LIBERTAS MATHEMATICA, VOL. 5, 1985

SHARP ERROR BOUNDS FOR A CLASS OF NEWTON-LIKE METHODS

Florian-Alexandru Potra

Abstract. We give sufficient convergence conditions as well as sharp error bounds for an iterative procedure which generalizes a wide class of known iterative methods for solving nonlinear equations in Banach spaces, among which Newton's method and the secant method.

1. Introduction.

Let f be a nonlinear operator defined on a convex subset D of a Banach space X and with values in a Banach space Y. A lot of iterative methods for solving the equation $f(x) = 0$ can be written under the form:

(1) $$x_{n+1} = x_n - T_n f(x_n) \qquad n=0,1,2,\ldots$$

where for each $n \in Z_+$, T_n is a bounded linear operator from Y into X (i.e. $T_n \in L(Y,X)$). The best known methods of type (1) are Newton's method, where $T_n = f'(x_n)^{-1}$, and the secant method, where $T_n = \delta f(x_n,x_{n-1})^{-1}$, δf being a consistent approximation of the Fréchet derivative of f. One has remarked that from the point of view of the numerical efficiency it is not advantageous to change the operator T_n at each step of the iterative algorithm. By keeping it piecewise constant one obtains more efficient iterative procedures. Optimal recepts can be prescribed according to the dimension of the space (see [21]). Such iterative procedures have been investigated by Traub [22], Schmidt and

Schwetlick [21], Bosarge and Falb [1], [2], Dennis [3], [4], etc.

In the present paper we will study the iterative procedure (1) in case for each n T_n can be taken either $\delta f(x_{p_n},x_{q_n})^{-1}$ or $\delta f(x_{q_n},x_{p_n})^{-1}$, i.e.:

(2) $$T_n \in \{\delta f(x_{p_n},x_{q_n})^{-1},\ \delta f(x_{q_n},x_{p_n})^{-1}\} \quad n=0,1,2,\ldots$$

where $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ are two nondecreasing sequences of integers satisfying the conditions

(3) $$q_o = -1,\ p_o = 0,\ q_n \leq p_n \leq n, \quad n=1,2,3,\ldots$$

All the iterative procedures mentioned above can be obtained from (1) - (2) with an appropriate choice of the sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$. A semilocal analysis of the iterative procedure (1) - (2) was given in [13]. In what follows we will prove that the convergence and uniqueness conditions obtained in that paper are optimal and that the error estimates obtained there are better than those given in the literature for different particular cases.

2. Convergence conditions and error estimates

In this section we shall study the iterative procedure (1) - (2) for triplets (f,x_o,x_{-1}) belonging to the class $C(a,b,c)$ defined as follows:

Definition 1. Let $a > 0$, $b \geq 0$, $c \geq 0$ be three real numbers satisfying the inequality

(4) $$ac + 2 \cdot \sqrt{ab} \leq 1.$$

We say that a triplet (f,x_o,x_{-1}) belongs to the class $C(a,b,c)$ if:

c_1) f is a nonlinear operator defined on a convex subset D of a Banach space X and with values in a Banach space Y.

c_2) x_o and x_{-1} are two points belonging to the interior $\mathring{D}$ of D and satisfying the inequality

(5) $$\|x_o-x_{-1}\| \leq c.$$

c_3) f is Fréchet differentiable on $\mathring{D}$ and there exists a mapping $\delta f\colon \mathring{D}\times\mathring{D} \to L(X,Y)$ such that:

(6) the linear operator P_o, where P_o is either $\delta f(x_o,x_{-1})$ or $\delta f(x_{-1},x_o)$, is invertible, its inverse $T_o = P_o^{-1}$ is bounded and:

(7) $$\|T_o f(x_o)\| \leq b;$$

(8) $\|T_o(\delta f(x,y)-f'(z))\| \leq a(\|x-z\| + \|y-z\|)$ for all $x,y,z \in D$.

c_4) the set $D_c = \{x \in D;\ f$ is continuous at $x\}$ contains the closed ball U with center $x_1 = x_o - T_o f(x_o)$ and radius $r_1 = \frac{1}{2a}[1-a(2b+c) - \sqrt{(1-ac)^2-4ab}]$.

It is convenient to associate with the class $C(a,b,c)$ the constant d and the sequence $(t_n)_{n\geq -1}$ given by the formulae:

(9) $$d = \frac{1}{2a}\sqrt{(1-ac)^2-4ab}$$

(10) $$t_{-1} = \frac{1+ac}{2a},\quad t_o = \frac{1-ac}{2a},\quad t_{n+1} = t_n - \frac{t_n^2-d^2}{t_{p_n}+t_{q_n}},\quad n=0,1,2,\ldots$$

Using this notation we can state the following theorem (see [13]):

<u>Theorem 1</u>. If $(f,x_o,x_{-1}) \in C(a,b,c)$ then the iterative algorithm (1) - (2) is well defined, the sequence $(x_n)_{n\geq 1}$ produced by it converges to a root $x^* \in U$ of the equation $f(x) = 0$ and the following estimates are satisfied:

(11) $$\|x_n-x^*\| \leq t_o - \|x_n-x_o\| - [(t_o-\|x_n-x_o\|)^2 - (\|x_n-x_{p_{n-1}}\| + \|x_{n-1}-x_{p_{n-1}}\| + \|x_{p_{n-1}}-x_{q_{n-1}}\|)\|x_n-x_{n-1}\|]^{1/2} \leq t_n - d;$$

(12) $$\|x_n - x^*\| \geq [(t_o - 2^{-1}(\|x_{p_n} - x_{q_n}\| + \|x_{p_n} - x_o\| + \|x_{q_n} - x_o\|) - \|x_n - x_{p_n}\|)^2 + (2t_o - \|x_{p_n} - x_o\| - \|x_{q_n} - x_o\|)\|x_n - x_{n+1}\|]^{1/2} - t_o + 2^{-1}(\|x_{p_n} - x_{q_n}\| + \|x_{p_n} - x_o\| + \|x_{q_n} - x_o\|) + \|x_n - x_{p_n}\|.$$

Proof. First let us observe that the linear operator $P = \delta f(u,v)$ is invertible for all $u,v \in \mathring{D}$ with

(13) $$\|u - x_o\| + \|v - x_o\| < 2t_o.$$

Indeed from (8) it follows that

$$\|I - T_o P\| = \|T_o(P_o - P)\| \leq \|T_o(P - f'(x_o))\| + \|T_o(f'(x_o) - P_o)\|$$
$$\leq a(\|u - x_o\| + \|v - x_o\| + \|x_o - x_{-1}\|) < 1$$

so that according to Banach's lemma P is invertible and

(14) $$\|(T_o P)^{-1}\| \leq [1 - a(\|u - x_o\| + \|v - x_o\| + C)]^{-1}.$$

Let us note that condition (8) implies the following Lipschitz condition for f'

(15) $$\|T_o(f'(u) - f'(v))\| \leq 2a\|u - v\|, \quad u,v \in \mathring{D}.$$

Using the integral representation

(16) $$f(x) - f(y) = [\int_0^1 f'(y + t(x-y))dt](x-y)$$

we deduce that

(17) $$\|T_o(f(x) - f(y) - f'(u)(x-y))\| \leq a(\|x-u\| + \|y-u\|)\|x-y\|$$

for all $x,y \in \mathring{D}$.

Finally from (8) and (17) we have

(18) $$\|T_o(f(x) - f(y) - \delta f(u,v)(x-y))\| \leq a(\|x-u\| + \|y-u\| + \|u-v\|)\|x-y\|$$

for all $x,y,u,v \in \mathring{D}$. By a continuity argument (16), (17) and (18) remain valid if x and/or y belong to D_c.

Using the above inequalities we shall prove that

$$\|x_n - x_{n+1}\| \le t_n - t_{n+1} \tag{19}$$

for $n = -1,0,1,2,\ldots$

It is easy to see that the sequence $(t_n)_{n\ge -1}$ given by (10) is decreasing and converges to d. In this case it follows that if (1) - (2) is well defined for $n=0,1,2,\ldots,k$ and if (19) holds for $n \le k$ then

$$\|x_o - x_n\| \le t_o - t_n < t_o - d$$

for $n \le k$. This shows that (13) is satisfied for $u = x_i$ and $v = x_j$ with $i,j \le k$. Thus (1) - (2) will be well defined for $n = k+1$ too.

For $n = -1$ and $n = 0$ (19) reduces to $\|x_{-1} - x_o\| \le c$ and $\|x_o - x_1\| \le b$ (compare with (5) and (7)). Suppose (19) holds for $n = -1,0,\ldots,k$, where $k \ge 0$. Denote $P_n = T_n^{-1}$, where T_n is given by (2). Observing that

$$f(x_{k+1}) = f(x_{k+1}) - f(x_k) - P_k(x_{k+1} - x_k) \tag{20}$$

and using (14) and (18) we may write

$$\|x_{k+1} - x_{k+2}\| = \|T_{k+1} f(x_{k+1})\| = \|(T_o P_{k+1})^{-1} T_o f(x_{k+1})\|$$

$$\le \frac{a(\|x_{k+1} - x_{p_k}\| + \|x_k - x_{p_k}\| + \|x_{p_k} - x_{q_k}\|)}{1 - a(\|x_{p_{k+1}} - x_o\| + \|x_{q_{k+1}} - x_o\| + c)} \|x_k - x_{k+1}\|$$

$$\le \frac{a(t_{p_k} - t_{k+1} + t_{p_k} - t_k + t_{q_k} - t_{p_k})}{1 - a(t_o - t_{p_{k+1}} + t_o - t_{q_{k+1}} + t_{-1} - t_o)} (t_k - t_{k+1})$$

$$= \frac{t_{p_k} + t_{q_k} - t_{k+1} - t_k}{t_{p_{k+1}} + t_{q_{k+1}}} (t_k - t_{k+1}) = t_{k+1} - t_{k+2}.$$

We have thus proved that (19) holds for all n. From the completeness of X it follows that the sequence $(x_n)_{n\geq 0}$ converges to a point x^* and that

$$\|x_n - x^*\| \leq t_n - d. \tag{21}$$

From (18) and (20) we obtain the inequality

$$\|T_o f(x_{k+1})\| \leq a(\|x_{k+1} - x_{p_k}\| + \|x_k - x_{p_k}\| + \|x_{p_k} - x_{q_k}\|)\|x_k - x_{k+1}\| , \tag{22}$$

wherefrom it follows that $f(x^*) = 0$.

Let us take now $x = x_n$ and $y = x^*$ in (16) and denote $A = \int_0^1 f'(x + t(x_n - x^*))dt$. Using (19) and (21) it follows that

$$\|x_n - x_o\| + \|x^* - x_o\| + \|x_o - x_{-1}\| \leq 2\|x_n - x_o\| + \|x_n - x^*\| + c$$

$$< 2(\|x_n - x_o\| + \|x_n - x^*\|) \leq 2(t_o - t_n + t_n - d) + c \leq 2t_o + c = 1/a.$$

Using (8) and Banach's lemma, one can prove that A is invertible and

$$\|(T_o A)^{-1}\| \leq [1 - a(2\|x_n - x_o\| + \|x_n - x^*\| + c)]^{-1} \tag{23}$$

(see also the proof of the inequality (14)) According to (22) and (23) we have

$$\|x_n - x^*\| = \|A^{-1} f(x_n)\| \leq \|(T_o A)^{-1}\| \|T_o f(x_n)\|$$

$$\leq \frac{a(\|x_n - x_{p_{n-1}}\| + \|x_{n-1} - x_{p_{n-1}}\| + \|x_{p_{n-1}} - x_{q_{n-1}}\|)}{1 - a(2\|x_n - x_o\| + \|x_n - x\| + c)} \|x_n - x_{n-1}\| ,$$

and it is easy to see that the above inequality together with the fact that $\|x_n - x^*\| < t_o$ implies the estimate (11).

Using the identity

$$x_{n+1} - x_n = x^* - x_n + (T_o P_n)^{-1} T_o (f(x^*) - f(x_n) - P_n(x^* - x_n))$$

and the inequalities (14) and (18) we obtain

$$\|x_{n+1}-x_n\| \leq \frac{a(2\|x_n-x_{p_n}\| + \|x^*-x_n\| + \|x_{p_n}-x_{q_n}\|)}{1-a(\|x_{p_n}-x_o\| + \|x_{q_n}-x_o\| + c)} \|x_n-x^*\| + \|x_n-x^*\|$$

This inequality implies the lower bound (12).□

As a consequence of the above theorem one obtains the following result about the existence and the uniqueness of the solution for nonlinear equations.

Corollary. If the triplet (f,x_o,x_{-1}) belongs to the class $C(a,b,c)$ then the equation $f(x) = 0$ has a root $x \in U$ and this is the unique solution of the equation in the set $V = \{x \in D_c;\ \|x-x_o\| < t_o + d\}$ if $d > 0$, or in the set $W = \{x \in D_c;\ \|x-x_o\| \leq d\}$ if $d = 0$.

Proof. The existence has been proved in the theorem. Suppose $d > 0$ and let $x^* \in U$ and $y^* \in V$ be two solutions of the equation $f(x) = 0$. Let us denote $A_* = \int_0^1 f'(y +t(x -y))dt$. According to (8) we have

$$\|I-T_oA_*\| = \|T_o(P_o-A_*)\| \leq a(\|y^*-x_o\| + \|x^*-x_o\| + \|x_o-x_{-1}\|)$$
$$< a(t_o+d+t_o-d+c) = 1.$$

It follows that A_* is invertible, and in this case from (16) we deduce that $x^* = y^*$.

Now suppose $d = 0$. If we take $p_n = 0$ and $q_n = -1$ for $n=0,1,2,...$ then the iterative algorithm (1) - (2) becomes

(24) $$x_{n+1} = x_n - T_of(x_n) \quad n=0,1,2,...$$

From Theorem 1 it follows that the sequence $(x_n)_{n\geq 0}$ given by (24) converges to a root x^* of the equation $f(x) = 0$. It also follows that

(25) $$\|x_n - x_{n+1}\| \leq t_n - t_{n+1}$$

where

(26) $$t_o = (b/a)^{1/2}, \quad t_{n+1} = t_n - at_n^2, \quad n=0,1,2,\ldots$$

It is easy to prove by induction that

(27) $$t_n \geq \frac{(b/a)^{1/2}}{n+1} \quad n=0,1,2,\ldots$$

Let $y^* \in W$ be a solution of the equation $f(x) = 0$ and denote $A_n = \int_0^1 f'(y^*+t(x_n-y^*))dt$. According to (8), (16), (24) and (25) we have

$$\|x_{n+1}-y^*\| = \|T_o(P_o-A_n)(x_n-y^*)\|$$
$$\leq a\|x_n-y^*\|(\|y^*-x_o\| + \|x_n-x_o\| + \|x_o-x_{-1}\|)$$
$$\leq \|x_n-y^*\|(1-at_n) \leq \ldots \leq \|x_1-y^*\| \cdot \prod_{j=1}^{n}(1-at_j).$$

The inequalities (27) imply that $\lim_{n\to\infty} \prod_{j=1}^{n}(1-at_j) = 0$; hence $y^* = \lim_{n\to\infty} x_n = x^*$. The proof is complete.□

In the following we want to show that the results obtained in this section are sharp within the class $C(a,b,c)$. Concerning the estimates (11) and (12) we can state

Proposition 1. If $a > 0$, $b \geq 0$, $c \geq 0$ are three constants satisfying inequality (4) then:

(i) There exist a function $f: \mathbb{R} \to \mathbb{R}$ and two points x_o, $x_{-1} \in \mathbb{R}$ such that the triplet (f,x_o,x_{-1}) belongs to $C(a,b,c)$ and for this triplet the estimates (11) are attained at each $n=1,2,3,\ldots$

(ii) For each $n=0,1,2,\ldots$ there exist a function $f_n: \mathbb{R} \to \mathbb{R}$ and two points x_o, $x_{-1} \in \mathbb{R}$ such that the triplet (f_n,x_o,x_{-1}) belongs to $C(a,b,c)$ and for this triplet (12) holds with equality.

Proof. (i) Take $f(x) = x^2 - d^2$, $x_o = t_o$, $x_{-1} = t_{-1}$;

$$\text{(ii) Take } f_n(x) = \begin{cases} x^2 - d^2, & \text{if } x \geq t_n \\ -x^2+4t_nx-2t_n^2-d^2, & \text{if } x < t_n, \end{cases}$$

$x_o = t_o$, $x_{-1} = t_{-1}$, where d and $(t_n)_{n\geq -1}$ are given by (9) and (10).

Concerning the domain of uniqueness of the solution x^* established in the corollary of Theorem 1 we have

Proposition 2. Let $a > 0$, $b \geq 0$, $c \geq 0$ be three numbers satisfying condition (4) and let d be the constant defined by (9).

(i) If $d > 0$ then there exist a function $f: \mathbb{R} \to \mathbb{R}$ and four points x_o, x_{-1}, x^*, $y^* \in \mathbb{R}$ such that $(f,x_o,x_{-1}) \in C(a,b,c)$, $f(x^*) = f(y^*) = 0$, $|x_o-x^*| = t_o-d$, $|x_o-y^*| = t_o+d$.

(ii) If $d = 0$, then for each $\varepsilon > 0$ there exist a function $f_\varepsilon: \mathbb{R} \to \mathbb{R}$ and four points x_o, x_{-1}, x^*, $y^*_\varepsilon \in \mathbb{R}$ such that $(f_\varepsilon,x_o,x_{-1}) \in C(a,b,c)$, $f(x^*) = f(y^*_\varepsilon) = 0$, $|x^*-x_o| = t_o$, $|y^*_\varepsilon-x_o| = t_o+\varepsilon$.

Proof. (i) Take $f(x) = x^2-d^2$, $x_o - t_o$, $x_{-1} = t_{-1}$, $x^* = d$, $y^* = -d$.

$$\text{(ii) Take } f_\varepsilon(x) = \begin{cases} x^2, & \text{if } x \geq \frac{-\varepsilon}{2+\sqrt{2}} \\ -x^2 - \frac{4\varepsilon}{2+\sqrt{2}}x - \left(\frac{\varepsilon}{1+\sqrt{2}}\right)^2, & \text{if } x < \frac{-\varepsilon}{2+\sqrt{2}} \end{cases}$$

$x_o = t_o$, $x_{-1} = t_{-1}$, $x^* = 0$, $y^* = -\varepsilon$.

3. Comments

In what follows we shall consider some particular cases of the iterative procedure (1) - (2) and shall compare the results obtained in the preceding section with some known results.

a_1) If $p_n = n$ and $q_n = n-1$ for $n=1,2,3,\dots$ then (1) - (2) reduces to the secant method. The error estimates (11) and (12) become in this case

(28) $$\|x_n - x^*\| \leq t_o - \|x_n - x_o\| - [(t_o - \|x_n - x_o\|)^2 - (\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|)\|x_n - x_{n-1}\|]^{1/2}$$

(29) $$\|x_n - x^*\| \geq [(t_o - 2^{-1}(\|x_n - x_{n-1}\| + \|x_n - x_o\| + \|x_{n-1} - x_o\|))^2 + (2t_o - \|x_n - x_o\| - \|x_{n-1} - x_o\|)\|x_n - x_{n+1}\|]^{1/2} - t_o + 2^{-1}(\|x_n - x_{n-1}\| + \|x_n - x_o\| + \|x_{n-1} - x_o\|)$$

These estimates improve the estimates obtained in [12] and [20].

a_2) If $p_n = 0$ and $q_n = -1$ for $n=0,1,2,\dots$ then (1) - (2) reduces to the simplified secant method. The result contained in Theorem 1 improves in this case the result from [12].

a_3) If $p_{km+j} = km$, $q_{km+j} = km-1$ ($q_{-1} = q_o = -1$), $j=0,1,\dots,m-1$, $k=0,1,2,\dots$, then (1) - (2) reduces to a procedure considered by J.F. Traub [22] for scalar equations. A local analysis for this procedure has been done by J.W. Schmidt and H. Schwetlick [21]. P. Laasonen [7] made a semilocal analysis for the case $m=2$. His result was improved in [16] (for $m=2$) and in [18] (for m arbitrary). J. Dennis [4] studied the iterative process (1) - (2) in case $p_n = q_n + 1$. The result contained in Theorem 1 improves all the above mentioned results. Let us note that by taking $y_n = x_{nm}$ one obtains a sequence $(y_n)_{n\geq 0}$ which converges to x^* with R-order $(m+\sqrt{m^2+4})/2$. The parameter m can be chosen according to the dimension of the space in order to maximize the numerical efficiency of the procedure (see [21]).

In the above examples we had $p_n \neq q_n$ for all n. If $x_{-1} = x_o$ and $p_n = q_n$ for $n=0,1,2,...$ then the iterative procedure (1) - (2) becomes

$$x_{n+1} = x_n - f'(x_{p_n})^{-1} f(x_n), \quad n=0,1,2,... \tag{30}$$

J. Dennis [3] has proved that this iterative procedure converges under the hypotheses of the Kantorovich theorem. This fact follows by taking $c=0$ in Theorem 1. To be more precise let us consider the class $C'(a,b)$ defined below.

Definition 2. Let $a > 0$ and $b \geq 0$ be two real numbers satisfying the inequality

$$4ab \leq 1. \tag{31}$$

We say that a pair (f,x_o) belongs to the class $C'(a,b)$, if

c_1') f is a nonlinear operator defined on a convex subset D of a Banach space X and with values in a Banach space Y.

c_2') x_o is a point belonging to the interior $\overset{\circ}{D}$ of D.

c_3') f is Fréchet differentiable on $\overset{\circ}{D}$, $f'(x_o)$ is boundedly invertible and

$$\| f'(x_o)^{-1}(f'(x)-f'(y))\| \leq 2a\| x-y\| \quad \text{for all } x,y \in D. \tag{32}$$

c_4') The set $D_c = \{x \in D,\ f \text{ is continuous at } x\}$ contains the closed ball U with center $x_1 = x_o - f'(x_o)^{-1}f(x_o)$ and radius $r_1 = \frac{1}{2a}(1-2ab-\sqrt{1-4ab})$.

It is easy to see that $(f,x_o) \in C'(a,b)$ if and only if $(f,x_o,x_o) \in C(a,b,o)$. In this case from Theorem 1 it follows that the iterative procedure (30) converges and the following estimates hold:

$$\| x_n - x^*\| \leq t_o - \| x_n - x_o\| - [(t_o - \| x_n - x_o\|)^2 - (\| x_n - x_{p_{n-1}}\| + \| x_{n-1} - x_{p_{n-1}}\|) \| x_n - x_{n-1}\|]^{1/2} \tag{33}$$

$$(34)\qquad \|x_n - x^*\| \geq [(t_o - \|x_{p_n} - x_o\| - \|x_n - x_{p_n}\|)^2 + 2(t_o - \|x_{p_n} - x_o\|)\|x_n - x_{n+1}\|]^{1/2}$$
$$- t_o + \|x_{p_n} - x_o\| + \|x_n - x_{p_n}\| .$$

In the above formulae we have denoted $t_o = 2^{-1}a^{-1}$ (see (10)). By analogy with $a_1) - a_3)$ we shall consider the following three particular cases of the iterative procedure (30).

$a_1')$ If $p_n = n$ for $n=0,1,2,\ldots$ then (30) reduces to Newton's method and the error estimates (33) and (34) become

$$(35)\qquad \|x_n - x^*\| \leq t_o - \|x_n - x_o\| - [(t_o - \|x_n - x_o\|)^2 - \|x_n - x_{n-1}\|^2]^{1/2}$$

$$(36)\qquad \|x_n - x^*\| \geq [(t_o - \|x_n - x_o\|)^2 + 2(t_o - \|x_n - x_o\|)\|x_n - x_{n+1}\|]^{1/2} - t_o + \|x_n - x_o\| .$$

The upper bounds (35) have been obtained in [14]. There it has been shown that they are better than those obtained by Gragg and Tapia [5], Miel [8] and Potra and Ptak [15]. The lower bounds (36) are new. It can be shown that they are better than the estimates from [5], [9] and [15].

$a_2')$ If $p_n = 0$ for $n=0,1,2,\ldots$ then the iterative procedure (30) reduces to the simplified Newton method. The error estimates (33) are in this case better than the estimates obtained in [6, Satz 2] and [11, Th. 4.1].

$a_3')$ If $p_{km+j} = km$ for $j=0,1,\ldots,m-1$, $k=0,1,2,\ldots$ then (30) reduces to an iterative procedure studied by Traub [2], Schmidt and Schwetlick [21], Bosarge and Falb [1], [2], Wolfe [23]. In [17] one gives computable aposteriori error bounds for this procedure. One can prove that the estimates (33) are in general more accurate than those ones. Taking $y_n = x_{mn}$ for $n=0,1,2,\ldots$ one obtains a sequence $(y_n)_{n\geq 0}$ which converges to x^* with R-order $m+1$. The same as for the iterative procedure considered in $a_3)$, m can be chosen according to the dimension of the space in order to maximize the numerical efficiency.

REFERENCES

[1] Bosarge, W.E. and Falb, P.L. "A multipoint method of third order", J. Optimiz. Theory Appl. 4 (1969), 156-166.

[2] Bosarge, W.E. and Falb, P.L. "Infinite dimensional multipoint methods and the solution of two point boundary value problems", Numer. Math. 14 (1970), 264-286.

[3] Dennis, J.E. "On the Kantorovich hypothesis for Newton's method", SIAM J. Numer. Anal. 6, 3(1969), 493-507.

[4] Dennis, J.E. "Toward a unified convergence theory for Newton-like methods". In *Nonlinear Functional Analysis and Applications*, L.B. Rall, Ed., (New York: Academic Press, 1971).

[5] Gragg, W.B. and Tapia, R.A. "Optimal error bounds for the Newton-Kantorovich theorem", SIAM J. Numer. Anal. 11, 1(1974), 10-13.

[6] Kornstaedt, H.-J. "Ein allgemeiner Konvergenzsatz für verschärfte Newton-Verfahren", ISNM 28, Birkhaüser Verlag, Basel and Stuttgart, 1975, 53-69.

[7] Laasonen, P. "Ein überquadratisch konvergenter iterativer algorithmus", Ann. Acad. Sci. Fenn. Ser I, 450(1969), 1-10.

[8] Miel, G.J. "The Kantorovich theorem with optimal error bounds", Amer. Math. Monthly, 86(1979), 212-215.

[9] Miel, G.J. "An updated version of the Kantorovich theorem for Newton's method", Technical Summary Report, Mathematical Research Center, University of Wisconsin, Madison, 1980.

[10] Ortega, J.M. and Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables* (New York: Academic Press, 1970).

[11] Potra, F.-A. "On a modified secant method", L'analyse numérique et la théorie de l'approximation 8, 2(1979), 203-214.

[12] Potra, F.-A. "An application of the induction method of V. Pták to the study of Regula Falsi", Aplikace Matematiky, 26(1981), 111-120.

[13] Potra, F.-A. "On the convergence of a class of Newton-like methods," *Iterative Solution of Nonlinear Systems of Equations*, Lecture Notes in Mathematics 953, (New York: Springer-Verlag, 1982).

[14] Potra, F.-A. "On the aposteriori error estimates for Newton's method", Beiträge Numer. Math., 12(1984), 125-138.

[15] Potra, F.-A. and Pták, V. "Sharp error bounds for Newton's method", Numer. Math. 34(1980), 63-72.

[16] Potra, F.-A. and Pták, V. "Nondiscrete induction and a double step secant method", Math. Scand. 46(1980), 236-250.

[17] Potra, F.-A. and Pták, V. "On a class of modified Newton processes", Numer. Func. Anal. and Optimiz. 2, 1(1980), 107-120.

[18] Potra, F.-A. and Pták, V. "A generalization of Regula Falsi", Numer. Math. 36(1981), 333-346.

[19] Schmidt, J.W. "Regula-Falsi Verfahren mit konsistenter Steigung und Majoranten Printzip", Periodica Mathematica Hungarica 5, 3(1974), 187-193.

[20] Schmidt, J.W. "Untere Fehlerschranken für Regula-Falsi Verfahren", Periodica Mathematica Hungarica 9, 3(1978), 241-247.

[21] Schmidt, J.W. and Schwetlick, H. "Ableitungsfreie Verfahren mit höherer Konvergenzgeschwindifkeit, Computing, 3(1968), 215-226.

[22] Traub, J.F. *Iterative Methods for the Solution of Equations* (Englewood Cliffs, New Jersey: Prentice Hall, 1964).

[23] Wolfe, M.A. "Extended iterative methods for the solution of operator equations", Numer. Math., 31(1978), 153-174.