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ON STABILITY PROPERTIES WHICH ARE EQUIVALENT TO RIEMANN HYPOTHESIS

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1. Introduction. In a previous paper [1] it has been shown that the location of the zeros of Riemann's zeta function is related to the rate of growth of the solutions of some dynamical systems of interest in control theory. The purpose of the present note is to introduce a class of systems whose stability (in a global sense, defined below) is equivalent to Riemann Hypothesis.

As in [1], the systems studied can serve as mathematical models for some new types of control systems which are made actual by today's technology. These mathematical models are of interest in system theory, where they raise new, challenging problems. Finding properties which are equivalent to Riemann Hypothesis is still another actual direction of research (see e.g. [2] for a recent contribution) and further progress in this respect is likely to lead to interactions of different fields.

2. The system. Given $\theta > 0$, introduce the linear operator

$$(Lx)(t) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\theta}} x(t-\log n) - \int_{-\infty}^{t} e^{(1-\theta)(t-\tau)} x(\tau)d\tau$$

$$- \int_{-\infty}^{t} \sum_{n=1}^{\infty} \frac{e^{-(2n+\theta)(t-\tau)}}{2n} e^{-\theta\tau} \frac{d}{d\tau}(e^{\theta\tau}x(\tau))d\tau + c\, x(t) \qquad (1)$$

where Λ is the "von Mangoldt's function" ($\Lambda(n) = \log p$ if n is equal to a prime p or to a power of the prime p and $\Lambda(n) = 0$ otherwise) and where

$$c = \frac{1}{2} (\log \pi + \gamma) \tag{2}$$

in which γ is the Euler-Mascheroni constant.

The domain of L will be defined as follows. Let t_0 be a fixed, strictly negative number. Let X_0 be the set of all C^1 functions from $(-\infty,0]$ into R, such that $x(t) = 0$ for all $t \leq t_0$. Let X be the set of all continuous functions from R into R whose restrictions to $(-\infty,0]$ are in X_0 and whose restrictions to $[0,\infty)$ are of class C^1. If $x \in X$ then Lx is well defined by (1).

Notation. If $x \in X$, then the restriction of x to X_0 will be denoted by x_0.

Let f be a globally Lipschitzian function from R into R and let $\lambda > 1$. The system studied has the form

$$(Lx)(t) = \lambda \dot{x}(t) + f(x(t)), \quad t > 0 \tag{3}$$

$$x(t) = \phi(t), \quad t \leq 0. \tag{4}$$

Here ϕ is interpreted as an initial function.

A solution of (3), for the initial condition (4) is a function x such that Lx is defined and equations (3) and (4) are satisfied. It can be shown - e.g. by the contraction mapping technique - that this system has a unique solution $x \in X$ for every $\phi \in X_0$.

3. The property of stability. If $\phi \in X_0$, let

$$\|\phi\| = \sup_{t \in [t_0,0]} (\max (|\phi(t)|,|\dot{\phi}(t)|).$$

If $\phi = 0$ then the system has the trivial solution $x = 0$. The problem is to study the stability of the trivial solution in the following sense:

<u>Definition</u>. The trivial solution $x = 0$ of (3), (4) is said to be <u>λ-stable</u> if: (i) for every $\phi \in X_0$ and every $\lambda > 1$ the corresponding solution $x \in X$ of (3), (4) is bounded and (ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $\lambda > 1$ and for every $\phi \in X_0$ which satisfies the condition

$$\|\phi\| < \delta$$

the corresponding solution x of (3),(4) satisfies the condition

$$|x(t)| < |x(0)| + \frac{\varepsilon}{\sqrt{\lambda}}, \quad \text{for every} \quad t > 0. \tag{5}$$

4. <u>The equivalence</u>. Let $\Theta = \sup\{\text{Re}\,\rho \,|\zeta(\rho) = 0\}$.

<u>Theorem</u>. Let $\theta > 0$. The following properties of θ are equivalent:

1^0 $\theta \geq \Theta$.

2^0 There exists $k > 0$ such that, for every $x \in X$,

$$\int_0^t x(\tau)(Lx)(\tau)d\tau \leq k\,\|x_0\|^2, \quad \text{for every} \quad t > 0.$$

3^0 For every globally Lipschitzian function f satisfying the condition "$f(r)\, r \geq 0$ for every real r" the trivial solution of (3),(4) is λ-stable.

4^0 For $f = 0$ the trivial solution of (3),(4) is λ-stable.

<u>Remarks</u>. (i) It is known that $1/2 \leq \Theta \leq 1$. Therefore from the theorem it follows that the trivial solution of (3),(4) is λ-stable if $\theta \geq 1$, if f satisfies the conditions specified. Moreover, if $\theta < 1/2$ then, under the same conditions on f, the trivial solution is certainly not λ-stable. Riemann Hypothesis is equivalent to the statement that the trivial solution of (3),(4) (e.g., with $f = 0$) is λ-stable for $\theta = 1/2$: the most optimistic conjecture. This is a very strong property of stability, which has not been contradicted so

far by any theoretical or experimental result. It is the plausibility of such strong properties of stability which makes these systems of special interest in system theory.

(ii) Property 3^0 has been introduced in order to show that the result is not restricted to linear systems. Property 2^0, in conjunction with the hyperstability approach [3] allows to find other nonlinear stability properties which are equivalent to Riemann Hypothesis.

5. The proof. The proof uses the remark that, if $\phi \in X_0$, the system can also be written in the equivalent "spectral form"

$$\mu(t) = \int_{t_0}^{t} x(\tau)d\tau \tag{6}$$

$$\dot{\nu}_\rho = (\rho-\theta)\nu_\rho + \mu(t) \tag{7}$$

$$\nu_\rho(t_0) = 0 \tag{8}$$

$$h_\rho(t) = -\frac{1}{2}\nu_\rho(t) - \frac{1}{2}\nu_{1-\rho}(t), \quad t > 0 \tag{9}$$

$$\lambda\, x(t) - \lambda\, x(0) + \int_0^t f(x(\tau))d\tau = \sum_\rho (h_\rho(t) - h_\rho(0)), \quad t > 0 \tag{10}$$

$$x(t) = \phi(t), \quad t \leq 0, \tag{11}$$

where ρ runs over all the nonreal zeros of ζ. (The series in (10) is absolutely convergent, uniformly on bounded intervals; there exist simpler forms of the system but they raise convergence problems.) This equivalence can be established by showing that, for every bounded $x \in X$ (and hence also for all $x \in X$)

$$\int_{t_0}^{t} (Lx)(\tau)d\tau = \sum_\rho h_\rho(t), \quad \text{for every } t > t_0. \tag{12}$$

This in turn follows from the fact that the members of (12) represent two continuous functions which have equal Laplace transforms for sufficiently large Re s. (The equality of these Laplace transforms follows from well known identities in the theory of Riemann's zeta function; e.g. [4])

From the new form of the system and from (12) the implication $1^0 \to 2^0$ easily follows. Moreover $2^0 \to 3^0$ is straightforward and $3^0 \to 4^0$ is obvious. (In a slightly different variant, the above part of the proof can be handled by the same approach as in [1].)

The implication $4^0 \to 1^0$ is proved by contradiction. Suppose that $\Theta > \theta$ and that 4^0 is true. Let ρ_0 be a zero of ζ such that $\mathrm{Re}\,\rho_0 - \theta > (\Theta-\theta)/2$. Let δ be the number from the definition in section 3, corresponding to $\varepsilon = 1$. One can find $x_0 \in X_0$ such that the conditions $|\nu_{\rho_0}(0)| > 0$, $\|x_0\| < \delta$, $x_0(0) = 0$ and $\mu(0) = 0$ are simultaneously satisfied. Then 4^0 implies that for every $\lambda > 1$,

$$|x(t)| < 1/\sqrt{\lambda}, \quad \text{for every} \quad t > 0. \tag{13}$$

From (10), with $f = 0$, it now follows that

$$|\sum_\rho h_\rho(t)| < \sqrt{\lambda} + |\sum_\rho h_\rho(0)| \tag{14}$$

and from (6) and $\mu(0) = 0$, that

$$|\mu(t)| < t/\sqrt{\lambda}, \quad \text{for every} \quad t > 0. \tag{15}$$

Let now $\varepsilon > 0$ be such that

$$\mathrm{Re}\,\rho_0 - \theta - \varepsilon > (\Theta - \theta + \varepsilon)/2. \tag{16}$$

From (3), with $f = 0$, it follows that there exists $k_0 > 0$ such that

$$\lambda\,|\dot{x}(t)| \le k_0\, e^{(\Theta-\theta+\varepsilon)t}\Big(\sup_{\tau \in [t_0,t]} |x(\tau)| + \sup_{\tau \in [t_0,t]} |\dot{x}(\tau)|\Big)$$

(this result is obtained by writing the sum in (1) as a Stieltjes integral, in terms of the function $\psi(x) = \sum_{n \le x} \Lambda(n)$ and by using the known estimates for

$\psi(x)-x$ in terms of Θ; e.g. [5], theorem 30).

It follows that there exists $k_1 > 0$ such that the condition

$$k_0 e^{(\Theta-\theta+\varepsilon)t} < \lambda/2 \tag{17}$$

implies that

$$\lambda \, |\dot{x}(t)| < k_1 \, e^{(\Theta-\theta+\varepsilon)t}. \tag{18}$$

The solution of (7), (8) is written as

$$\nu_\rho(t) = \chi_\rho(t) + \xi_\rho(t) \tag{19}$$

where

$$\chi_\rho(t) = e^{(\rho-\theta)t}\nu_\rho(0) \tag{20}$$

$$\xi_\rho(t) = \int_0^t e^{(\rho-\theta)(t-\tau)}\mu(\tau)d\tau, \quad \text{for every} \quad t > 0. \tag{21}$$

Integrating by parts twice in (21) and using the estimates (18), (13), (15), one finds that there exists $k_2 > 0$ such that

$$\left|\frac{1}{2}\sum_\rho (\xi_\rho(t) + \xi_{1-\rho}(t))\right| \le k_2\left(\frac{t+1}{\sqrt{\lambda}} + \frac{1}{\lambda}\, e^{(\Theta-\theta+\varepsilon)t}\right) \tag{22}$$

as long as λ and t satisfy (17).

On the other hand, since $\nu_{\rho_0}(0) \neq 0$, one can find a sequence t_n such that $\lim_{n\to\infty} t_n = \infty$ and

$$\left|\sum_\rho \chi_\rho(t_n)\right| > e^{(\mathrm{Re}\,\rho_0-\theta-\varepsilon)t_n} \tag{23}$$

(this can be shown by an argument which is similar to, but simpler than the proof of Theorems 32 and 33 in [5]).

Now from (14) (19), (22) and (23) it follows that there exists $k_3 > 0$ such that

$$e^{(\mathrm{Re}\,\rho_0-\theta-\varepsilon)t_n} < \sqrt{\lambda} + k_3\left(1 + \frac{t+1}{\sqrt{\lambda}} + \frac{1}{\lambda}\, e^{(\Theta-\theta+\varepsilon)t_n}\right) \tag{24}$$

provided (17) is satisfied by λ, for t replaced by t_n.

The proof ends with the remark that, by virtue of (16), one can find $\lambda > 1$ and t_n such that (17) holds but relation (24) is violated.

REFERENCES

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