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ULTIMATE BEHAVIOR IN CERTAIN NONLINEAR FEEDBACK SYSTEMS

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In this paper we are concerned with the ultimate behavior of solutions of certain nonlinear integrodifferential systems of the form

$$(1)\qquad \begin{aligned} \dot{x}(t) &= (Ax)(t) + C\dot{y}(t) , \\ \dot{y}(t) &= Dx(t) + f(y(t)) , \end{aligned}$$

where $x \in R^n$, $y \in R^m$, C and D are constant matrices of convenient dimensions , $f: R^m \to R^m$, and A stands for an operator that can be (formally) represented by means of the formula

$$(2)\qquad (Ax)(t) = \sum_{j=o}^{\infty} A_j x(t - t_j) + \int_o^t B(t - s)x(s)ds , \quad t \in R_+ ,$$

where t_j , $j = 0,1,2,\ldots$, are nonnegative numbers , and A_j , B are such that

$$(3)\qquad \sum_{j=o}^{\infty} \|A_j\| < + \infty , \quad \|B(t)\| \in L^1(R_+) .$$

A somewhat more general system will be also dealt with , namely

$$(4)\qquad \begin{aligned} \dot{x}(t) &= (Ax)(t) + C(t)\dot{y}(t) , \\ \dot{y}(t) &= Dx(t) + f(y(t)) , \end{aligned}$$

whoch differs with respect to the system (1) by assuming that C depends on t . Therefore , (4) is a time varying system , while (1) is time invariant .

In order to investigate the ultimate behavior of solutions of the systems (1) or (4) , we shall transform these systems in equivalent integrodifferential systems which also involve the initial data . Such approach has been already used by J. Moser [5] (see also [1] , Theorem 7.1) . The operator A has been the usual multiplication by a constant matrix . In [3], [4] Moser's result has generalized in order to cover time varying systems .

While the original Moser's result is applicable to the case of systems of the form (1) , after reduction to an integrodifferential equation , the investigation of ultimate behavior fot systems of the form (4) will be based on the application of our result in [3] for time varying systems (with nonconvolution kernel) .

Let us notice that the initial data to be associated with (1) or (4) must be of the form

(5) $$x(t) = h(t)\ ,\ t < 0\ ,\quad x(0) = x^o \in R^n\ ,\quad y(0) = y^o \in R^m\ .$$

Further conditions will be specified in regard to h(t) .

We shall first consider the system (1) , under initial data (5) . Under rather general assumptions on h , say $h \in L^1(R_-)$, the first equation of the system (1) leads to the following integral formula [2] :

(6) $$x(t) = X(t)x^o + (Yh)(t) + \int_o^t X(t-s)C\dot{y}(s)ds\ ,\ t \in R_+\ .$$

where X(t) is the unique (a.e.) solution of $\dot{X}(t) = (AX)(t)$, $t > 0$, $X(0+) = I$ (the unit matrix) , $X(t) = 0$ (the zero matrix) for $t < 0$, and the operator Y is defined by the formula

(7) $$(Yh)(t) = \sum_{j=o}^{\infty} \int_{-t_j}^{o} X(t-t_j-u)A_jh(u)du\ ,\ t \in R_+\ .$$

The convergence of the series in the riht hand side of (7) is uniform on any compact interval of R_+ , and in case of asymptotically stable systems (i.e. $\dot{x}(t) = (Ax)(t)$ is asymptotically stable) one obtains uniform convergence on R_+ (because stability is equivalent to $X(t) \in L^1(R_+)$ [2]) .

If x(t) given by (6) is substituted in the second equation (1) , we obtain

(8) $$\dot{y}(t) = \int_o^t DX(t-s)C\dot{y}(s)ds + f(y(t)) + g(t)\ ,\ t \in R_+\ .$$

where

(9) $$g(t) = DX(t)x^o + D(Yh)(t)\ ,\ t \in R_+\ .$$

Of course , we have to associate with the equation (8) the initial condition in (5) related to y , namely $y(0) = y^o \in R^m$.

The integrodifferential equation (8) is of the type investigated by Moser in [5] (see also [1] , Theorem 7.1) , and the only fact that has to be accomplished in order to obtain the result on ultimate behavior is to check the validity of the conditions required by Moser's result . Of course , such conditions will be satisfied only if we make the right assumptions on the system (1) .

First of all , the kernel of the equation (8)

$$k(t) = DX(t)C \ , \ t \in R_+ \ , \tag{10}$$

has to belong to $L^2(R_+)$. Since C and D are by assumption constant matrices , it will suffice to assure $\|X(t)\| \in L^2(R_+)$. This last condition holds true for systems $\dot{x}(t) = (Ax)(t)$ whic are asymptotically stable [2] .

Another condition related to the kernel is the frequency domain condition which we shall write slightly differently than it appears in [5] ,

$$\mathrm{Re}\ (\ \|\eta\|^2 + \langle \bar{\eta}\ ,\ \tilde{k}(s)\eta \rangle) \geq \delta\|\eta\|^2 \ ,$$

for any real s , and for any complex n-vector η . Taking into account (10) , as well as the formula [2]

$$\tilde{X}(s) = [\,sI - A(s)]^{-1} \ , \ \mathrm{Re}\ s \geq 0 \ ,$$

where

$$A(s) = \sum_{j=o}^{\infty} A_j \exp(-t_j s) + \int_o^{\infty} B(t)\exp(-ts)dt \ , \tag{11}$$

one obtains

$$\mathrm{Re}\ \langle \bar{\eta}\ ,\ D[\,isI - A(is)]^{-1}\eta \rangle \geq \delta\|\eta\|^2 \ , \quad s \in R \ , \tag{12}$$

for some $\delta < 1$, and any complex n-vector η .

One more condition to be checked is

$$g(t) \in C_o(R_+) \cap L^2(R_+) \ , \tag{13}$$

with g(t) given by (9) . Since $X(t) \in C_o(R_+) \cap L^2(R_+)$ [2] , it remains to prove the same inclusion for (Yh)(t) given by (7) . If we notice that

$$(Yh)(t) = \sum_{j=o}^{\infty} \int_o^{t_j} X(t-u)A_j h(u-t_j)du \ , \quad t \in R_+ \ ,$$

the stability of the system $\dot{x}(t) = (Ax)(t)$, $h(t) \in L^1(R_+)$, and the first condition (3) lead easily to the needed result .

Finally , in regard to f(y) we have only to state the hypotheses under which Moser's result has been obtained :

$$f(y) = -\ \mathrm{grad}\ U(y) \ , \ y \in R^m \ , \tag{14}$$

where $U \in C^{(1)}(R^m, R)$, and

(15) $$\lim U(y) = \infty \ , \quad \text{as} \quad \|y\| \to \infty \ .$$

Summarizing the above discussion in regard to the system (1) , we can state the following result :

Theorem 1 . Consider the system (1) , under conditions (2) , (3) , and the initial condition (5) , and further assume :

a) the system $\dot{x}(t) = (Ax)(t)$ is asymptotically stable , i.e.

(16) $$\det[sI - A(s)] \neq 0 \quad \text{for} \quad \operatorname{Re} s \geq 0 \ ;$$

b) there exists a real number δ , $\delta < 1$, such that condition (12) holds true for any complex n-vector η ;

c) $f : R^m \to R^m$ is given by (14) , with $U \in C^{(1)}(R^m, R)$ satisfying condition (15) ;

d) $h(t) \in L^1(R_-)$.

Then any solution of system (1) , satisfying (5) , is bounded on R_+ and its limit set coincides with the limit set of a convenient solution of the ordinary differential system

(17) $$\dot{z}(t) = f(z(t)) \ .$$

We shall proceed in the same manner with the time varying system (4) , but instead of Moser's result we the result we have obtained in [3 ; Theorem 1] .

Let us notice , first of all , that the integrodifferential equation equivalent to (4) , (5) will be now

(18) $$\dot{y}(t) = \int_0^t DX(t-s)C(s)\dot{y}(s)ds + f(y(t)) + g(t) \ ,$$

with exactly the same meaning as in (8) for $X(t)$ and $g(t)$. Of course , one has to substitute $x(t)$ given by (6) in the second equation of (4) , in order to ob - tain (18) . The kernel of the integrodifferential equation (18) is

(19) $$k(t,s) = DX(t-s)C(s) \ ,$$

and as one can see it is a nonconvolution kernel if $C(s) \neq \text{Const.}$

We have to check now the validity of the conditions in Theorem 1 in [4], in case of the kernel given by (19) . The other conditions , regarding $f(y)$ and $g(t)$, do not require different considerations than those already encountered in this paper (in view of obtaining Theorem 1) . More precisely , we need to show that the kernel $k(t,s)$ given by (19) verifies the following conditions listed in the statement of Theorem 1 in [4] :

1) there exists $A > 0$, such that

(20) $$\int_0^t \|k(t,s)\|^2 ds \le A^2 \text{ , for any } t \in R_+ \text{ ;}$$

2) $$\lim \int_0^T \|k(t,s)\|^2 = 0 \text{ , as } t \to \infty \text{ ;}$$

for every fixed $T > 0$;

3) for every $t \in R_+$, one has

(21) $$\lim_{h \to 0} \left(\int_0^{t+h} \|k(t+h,s)\|^2 ds + \int_0^t \|k(t+h,s) - k(t,s)\|^2 ds\right) = 0 \text{ ;}$$

4) there exists $\delta < 1$, such that

(22) $$\int_0^t \langle (Kx)(s) + \delta x(s) , x(s) \rangle \, ds \ge 0 \text{ ,}$$

for every $x \in L^2(R_+)$, and $t \in R_+$, where K stands for the Volterra linear operator generated by the kernel $k(t,s)$:

(23) $$(Kx)(t) = \int_0^t k(t,s)x(s)ds \text{ .}$$

Let us proceed now in checking that , under adequate conditions on (4) , the kernel $k(t,s)$ given by (19) verifies all four conditions stated above .

First of all , we will make the assumption that $C(t)$ is bounded on R_+ :

(24) $$\|C(t)\| \le C_o \text{ , } t \in R_+ \text{ .}$$

If we admit again , as in case of Theorem 1 above , that the system $\dot{x}(t) = (Ax)(t)$ is asymptotically stable , then $X(t) \in L^2(R_+)$. For details see Moser [5] or [3 ; Lemma 1] . This property of $X(t)$, together with (24) , immediately takes care of conditions 1) and 2) formulated above .

Property 3) for the kernel $k(t,s)$ given by (19) also follows from the fact $X(t) \in L^2(R_+)$, and condition (24) imposed on $C(t)$, the details bein left to the reader .

Finally , condition (4) has to be imposed on the kernel $k(t,s)$ given by (19) . As remerked earlier [3] , this condition simply states that the operator $K + \delta I$ is a positive (or monotone) operator on the Hilbert space $L^2(R_+)$.

We are now prepared to state the result concerning the ultimate behavior of solutions of the time varying system (4) .

Theorem 2 . Consider the system (4) , under conditions (2) and (3) for the operator A , with initial data (5) , and such that (24) takes place for the continuous matrix C(t) . We will further assume :

a) the same as in Theorem 1 above ;

b) $k(t,s)$ given by (19) verifies the positivness condition (22) , for some real number $\delta < 1$;

c) and d) the same as in Theorem 1 above .

Then any solution of the problem is bounded on the positive half-axis , and its limit set agrees with the limit set of a convenient solution of the ordinary differential system (17) .

Both Theorems 1 and 2 provide conditions under which the asymptotic behavior of solutions of some integrodofferential equations (with infinite delay) can be determined by the asymptotic behavior of solutions of certain ordinary differential systems . How general is this situation ? Or how representative is it for the whole class of integrodifferential equations in finite dimensional spaces ?

If we take into consideration the fact that the trajectories of integrodifferential equations are curves in the Euclidean space R^n , as well as the trajectories of the ordinary differential systems (of same dimension) such as (17) , it would not be , perhaps , very surprising to find out that for wider classes of integrodifferential equations the limit behavior of solutions can be adequately investigated by means of certain associated ordinary differential equations .

Without trying now to answer the above problem , let us mention that somewhat more general results than those obtained above could be derived if one uses more sophisticated tools of investigation . For instance , if one considers the system similar to (1)

$$\dot{x}(t) = \int_{-\infty}^{0} [\, d\eta(s)]\, x(t+s) + C\dot{y}(t) \quad , \quad \dot{y}(t) = Dx(t) + f(y(t)) \, , \tag{25}$$

where $\eta(s)$ stands for a matrix-valued function with bounded variation on R_- , while other quantities have the same meaning as in (1) , then the approach used above can be applied successfully . Indeed , the system (25) is more general than the system (1) by the fact that the measure η can have a singular part . But variation of constants formula are available for such kind of system [6] , [7] , and the stability problem is also solved . This approach will require the use of semigroup theory .

Let us also notice that the case of time varying systems which can be obtained from (25) by considering $C = C(t)$, can be also dealt with in the same manner we dealt with the system (4) .

In regard to the case when more complicated time varying system are considered , say for instance

$$\dot{x}(t) = A(t)x(t) + C(t)\dot{y}(t) \quad , \quad \dot{y}(t) = D(t)x(t) + f(t,y(t)) \ , \tag{26}$$

some results in [3] and [4] can be applied . Nevertheless , this case is much more difficult than the cases encompassed by Theorems 1 and 2 above , and only further study of the nonconvolution type integrodifferential equations will probably lead toward more satisfactory solutions .

R E F E R E N C E S

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