqrt{3}/8 = \alpha < 1.$$

Then boundary value problem (1.2), (1.3) has a unique solution. Proof is the same as the proof of Theorem 2.2.

Remark 2.1. Conditions (2.3) and (2.7) are not comparable.

Theorem 2.4. Let f satisfy the Lipschitz condition on

$$D = \{(x,y,u,v): \ a \le x \le b, \ |y| \le (b-a)N, \ |u| \le N, \ |v| \le 2N/(b-a)\}.$$

Let

$$m = \max_{a\le x\le b} |f(x,0,0,0)| \quad \text{and} \quad Q = \max_{D} |f(x,y,u,v)|.$$

Then if

$$K(b-a)^3/3 + L(b-a)^2/2 + M(b-a) = \alpha < 1 \tag{2.8}$$

and either

$$m(b-a)^2 \le 2N(1-\alpha) \tag{2.9}$$

or

$$Q(b-a)^2 \le 2N \tag{2.10}$$

the boundary value problem (1.2), (1.4) has a unique solution. The proof is similar to that of Theorem 2.2.

3. Convergence of Successive Approximations.

In this section we use successive approximations defined as

$$u_0(x) = 0, \quad u_{n+1}(x) = \int_a^b G(x,t) f\left(t, \int_a^t u_n(s)ds, u_n(t), u'_n(t)\right)dt \tag{3.1}$$

To find existence and uniqueness results for the boundary value problems defined in section 2. In (3.1) $G(x,t)$ is the Green's function for the concerned homogeneous boundary problem.

Theorem 3.1. Let f be continuous and satisfy Lipschitz condition (2.2) on

$$D = \{(x,y,u,v): \ a \le x \le b, \ y \in R_1, \ u \in R_2, \ v \in R_3\}$$

with

$$R_1 = \{y(x): \ |y(x)-y_0(x)| \le Q(x-a)^2(3b-a-2x)/12(1-\alpha)\}$$

Now, we shall prove that the solution is unique. Assume that there are two solutions $u(x)$ and $v(x) \in D$, then

$$(3.19)\quad |u(x)-v(x)| \le \int_a^b |G(x,t)|[K(t-a)+L)|u(t)-v(t)|+M|u'(t)-v'(t)|]dt.$$

Let

$$P = \max_{a \le t \le b} (K(b-a)|u(t)-v(t)| + L|u(t)-v(t)| + M|u'(t)v'(t)|)$$

then

$$|u(x)-v(x)| \le P(b-x)(x-a)/2$$
$$|u'(x)-v'(x)| \le P((x-a)^2 + (b-x)^2)/2(b-a)$$

From (3.19) and above results, we find

$$|u(x)-v(x)| \le \alpha P(b-x)(x-a)/2$$
$$|u'(x)-v'(x)| \le \alpha P((x-a)^2 + (b-x^2)/2(b-a)$$

Continuing this way, we get

$$|u(x)-v(x)| \le \alpha^n P(b-x)(x-a)$$
$$|u'(x)-v'(x)| \le \alpha^n P((x-a)^2 + (b-x)^2)/2(b-a).$$

Since $\alpha < 1$, the results follows immediately.

Theorem 3.2. Let f be continuous and satisfy the Lipschitz condition (2.2) on

$$D = \{(x,y,u,v):\ a \le x \le b,\ y \in R_1,\ u \in R_2,\ v \in R_3$$

with

$$R_1 = \{y(x):\ |y(x)-y_0(x)| \le Q(x-a)^2(12b-7a-5x)\}$$
$$R_2 = \{u(x):\ |u(x)-u_0(x)| \le Q(x-a)(2b-a-x)/2(1-\beta)\}$$
$$R_3 = \{v(x):\ |v(x)-u_0'(x)| \le Q(b-x)/(1-\beta)\}$$

and

$$(3.20)\qquad K(b-a)^3 3/10 + L(b-a)^2 5/12 + M(b-a)/2 = \beta < 1.$$

Then boundary value problem (1.2), (1.4) has a unique solution. Proof is similar to that of Theorem 3.1.

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