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THE BOUNDARY INTEGRAL EQUATION METHOD IN THE PROBLEM OF BENDING OF THIN PLATES

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I. INTRODUCTION

Theoretical considerations corroborated by experimental data indicate that a good approximation to the field equations governing the bending of thin plates is obtained when the three-dimensional equations of elastostatics are simplified by means of Kirchhoff's kinematic hypothesis regarding the form of the displacements [1]:

$$\begin{aligned} u_\alpha &= x_3 v_\alpha(x_1,x_2), \quad \alpha = 1,2, \\ u_3 &= v_3(x_1,x_2). \end{aligned} \tag{1}$$

Unfortunately, the Dirichlet and Neumann problems associated with the system derived on the basis of (1) cannot in general be solved by the boundary integral equation method both in the interior and in the exterior of a bounded domain, on account of the rapid growth at infinity of the matrix of fundamental solutions. The aim of this paper is to show that the above technique can still be applied provided that the solution is sought in certain classes of functions with finite energy.

II. PRELIMINARIES

Unless otherwise specified, throughout what follows Greek and Latin suffixes take the values 1,2 and 1,2,3 respectively, and the convention of summation over repeated indices is understood.

Suppose that a homogeneous and isotropic elastic plate with Lamé coefficients λ and μ occupies a cylindrical region $\bar{\Omega}_{in} \times [-h_o/2,h_o/2]$ or $\bar{\Omega}_{ex} \times [-h_o/2,h_o/2]$ in $R^3(x_1,x_2,x_3)$, where Ω_{in} is a domain in $R^2(x_1,x_2)$ bounded by a closed Lyapunov curve $\partial\Omega, \Omega_{ex} = R^2 \setminus \bar{\Omega}_{in}$, and $h_o > 0$ is a constant.

We denote by $x = (x_1,x_2,x_3)$ a generic point in R^3 and by H_k and J_k $(k = 0,1)$ the averaging operators

$$H_k w(x) = (1/h_o)[x_3^k w(x)]_{x_3=-h_o/2}^{x_3=h_o/2},$$

$$J_k w(x) = (1/h_o) \int_{-h_o/2}^{h_o/2} x_3^k w(x) dx_3.$$

If t_{ij} are the stresses, f_i the body forces and

$$F_\alpha = -(J_1 f_\alpha + H_1 t_{3\alpha}),$$
$$F_3 = -(J_o f_3 + H_o t_{33}),$$

then the linear equilibrium system obtained on the basis of (1) can be written in the form

$$L(\partial_x)v(x) = F, \tag{2}$$

where $L(\partial_x) = L(\partial/\partial x_1, \partial/\partial x_2) = (L_{ij}(\partial_x))$,

$$L(\xi_1,\xi_2)$$
$$= \begin{bmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{bmatrix}, \tag{3}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \Delta = \xi_1^2 + \xi_2^2, \quad h^2 = h_o^2/12.$$

In conjunction with (2) we consider the boundary stress operator $T(\partial_x) = T(\partial/\partial x_1, \partial/\partial x_2) = (T_{ij}(\partial_x))$ defined by

$$T(\xi_1,\xi_2)$$

$$= \begin{bmatrix} h^2(\lambda+2\mu)n_1\xi_1+h^2\mu n_2\xi_2 & h^2\mu n_2\xi_1+h^2\lambda n_1\xi_2 & 0 \\ h^2\mu n_2\xi_1+h^2\mu n_1\xi_2 & h^2\mu n_1\xi_1+h^2(\lambda+2\mu)n_2\xi_2 & 0 \\ \mu n_1 & \mu n_2 & \mu(n_1\xi_1+n_2\xi_2) \end{bmatrix}, \tag{4}$$

where $n(n_1,n_2)$ is the unit outward normal to $\partial\Omega$. It is not difficult to show that $T(\partial_x)v(x)$ is the normal boundary stress (in the case of a finite plate).

In what follows we assume that λ and μ satisfy the inequalities

$$\lambda + \mu > 0, \quad \mu > 0.$$

This ensures that (2) is an elliptic system and that the internal energy density $h_o E(v,v)$, where

$$\begin{aligned} 2E(v,v) &= h^2[\lambda v_{\alpha,\alpha}v_{\beta,\beta} + \mu(v_{\alpha,\beta} + v_{\beta,\alpha})v_{\alpha,\beta}] \\ &\quad + \mu(v_{3,\alpha}+v_\alpha)(v_{3,\alpha}+v_\alpha), \quad (\ldots)_{,\alpha} = \partial(\ldots)/\partial x_\alpha, \end{aligned} \tag{5}$$

is a positive quadratic form.

We also assume that all functions involved in the subsequent analysis are as smooth as required.

III. SINGULAR SOLUTIONS

Since due to our assumptions on λ and μ the matrix $L(\xi_1,\xi_2)$ is invertible, we can write

$$L^{-1}(\xi_1,\xi_2) = [\det L(\xi_1,\xi_2)]^{-1}B(\xi_1,\xi_2) \tag{6}$$

and obtain for the solution of (2) the Galerkin representation

$$v = B(\partial_x)\tau,$$

where τ satisfies

$$\det L(\partial_x)\tau = F.$$

From (3),

$$\det L(\xi) = h^2\mu^2(\lambda+2\mu)\Delta^2(h^2\Delta-1).$$

Denoting by $t(x,y)$ the solution of the equation

$$\det L(\partial_x)t = -h^2\mu^2(\lambda+2\mu)\delta(x-y),$$

where $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $\delta(x-y)$ is the Dirac distribution, and setting in turn each component of F equal to $-\delta(x-y)$ and the remaining ones equal to zero, we obtain the matrix of fundamental solutions

$$D(x,y) = B(\partial_x)t(x,y) \tag{7}$$

In this case,

$$t(x,y) = t(r) = a[r^2\ln r + 4h^2\ln r + 4h^2K_o(r/h)], \tag{8}$$

where

$$a = [8\pi h^2\mu^2(\lambda+2\mu)]^{-1},$$
$$r = |x-y| = [(x_1-y_1)^2+(x_2-y_2)^2]^{1/2},$$

and $K_o(\xi)$ is the modified Bessel function of order zero.

From (6), (8) and (3) we find that

$$\begin{aligned} B_{\alpha\alpha}(\xi) &= h^2\mu(\lambda+2\mu)\Delta^2 - h^2\mu(\lambda+\mu)\Delta\xi_\alpha^2 - \mu^2\xi_\alpha^2, \quad (\alpha \text{ not summed})\\ B_{33}(\xi) &= h^4\mu(\lambda+2\mu)\Delta^2 - h^2\mu(\lambda+3\mu)\Delta + \mu^2,\\ B_{12}(\xi) &= B_{21}(\xi) = -\mu\xi_1\xi_2[h^2(\lambda+\mu)\Delta + \mu],\\ B_{\alpha 3}(\xi) &= -B_{3\alpha}(\xi) = \mu^2\xi_\alpha(h^2\Delta - 1). \end{aligned} \tag{9}$$

Taking into account the series expansion of $K_o(\xi)$ as $\xi \to 0$ [2], we deduce that in the neighbourhood of $r = 0$

$$t(r) = -(a/16h^2)r^4\ln r + O(r^5\ln r),$$

hence

$$D(x,y) = -(1/2\pi)\begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1/\mu \end{bmatrix} \ln r + D_1(x,y), \tag{10}$$

where

$$b = (\lambda+3\mu)/2h^2\mu(\lambda+2\mu)$$

and $D_1(x,y)$ has no singularity at $x = y$.

We also consider the matrix

$$P(x,y) = [T(\partial_y)D(y,x)]' \tag{11}$$

(the accent denotes matrix transposition).

Proceeding as in the classical theory (see, for example, [3]), we can derive the equalities

$$\begin{aligned} &\int_{\partial\Omega} T_{ik}(\partial_x)D_{kj}(x,y)ds_x = \phi(y)\delta_{ij}, \\ &\int_{\partial\Omega} x_\alpha T_{3k}(\partial_x)D_{kj}(x,y)ds_x = y_\alpha\phi(y)\delta_{3j}, \end{aligned} \tag{12}$$

where

$$\phi(y) = \begin{cases} -1, & y \in \Omega_{in}, \\ -1/2, & y \in \partial\Omega, \\ 0, & y \in \Omega_{ex} \end{cases}$$

and δ_{ij} are the Kronecker delta, the Betti formula

$$\int_{\Omega_{in}} v'Lv\, d\sigma + \int_{\Omega_{in}} 2E(v,v)d\sigma = \int_{\partial\Omega} v'Tv\, ds, \tag{13}$$

and the Somigliana relation

$$\chi_1(x)v(x) = \int_{\partial\Omega}[D(x,y)T(\partial_y)v(y) - P(x,y)v(y)]ds_y \\ - \int_{\Omega_{in}} D(x,y)L(\partial_y)v(y)ds_y, \tag{14}$$

where

$$\chi_1(x) = \begin{cases} 1, & x \in \Omega_{in}, \\ 0, & x \in \Omega_{ex}. \end{cases}$$

Let y be fixed. Then for large values of $|x|$,

$$|x-y|^{-2} = |x|^{-2} + 2(x,y)|x|^{-4} - |y|^2|x|^{-4} + 4(x,y)^2|x|^{-6} + O(|x|^{-5}),$$

$$\ln|x-y| = \ln|x| - (x,y)|x|^{-2} + (1/2)|y|^2|x|^{-2} - (x,y)^2|x|^{-4} \\ + (x,y)|y|^2|x|^{-4} - (4/3)(x,y)^3|x|^{-6} + O(|x|^{-4}), \tag{15}$$

$$(x,y) = x_1y_1 + x_2y_2,$$

and from (4), (7) - (9) and (11) we find that, as $r \to \infty$,

$$D_{33} = O(r^2\ln r),\ D_{\alpha 3}, D_{3\alpha} = O(r\ln r),$$

$$D_{11}, D_{22} = O(\ln r),\ D_{12}, D_{21} = O(1),$$

$$P_{3\alpha} = O(\ln r),\ P_{\alpha\beta}, P_{33} = O(r^{-1}),\ P_{\alpha 3} = O(r^{-2}).$$

The asymptotics of $D(x,y)$ and $P(x,y)$ indicate that the boundary integral equation method is inoperative in the case of the exterior Dirichlet and Neumann problems, due to the lack of appropriate uniqueness results. However, this obstacle can be removed if the behaviour of the solution at infinity is suitably restricted.

Let A be the set of (3×1) - matrices v in Ω_{ex} admitting an asymptotic expansion of the form

$$\begin{aligned} v_1(r,\theta) = {} & r^{-1}[\alpha_o \sin\theta + 2\alpha_1\cos\theta - \alpha_o \sin 3\theta + (\alpha_2-\alpha_1)\cos 3\theta] \\ & + r^{-2}[(2\beta_1+\gamma_1)\sin 2\theta + \gamma_2\cos 2\theta - 2\beta_1\sin 4\theta + 2\beta_2\cos 4\theta] \\ & + r^{-3}[2\rho_1\sin 3\theta + 2\pi_1\cos 3\theta + 3(\rho_2-\rho_1)\sin 5\theta + 3(\pi_2-\pi_1)\cos 5\theta] \\ & \qquad + 0(r^{-4}), \\ v_2(r,\theta) = {} & r^{-1}[2\alpha_2\sin\theta + \alpha_o\cos\theta + (\alpha_2-\alpha_1)\sin 3\theta + \alpha_o\cos 3\theta] \\ & + r^{-2}(2\beta_2+\gamma_2)\sin 2\theta - \gamma_1\cos 2\theta + 2\beta_2\sin 4\theta + 2\beta_1\cos 4\theta] \\ & + r^{-3}[2\pi_2\sin 3\theta - 2\rho_2\cos 3\theta + 3(\pi_2-\pi_1)\sin 5\theta + 3(\rho_1-\rho_2)\cos 5\theta] \\ & \qquad + 0(r^{-4}), \\ v_3(r,\theta) = {} & -(\alpha_1+\alpha_2)\ln r - [\alpha_1+\alpha_2+\alpha_o\sin 2\theta + (\alpha_1-\alpha_2)\cos 2\theta] \\ & + r^{-1}[(\beta_1+\gamma_1)\sin\theta + (\beta_2+\gamma_2)\cos\theta - \beta_1\sin 3\theta + \beta_2\cos 3\theta] \\ & + r^{-2}[\sigma_1\sin 2\theta + \sigma_2\cos 2\theta + (\rho_2-\rho_1)\sin 4\theta + (\pi_2-\pi_1)\cos 4\theta] \\ & \qquad + 0(r^{-3}), \end{aligned} \tag{16}$$

where α_o, α_ν, β_ν, γ_ν, ρ_ν, π_ν, and σ_ν are arbitrary constants. We also consider the set A^* of (3×1) - matrices $v^* = v+v_o$, where $v \in A$ and

$$v'_o = (c_1, c_2, c-c_1x_1-c_2x_2), \quad c, c_1, c_2 = \text{const.} \tag{17}$$

<u>Remark</u>. Taking (1) into account, it is easy to see that (17) represents a rigid displacement.

<u>Theorem 1</u>. If $v \in A$ is a solution in Ω_{ex} of the homogeneous system (2), then

$$\chi_2(x)v(x) = -\int_{\partial\Omega}[D(x,y)T(\partial_y)v(y) - P(x,y)v(y)]ds_y, \tag{18}$$

where

$$\chi_2(x) = \begin{cases} 0, & x \in \Omega_{in}, \\ 1, & x \in \Omega_{ex}. \end{cases}$$

Proof. Consider a circle K_R with the centre at x and radius R sufficiently large so that $\bar{\Omega}_{in} \subset K_R$. Changing to polar coordinates with the origin at x and using (4), (7) - (9), (11), (15) and (16), we arrive at the asymptotic relations

$$T_{3j}v_j = R^{-3}[(\rho_1+\rho_2-2\sigma_1)\sin 2\theta + (\pi_1+\pi_2-2\sigma_2)\cos 2\theta] + O(R^{-4}),$$
$$(D_{3\alpha}T_{\alpha i}-P_{3i})v_i = 2h^2a\mu^2R^{-1}(4\lambda \ln R + 3\lambda + 2\mu)[\alpha_o \sin 2\theta + 2(\alpha_2-\alpha_1)\cos 2\theta] + O(R^{-2}\ln R).$$
$$(D_{\alpha i}T_{ij}-P_{\alpha j})v_j = O(R^{-2}\ln R), \quad (R,\theta) \in \partial K_R.$$

Hence,

$$\int_{\partial K_R}[D(x,y)T(\partial_y)v(y) - P(x,y)v(y)]ds_y = O(R^{-1}\ln R).$$

Formula (18) is now established by applying (14) to v in $\Omega_{ex} \cap K_R$ and letting $R \to \infty$.

Theorem 2. *If $v \in A^*$ is a solution in Ω_{ex} of the homogeneous system (2), then*

$$\int_{\Omega_{ex}} 2E(v,v)d\sigma = - \int_{\partial\Omega} v'Tv\, ds. \tag{19}$$

Proof. If $v^* = v+v_o \in A^*$, where $v \in A$ and v_o' is given by (17), then $Tv^* = Tv$. Proceeding as above, we find that for large values of R

$$T_{\alpha j}v_j = O(R^{-2}),$$
$$T_{3j}v_j = O(R^{-3}), \tag{20}$$

which means that

$$\int_{\partial K_R} v^{*\prime}Tv^*\, ds = R\int_0^{2\pi} v^{*\prime}Tv^*\, d\theta = O(R^{-1}).$$

To obtain (19), we apply (13) to v^* in $\Omega_{ex} \cap K_R$ and let $R \to \infty$.

IV. UNIQUENESS THEOREM

We denote by $C^k_{(3\times1)}(\Omega)$ the space of (3×1) - matrices that are k times continuously differentiable in Ω. Let $A(x)$, $B(x)$, $R(x)$, and $S(x)$ be Hölder continuous (3×1) - matrices prescribed on $\partial\Omega$. For the homogeneous system (2)

$$L(\partial_x)v(x) = 0 \qquad (21)$$

we consider the following Dirichlet and Neumann problems:

(D_{in}) Find $v \in C^2_{(3\times1)}(\Omega_{in}) \cap C^0_{(3\times1)}(\bar{\Omega}_{in})$ satisfying (21) in Ω_{in} and

$$v(x) = A(x), \quad x \in \partial\Omega. \qquad (22)$$

(N_{in}) Find $v \in C^2_{(3\times1)}(\Omega_{in}) \cap C^1_{(3\times1)}(\bar{\Omega}_{in})$ satisfying (21) in Ω_{in} and

$$T(\partial_x)v(x) = B(x), \quad x \in \partial\Omega. \qquad (23)$$

(D_{ex}) Find $v \in C^2_{(3\times1)}(\Omega_{ex}) \cap C^0_{(3\times1)}(\bar{\Omega}_{ex}) \cap A^*$ satisfying (21) in Ω_{ex} and

$$v(x) = R(x), \quad x \in \partial\Omega. \qquad (24)$$

(N_{ex}) Find $v \in C^2_{(3\times1)}(\Omega_{ex}) \cap C^1_{(3\times1)}(\bar{\Omega}_{ex}) \cap A$ satisfying (21) in Ω_{ex} and

$$T(\partial_x)v(x) = S(x), \quad x \in \partial\Omega. \qquad (25)$$

Theorem 3. (i) (D_{in}), (D_{ex}) and (N_{ex}) have at most one solution.

(ii) Any two solutions of (N_{in}) differ by a matrix of the form (17).

Proof. The results for Ω_{in} are proved as in the classical theory, noting that $E(v,v) = 0$ if and only if v is a rigid displacement (17).

The difference v of two solutions of (D_{ex}) satisfies

$$L(\partial_x)v(x) = 0, \quad x \in \Omega_{ex},$$
$$v(x) = 0, \quad x \in \partial\Omega,$$
$$v \in A^*.$$

By Theorem 2, v is of the form (17), and since it vanishes on $\partial\Omega$, it follows that $v(x) = 0$, $x \in \Omega_{ex}$.

The difference v of two solutions of (N_{ex}) satisfies

$$L(\partial_x)v(x) = 0, \quad x \in \Omega_{ex},$$
$$T(\partial_x)v(x) = 0, \quad x \in \partial\Omega,$$
$$v \in A.$$

Again by Theorem 2, v is a rigid displacement. But $v \in A$, hence $v(x) = 0$, $x \in \Omega_{ex}$.

V. EXISTENCE OF THE SOLUTION

We define the single layer potential

$$V(x) = \int_{\partial\Omega} D(x,y)z(y)ds_y \tag{26}$$

and the double layer potential

$$W(x) = \int_{\partial\Omega} P(x,y)z(y)ds_y, \tag{27}$$

where z is a (3×1) - matrix.

The following properties of V and W are established by classical methods:

<u>Theorem 4.</u> (i) $L(\partial_x)V(x) = L(\partial_x)W(x) = 0$, $x \notin \partial\Omega$.

(ii) If $z \in C^o_{(3\times1)}(\partial\Omega)$, then $V \in C^o_{(3\times1)}(R^2)$.

(iii) If z is Hölder continuous on $\partial\Omega$, then W tends to finite limits as $x \to x_o \in \partial\Omega$ from both Ω_{in} and Ω_{ex}, these limits being, respectively,

$$\begin{aligned} W_{in}(x_o) &= -\frac{1}{2}\,z(x_o) + \int_{\partial\Omega} P(x_o,y)z(y)ds_y, \\ W_{ex}(x_o) &= \frac{1}{2}\,z(x_o) + \int_{\partial\Omega} P(x_o,y)z(y)ds_y. \end{aligned} \tag{28}$$

(iv) If z is Hölder continuous on $\partial\Omega$, then TV tends to finite limits as $x \to x_o \in \partial\Omega$ from both Ω_{in} and Ω_{ex}, these limits being, respectively,

$$\begin{aligned}(TV)_{in}(x_o) &= \frac{1}{2} z(x_o) + \int_{\partial\Omega} T(\partial_{x_o})D(x_o,y)z(y)ds_y,\\ (TV)_{ex}(x_o) &= -\frac{1}{2} z(x_o) + \int_{\partial\Omega} T(\partial_{x_o})D(x_o,y)z(y)ds_y.\end{aligned} \tag{29}$$

(v) If either of the limits $(TW)_{in}(x_o)$ or $(TW)_{ex}(x_o)$ of TW as $x \to x_o \in \partial\Omega$ from Ω_{in} or Ω_{ex}, respectively, exists, then so does the other, and the two limits are equal.

Theorem 5. If $z \in C^o_{(3\times1)}(\partial\Omega)$, then

(i) $W \in A$;

(ii) $V \in A$ if and only if

$$M = \int_{\partial\Omega} z_3 ds = 0,$$

$$M_\alpha = \int_{\partial\Omega}(z_\alpha - x_\alpha z_3)\, ds = 0.$$

Proof. The asymptotics (16) are obtained for W from (27), (7) - (9) and (15).

Similarly, from (26) we find that for large values of r

$$\begin{aligned}V_1(r,\theta) &= -a\mu^2[Mr(2\ln r + 1)\cos\theta + M_1(2\ln r + 2 + \cos 2\theta) + M_2\sin 2\theta]\\ &\qquad + \tilde{V}_1(r,\theta),\\ V_2(r,\theta) &= -a\mu^2[Mr(2\ln r + 1)\sin\theta + M_2(2\ln r + 2 - \cos 2\theta) + M_1\sin 2\theta]\\ &\qquad + \tilde{V}_2(r,\theta),\\ V_3(r,\theta) &= a\mu M[\mu r^2\ln r - 4h^2(\lambda+2\mu)\ln r - 4h^2(\lambda+3\mu)]\\ &\qquad + a\mu^2(M_1\cos\theta + M_2\sin\theta)r(2\ln r + 1) + \tilde{V}_3(r,\theta),\end{aligned}$$

where $\tilde{V} \in A$. This proves the assertion (ii).

We can now formulate and prove the main result concerning the solvability of the four problems defined above.

Theorem 6. (i) (D_{in}) has a unique solution that can be expressed as a double layer potential.

(ii) (N_{in}) is solvable if and only if

$$\int_{\partial\Omega} B_3 ds = \int_{\partial\Omega}(B_\alpha - x_\alpha B_3) ds = 0. \tag{30}$$

The solution can be expressed as a single layer potential and is unique up to an additive rigid displacement (17).

(iii) (D_{ex}) has a unique solution that can be represented as the sum of a double layer potential and a rigid displacement (17).

(iv) (N_{ex}) has a unique solution if and only if

$$\int_{\partial\Omega} S_3 ds = \int_{\partial\Omega}(S_\alpha - x_\alpha S_3) ds = 0. \tag{31}$$

The solution can be represented as a single layer potential.

Proof. (i), (iv) Let $v \in A$ be a solution of (N_{ex}) and K_R the circle considered in the proof of Theorem 1. By direct calculation,

$$\int_{\partial\Omega} S_3 ds = \int_{\partial\Omega} T_{3i} v_i ds = -\int_{\Omega_{ex}\cap K_R} L_{3i} v_i d\sigma + \int_{\partial K_R} T_{3i} v_i ds = \int_{\partial K_R} T_{3i} v_i ds,$$

$$\begin{aligned}\int_{\partial\Omega}(S_\alpha - x_\alpha S_3) ds &= \int_{\partial\Omega}(T_{\alpha i} - x_\alpha T_{3i}) v_i ds \\ &= -\int_{\Omega_{ex}\cap K_R} (L_{\alpha i} - x_\alpha L_{3i}) v_i d\sigma + \int_{\partial K_R}(T_{\alpha i} - x_\alpha T_{3i}) v_i\, ds,\end{aligned}$$

and the equalities (30) follow from (20).

We now assume that (30) hold and seek the solutions of (D_{in}) and (N_{ex}) in the form of W and V, respectively, interpreting (22) and (25) as

$$W_{in}(x) = A(x), \quad x \in \partial\Omega,$$
$$(TV)_{ex}(x) = S(x), \quad x \in \partial\Omega.$$

By Theorem 4 (iii,iv), z must satisfy

$$-\frac{1}{2}z(x) + \int_{\partial\Omega} P(x,y)z(y)ds_y = A(x), \quad x \in \partial\Omega \tag{32}$$

(in the case of (D_{in})) and

$$-\frac{1}{2}z(x) + \int_{\partial\Omega} T(\partial_x)D(x,y)z(y)ds_y = S(x), \quad x \in \partial\Omega \tag{33}$$

(in the case of (N_{ex})). These are mutually adjoint integral equations whose kernels, in view of (10), are weakly singular. Since the index [4] of both (32) and (33) is zero, Fredholm's theorems can be applied to them.

We multiply $(33)_3$ and $(33)_\alpha - x_\alpha(33)_3$ by ds_x and integrate over $\partial\Omega$. In view of (12), we arrive at

$$\begin{aligned}\int_{\partial\Omega} z_3 ds &= -\int_{\partial\Omega} S_3 ds,\\ \int_{\partial\Omega}(z_\alpha - x_\alpha z_3)ds &= -\int_{\partial\Omega}(S_\alpha - x_\alpha S_3)ds.\end{aligned} \tag{34}$$

Suppose that the homogeneous equation (33) has a non-trivial solution $z^o(x)$:

$$-\frac{1}{2}z^o(x) + \int_{\partial\Omega} T(\partial_x)D(x,y)z^o(y)ds_y = 0, \quad x \in \partial\Omega.$$

Then, according to Theorem 4 (i,iv), (34) and Theorem 5 (ii), the single layer potential

$$V^o(x) = \int_{\partial\Omega} D(x,y)z^o(y)ds_y$$

satisfies

$$\begin{aligned}&L(\partial_x)V^o(x) = 0, \quad x \in \Omega_{ex},\\ &(TV^o)_{ex}(x) = 0, \quad x \in \partial\Omega,\\ &V^o \in A.\end{aligned}$$

By Theorem 3(i), $V^o(x) = 0$, $x \in \Omega_{ex}$, and from Theorem 4(ii) it follows that $V^o(x) = 0$, $x \in \partial\Omega$. Since $L(\partial_x)V^o(x) = 0$, $x \in \Omega_{in}$, the uniqueness theorem for

(D_{in}) yields $V^o(x) = 0$, $x \in R^2$, and by (29),

$$z^o(x) = (TV^o)_{in}(x) - (TV^o)_{ex}(x) = 0, \quad x \in \partial\Omega.$$

Hence, the homogeneous equation (32) also has only the trivial solution, and we conclude that both (32) and (33), therefore (D_{in}) and (N_{ex}), are uniquely solvable. Furthermore, by (31), (34) and Theorem 5(ii), the solution V of (N_{ex}) belongs to $\mathcal{A}$.

(ii) Seeking the solution of (N_{in}) in the form of V and interpreting (23) as

$$(TV)_{in}(x) = B(x), \quad x \in \partial\Omega,$$

from Theorem 4(iv) we deduce that $z(x)$ satisfies

$$\frac{1}{2} z(x) + \int_{\partial\Omega} T(\partial_x)D(x,y)z(y)ds_y = B(x), \quad x \in \partial\Omega. \tag{35}$$

As above, the index of (35) is zero.

Clearly, a rigid displacement v_o given by (17) satisfies

$$L(\partial_x)v_o(x) = 0, \quad x \in \Omega_{in},$$
$$T(\partial_x)v_o(x) = 0, \quad x \in \partial\Omega.$$

By (14),

$$v_o(x) = -\int_{\partial\Omega} P(x,y)v_o(y)ds_y, \quad x \in \Omega_{in}.$$

The right-hand side of this equality is a double layer potential, so letting $x \to x_o \in \partial\Omega$, from $(28)_1$ we obtain

$$\frac{1}{2} v_o(x) + \int_{\partial\Omega} P(x_o,y)v_o(y)ds_y = 0, \tag{36}$$

which indicates that v_o is a solution of the adjoint homogeneous equation. It is easy to check that

$$\nu^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -x_1 \end{bmatrix}, \quad \nu^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -x_2 \end{bmatrix}, \quad \nu^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent solutions of the latter, therefore the homogeneous equation (35) has at least three linearly independent solutions $\mu^{(i)}$. As in [3], it can be shown that $\{\mu^{(i)}\}$ is a complete system, which implies that so is $\{\nu^{(i)}\}$ for (36). Hence, (35) is solvable if and only if

$$\int_{\partial\Omega} \nu^{(i)\prime} B \, ds = 0,$$

which is equivalent to (30).

(iii) Suppose that the systems $\{\mu^{(i)}\}$ and $\{\nu^{(i)}\}$ above have been biorthonormalized. Seeking the solution of (D_{ex}) in the form $W+C_i\nu^{(i)}$, where

$$C_i = \int_{\partial\Omega} \mu^{(i)\prime} R \, ds, \tag{37}$$

and interpreting (24) as

$$W_{ex}(x)+C_i\nu^{(i)}(x) = R(x), \quad x \in \partial\Omega.$$

from $(28)_2$ we find that the unknown density z of W must satisfy the integral equation

$$\frac{1}{2} z(x) + \int_{\partial\Omega} P(x,y) z(y) ds_y = R(x) - C_i\nu^{(i)}(x), \quad x \in \partial\Omega, \tag{38}$$

whose index is zero because it is adjoint to (35). By (37),

$$\int_{\partial\Omega} \mu^{(k)\prime} (R - C_i\nu^{(i)}) ds = 0,$$

so (38) is solvable. The fact that the solution $W+C_i\nu^{(i)}$ belongs to A^* follows from Theorem 5(i), (17) and the definition of A^*.

<u>Remarks</u>. (i) On account of (1) and the averaging procedure employed to derive the system (2), the solvability conditions (30) and (31) imply that the total force and

moment across $\partial\Omega$ must be zero.

(ii) The logarithmic term in $(16)_3$ is acceptable since, in view of (5), for any solution $v \in A$ (or $v \in A^*$) of (2) the internal energy remains finite.

VI. EXAMPLE

We consider an infinite plate with a circular hole of radius q, whose lateral surface $\partial\Omega \times (-h_o/2, h_o/2)$ is acted upon by a normal force px_3, $p = \text{const.}$ Suppose, for simplicity, that the body forces and moments, and the forces and moments on the faces $x_3 = \pm h_o/2$, are negligible.

Choosing the origin at the centre of the hole and noting that the appropriate boundary conditions of the problem are

$$\begin{aligned} T_{ij}(\partial_x)v_j(x) &= (ph^2/q)x_\alpha \delta_{\alpha i}, \quad x \in \partial\Omega, \\ T_{ij}(\partial_x)v_j(x) &\to 0 \quad \text{as} \quad |x| \to \infty, \end{aligned} \tag{39}$$

it is not difficult to check that (21) and (39) are satisfied by $v = \tilde{v} + v_o$, where

$$\begin{aligned} \tilde{v}_\alpha(x) &= -(pq^2/2\mu)(x_\alpha/|x|^2), \\ \tilde{v}_3(x) &= (pq^2/2\mu)\ln|x|, \end{aligned}$$

and v_o is an arbitrary rigid displacement (17). Hence we conclude that in general the solution of this Neumann problem in Ω_{ex} is not unique. However, its solution in A is unique because, by (16), $v \in A$ only if $c = pq^2/4\mu$ and $c_1 = c_2 = 0$ in (17), that is,

$$v_o' = (0, 0, pq^2/4\mu)$$

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