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TWO - DIMENSIONAL ANALYSIS OF A HYDRAULIC TURBINE WITH HELICOIDAL BLADES

Ion Paraschivoiu

ABSTRACT

The complex potential flow method is applied to the case of a hydraulic turbine with helicoidal blades. The vortex flow in the central part of the rotor is considered in particular in the analysis undertaken in this work.

Introduction

The complex potential flow method was originally developed by König [1] and later applied to the case of hydraulic rotors with helicoidal blades by Sörensen [2] who neglected the vortex flow in the central part of the rotor. However, in view of the interest in converting the kinetic energy of the vortex flow developed in the center of the rotor, the vortex distribution through the rotor has been calculated as a function of the number of helicoidal blades. The working fluid is assumed inviscid and incompressible and the flow is two-dimensional.

1. Complex potential flow

The flow of a fluid around an obstacle can be represented by a plane circular flow around systems of sources (or sinks) and vortices. In the case of a cascade, once the blades are represented in the z plane, they are also represented in the ξ plane with the real axis $O'\xi$ used for the blades and the upper half-plane comprising all the points outside the cascade. This type of transformation, which is known as a conformal transformation function, is expressed as follows:

$$z = e^{i\alpha}\,\ell n\,\frac{\zeta - \zeta_a}{\zeta - \zeta_b} + e^{-i\alpha}\,\ell n\,\frac{\zeta - \bar{\zeta}_a}{\zeta - \bar{\zeta}_b} \quad , \tag{1}$$

where

$$\zeta = \xi + i\eta \quad ,$$
$$\zeta_a = \xi_a + i\eta_a = \rho_a e^{i\nu_a} \quad ,$$
$$\zeta_b = \xi_b + i\eta_b = \rho_b e^{i\nu_b} \quad .$$

In this expression ξ_a and ξ_b are the complex numbers of the tips of a blade bearing the sources and vortices and $\bar{\xi}_a$, and $\bar{\xi}_b$ their inverse elements with respect to $0'\xi$, representing the streamline for the flow in the ξ plane. The following equations are obtained in this plane:

a) $$\pi \cdot \frac{\ell}{h} = \ell n\,\frac{\sin(\nu_b + \alpha - \pi/2)}{\sin(\nu_b + \pi/2 - \alpha)} - (\pi - 2\nu_b)\tan\alpha \tag{2}$$

b) $$\nu_a + \nu_b = \pi \tag{3}$$

c) $$\frac{\rho_a}{\rho_b} = \frac{\sin(\nu_b + \alpha - \pi/2)}{\sin(\nu_b + \pi/2 - \alpha)} \tag{4}$$

with $$\left.\begin{aligned} h &= 2\pi \cos\alpha \\ h &= t \cos\alpha \end{aligned}\right\} t = 2\pi$$

The z_1 plane is obtained for a clockwise rotation of angle α around the origin, namely

$$z_1 = ze^{-i\alpha} \tag{5}$$

To represent the domain of the width t in a z_2 plane, the transformed plane

$$z_2 = e^{z_1} \tag{6}$$

is used, which allows us to obtain the blades on the arch of a logarithmic spiral. Finally, in the physical plane we have

$$Z = z_2^{1/n} \ , \tag{7}$$

whence

$$\begin{aligned}
&z_2 = e^{z_1} \quad \text{or} \quad z_2 = e^{ze^{-i\alpha}} , \\
&Z = z_2^{1/n} = (e^z)^{e^{-i\alpha}} , \\
&z = e^{i\alpha} \ell n \frac{\zeta - \zeta_a}{\zeta - \zeta_b} + e^{-i\alpha} \ell n \frac{\zeta - \bar{\zeta}_a}{\zeta - \bar{\zeta}_b} , \\
&e^z = \left(\frac{\zeta - \zeta_a}{\zeta - \zeta_b}\right)^{e^{i\alpha}} \cdot \left(\frac{\zeta - \bar{\zeta}_a}{\zeta - \bar{\zeta}_b}\right)^{e^{-i\alpha}} , \\
&Z(\zeta) = \left[\left(\frac{\zeta - \zeta_a}{\zeta - \zeta_b}\right)^{e^{i\alpha}} \cdot \left(\frac{\zeta - \bar{\zeta}_a}{\zeta - \bar{\zeta}_b}\right)^{e^{-i\alpha}}\right]^{\frac{1}{n} e^{-i\alpha}}
\end{aligned} \tag{8}$$

It is assumed that for

$Z = 0$ → we have a source

$Z = \infty$ → we have a sink

which means that in ξ_a and $\bar{\xi}_a$ we put a source and in ξ_b and $\bar{\xi}_b$ a sink (negative source).

In this case, the complex potential of the flow $\phi_1(\xi)$ can be written:

$$\phi_1(\zeta) = \frac{Q}{2\pi}\,\ell n\,\frac{(\zeta - \zeta_a)}{(\zeta - \zeta_b)} \cdot \frac{(\zeta - \bar{\zeta}_a)}{(\zeta - \bar{\zeta}_b)} + i\,\frac{\Gamma o}{2\pi}\,\ell n\,\frac{\zeta - \zeta_a}{\zeta - \bar{\zeta}_a}$$

$$+ i\,\frac{\Gamma o + \Gamma}{2\pi}\,\ell n\,\frac{\zeta - \zeta_b}{\zeta - \bar{\zeta}_b} \;, \qquad (9)$$

and $\phi_2(\xi)$, due to blade rotation, is given by [3]

$$\phi_2(\zeta) = \frac{1}{\pi}\int_{-\infty}^{+\infty} Cn(\ell)\ \ell n\ (\zeta - \ell)d\ell\ . \qquad (10)$$

If $\phi(\xi)$ is designated as representing the total complex flow potential, then

$$\Phi(\zeta) = \Phi_1(\zeta) + \Phi_2(\zeta)\ , \qquad (11)$$

where

$$\Phi(\zeta) = \frac{Q}{2\pi}\,\ell n\,\frac{(\zeta - \zeta_a)(\zeta - \bar{\zeta}_a)}{(\zeta - \zeta_b)(\zeta - \bar{\zeta}_b)} + i\,\frac{\Gamma o}{2\pi}\,\ell n\,\frac{\zeta - \zeta_a}{\zeta - \bar{\zeta}_a}$$

$$+ i\,\frac{\Gamma o + \Gamma}{2\pi}\,\ell n\,\frac{\zeta - \zeta_b}{\zeta - \bar{\zeta}_b} + \frac{1}{\pi}\int_{-\infty}^{+\infty} C_n(\ell)\ell n(\zeta - \ell)d\ell . \qquad (12)$$

However, the tangential velocity at each point of the blade in the complex plane is given by

$$C_n(\ell)\Big|_{\zeta \doteq \ell} = v_n \left|\frac{dZ}{d\zeta}\right|_{\zeta=\ell} = \omega|Z|\ \cos\,\alpha \cdot \left|\frac{dZ}{d\zeta}\right|\ (\zeta=\ell) \qquad (13)$$

Since $|e^{i\alpha}| = 1$, we obtain the following equation

$$C_n(\ell) = \frac{1}{n}\,\omega\cos\alpha \left| \left[\left(\frac{\ell-\zeta_a}{\ell-\zeta_b}\right)^{e^{i\alpha}} \cdot \left(\frac{\ell-\bar{\zeta}_a}{\ell-\bar{\zeta}_b}\right)^{e^{-i\alpha}} \right]^{\frac{2}{n}e^{-i\alpha}} \right|$$

$$\cdot \left| \frac{(\zeta_a-\zeta_b)e^{i\alpha}}{(\ell-\zeta_a)(\ell-\zeta_b)} + \frac{(\bar{\zeta}_a-\bar{\zeta}_b)e^{-i\alpha}}{(\ell-\bar{\zeta}_a)(\ell-\bar{\zeta}_b)} \right| \quad . \tag{14}$$

Let

$$\frac{\ell-\zeta_a}{\ell-\zeta_b} = \mu e^{i\nu} \quad ,$$
$$\frac{\ell-\bar{\zeta}_a}{\ell-\bar{\zeta}_a} = \mu e^{-i\nu} \quad . \tag{15}$$

The expressions $\mu(\ell)$ and $\nu(\ell)$ are given by

$$\mu(\ell) = \tfrac{1}{2}\sqrt{\frac{[(\ell-\xi_a)(\ell-\xi_b)+\eta_a\eta_b]^2+[(\ell-\xi_a)\eta_b-(\ell-\xi_b)\eta_a]^2}{(\ell-\xi_b)^2+\eta_b^{\,2}}} \tag{16}$$

$$\tan\nu = \frac{(\ell-\xi_a)\eta_b-(\ell-\xi_b)\eta_a}{(\ell-\xi_a)(\ell-\xi_b)+\eta_a\eta_b} \quad . \tag{17}$$

Since

$$a^x = e^{x\ell n a} \quad , \tag{18}$$

then

$$\left| \left[\left(\frac{\ell-\zeta_a}{\ell-\zeta_b}\right)^{e^{i\alpha}} \cdot \left(\frac{\ell-\bar{\zeta}_a}{\ell-\bar{\zeta}_b}\right)^{e^{-i\alpha}} \right]^{\frac{2}{n}e^{-i\alpha}} \right| = \mu^{2/n(1+\cos 2\alpha)} \cdot e^{-2/n\nu\sin 2\alpha} \tag{19}$$

and

$$\frac{(\zeta_a - \zeta_b)e^{i\alpha}}{(\ell - \zeta_a)(\ell - \zeta_b)} + \frac{(\bar{\zeta}_a - \bar{\zeta}_b)e^{-i\alpha}}{(\ell - \bar{\zeta}_a)(\ell - \bar{\zeta}_b)} = \frac{2\ell[\cos\alpha(\xi_b^2 - \xi_a^2 + \eta_b^2 - \eta_a^2) - 2\sin\alpha(\xi_a\eta_b - \xi_b\eta_a)]}{[(\ell - \xi_a)^2 + \eta_a^2][(\ell - \xi_b)^2 + \eta_b^2]} \tag{20}$$

whence

$$C_n(\ell) = \frac{1}{n}\ \omega\cos\alpha\ \mu^{2/n(1+\cos 2\alpha)} \cdot e^{-2/n \sin 2\alpha} \cdot \frac{K \cdot \ell}{[(\ell - \xi_a)^2 + \eta_a^2][(\ell - \xi_b)^2 + \eta_b^2]} \tag{21}$$

where

$$K = 2[\cos\alpha(\xi_b^2 - \xi_a^2 - \eta_b^2 - \eta_a^2) - 2\sin\alpha(\xi_a\eta_b - \xi_b\eta_a)] \tag{22}$$

2. <u>Calculation of circulation through the turbine (Γo) and around the blades (Γ)</u>

The velocities at the blade tips in the z plane are observed to be finite. Consequently the corresponding velocities in the ξ plane must be zero. Hence

$$\frac{d\Phi}{dZ} = \frac{d\Phi}{d\zeta} \cdot \frac{d\zeta}{dZ} = 0 = u + iv \tag{23}$$

in other words $u = 0$

$v = 0$

The blade tips are given for $\xi = 0$ and $\xi = \infty$. Therefore:

$$\left.\frac{d\Phi}{dZ}\right|_{\substack{\zeta=0\\ \zeta=\infty}} = \frac{d\Phi}{d\zeta} \cdot \left.\frac{d\zeta}{dZ}\right|_{\substack{\zeta=0\\ \zeta=\infty}} = 0 \tag{24}$$

a) When $\xi = 0 \rightarrow$, it can be shown that $\frac{d\xi}{dZ} \neq 0$

therefore $\frac{d\phi}{d\xi} = 0$ (25)

where $\frac{d\phi_1}{d\zeta} + \frac{d\phi_2}{d\zeta} = 0$ at $\xi = 0$ (26)

such that

$$\left.\frac{d\phi_1}{d\zeta}\right|_{\zeta=0} = - \frac{Q}{2\pi}(2\xi_a) - \frac{\Gamma o}{2\pi}(2\eta_a) + \frac{(2\xi_b)Q-(\Gamma o+\Gamma)(2\eta_b)}{2\pi\rho_b^2} \tag{27}$$

and

$$\left.\frac{d\phi_2}{d\zeta}\right|_{\zeta=0} = - \frac{1}{\pi}\int_{-\infty}^{+\infty} C_n(\ell)\frac{d\ell}{\ell} \tag{28}$$

The general expression for $\left.\frac{d\phi}{d\xi}\right|_{\xi=0} = 0$ becomes

$$Q\xi_a + \Gamma_o\eta_a - \frac{Q\xi_b - (\Gamma o+\Gamma)\eta_b}{\rho_b^2} + \int_{-\infty}^{+\infty} C_n(\ell)\frac{d\ell}{\ell} = 0 \tag{29}$$

b) If $\xi = \infty$, we have

$$\left.\frac{d\phi_1}{d\zeta}\right|_{\zeta\to\infty} = \frac{Q}{2\pi}\left\{\frac{1}{\zeta-\zeta_a} + \frac{1}{\zeta-\bar{\zeta}_a} - \frac{1}{\zeta-\zeta_b} - \frac{1}{\zeta-\bar{\zeta}_b}\right\} + \frac{\Gamma o}{2\pi}\left\{\frac{1}{\zeta-\zeta_a} - \frac{1}{\zeta-\bar{\zeta}_a}\right\}$$

$$+ i\,\frac{\Gamma o+\Gamma}{2\pi}\left\{\frac{1}{\zeta-\zeta_b} - \frac{1}{\zeta-\bar{\zeta}_b}\right\} \tag{30}$$

Taking series development at a point (for $\xi = \infty$), we obtain the following expression

$$\left.\frac{d\phi_1}{d\zeta}\right|_{\zeta\to\infty} = \frac{1}{2\pi}\cdot\frac{(-2)}{\zeta^2}\left[\xi_a Q + \eta_a\Gamma_o - \xi_b Q + \eta_b(\Gamma_o+\Gamma)\right] \tag{31}$$

and, also,

$$\frac{d\Phi_2}{d\zeta} = \frac{1}{\pi}\int_{-\infty}^{+\infty} C_n(\ell)\,\frac{d\ell}{\zeta-\ell}$$

In serial form, we can write

$$\frac{1}{\zeta - \ell} = \frac{1}{\zeta} + \frac{\ell}{\zeta^2} + \frac{\ell^2}{\zeta^2} \cdot \frac{1}{\zeta - \ell} \tag{32}$$

$$\frac{d\Phi_2}{d\zeta} = \frac{1}{\pi}\left\{\frac{1}{\zeta}\int_{-\infty}^{+\infty} C_n(\ell)d\ell + \frac{1}{\zeta^2}\int_{-\infty}^{+\infty} C_n(\ell)\ell\, d\ell + \frac{1}{\zeta^2}\int_{-\infty}^{+\infty} C_n(\ell)\,\frac{\ell^2}{\zeta - \ell}\,d\ell\right\} . \tag{33}$$

We can show on the basis of a few mathematical concepts that

$$\frac{C_n(\ell)\ \ell^2}{\zeta - \ell} = \sum_{n=1}^{\infty} A_n\left[\frac{1}{\zeta^n(\zeta - \ell)} + \frac{1}{\zeta^n \ell} + \dots + \frac{1}{\zeta \ell^n}\right] \tag{34}$$

for ℓ about infinity , and

$$\frac{C_n(\ell)\ \ell^2}{\zeta - \ell} = \left[\frac{1}{\zeta - \ell} + \frac{1}{\ell}\right] \cdot \sum_{n=1}^{\infty} A_n \cdot \frac{1}{\zeta^n} + \lambda_r ,$$

$$\lambda_r = \text{residue} .$$

Whence , for $\zeta \to \infty$

$$\frac{C_n(\ell)\ \ell^2}{\zeta - \ell} = 0 . \tag{35}$$

As $\left.\frac{d\Phi_2}{d\zeta}\right|_{\zeta\to\infty} = \frac{1}{\pi}\left\{\frac{1}{\zeta}\int_{-\infty}^{+\infty} C_n(\ell)d\ell + \frac{1}{\zeta^2}\int_{-\infty}^{+\infty} C_n(\ell)\ell\ d\ell\right\}$

then $\left.\frac{d\Phi_1}{d\zeta}\right|_{\zeta\to\infty} + \left.\frac{d\Phi_2}{d\zeta}\right|_{\zeta\to\infty} = 0$ (36)

$$\frac{1}{\pi} \cdot \frac{1}{\zeta}\int_{-\infty}^{+\infty} C_n(\ell)\ d\ell = 0 \to \int_{-\infty}^{+\infty} C_n(\ell)\ d\ell = 0 \tag{37}$$

$$\frac{-2}{2\pi\zeta^2}\left[\xi_a Q + \eta_a\Gamma_o - \xi_b Q + \eta_b(\Gamma_o + \Gamma)\right] + \frac{1}{\pi} \cdot \frac{1}{\zeta^2}\int_{-\infty}^{+\infty} C_n(\ell)\ell\ d\ell = 0 \tag{38}$$

$$-\xi_a Q - \eta_a\Gamma_o + \xi_b Q - \eta_b(\Gamma_o + \Gamma) + \int_{-\infty}^{+\infty} C_n(\ell)\ell\ d\ell = 0$$

The following equations can be used to find Γ_o and Γ:

$$Q\xi_a + \Gamma_o\eta_a - \frac{Q\xi_b - (\Gamma_o + \Gamma)\eta_b}{\rho_b^2} + \int_{-\infty}^{+\infty} c_n(\ell)\,\frac{d\ell}{\ell} = 0 \tag{39}$$

$$-Q\xi_a - \Gamma_o\eta_a + Q\xi_b - \eta_b(\Gamma_o + \Gamma) + \int_{-\infty}^{+\infty} c_n(\ell)\,\ell\,d\ell = 0 \tag{40}$$

whence the following values can be derived:

$$\Gamma_o = -\left[Q\,\frac{\xi a}{\eta_a} + \frac{J_1 + J_2}{(\rho_b^2 - 1)\eta_a}\right] \tag{41}$$

$$\Gamma = \Gamma_o + Q\,\frac{\xi b}{\eta_b} + \frac{\rho_b^2 J_1 + J_2}{(\rho_b^2 - 1)\eta_b} \tag{42}$$

with

$$\begin{aligned} J_1 &= \int_{-\infty}^{+\infty} c_n(\ell)\cdot\ell\cdot d\ell \\ J_2 &= \int_{-\infty}^{+\infty} c_n(\ell)\cdot\frac{d\ell}{\ell} \end{aligned} \tag{43}$$

These two integrals are calculated in the next section.

3. Numerical application

The following example [4] is to illustrate the equations developed above:

n_1 = 1,2,3,4,5,12,24 (number of blades)

V_∞ = 5 m/s (flow velocity)

H = 1 m (height of turbine with helicoidal blades)

D = 2 m (diameter of turbine with helicoidal blades)

$n_1 Q = V_\infty \times HD$ (total flow)

The integrals (Eq. 43)

$$J_1 = \int_{-\infty}^{+\infty} Cn(\ell) \cdot \ell \cdot d\ell$$

$$J_2 = \int_{-\infty}^{+\infty} Cn(\ell) \frac{d\ell}{\ell}$$

were calculated numerically for each value of n_1, and $n_1 = 6$ was illustrated graphically (Figs. 1 and 2) using Romberg's and Tchebichev's methods.

Circulations Γ_o, Γ for $n_1 = 6$ are represented in Fig. 3. The ratio of the radii $R_1/R(R_1$: internal radius; R: external radius of the turbine) as a function of n_1 is shown in Fig. 4 for $\alpha = 60^\circ$ and $\nu_b = 30.13^\circ$.

By way of conclusion, it may be said that the kinetic energy of the vortex created at the center of a turbine with helicoidal blades corresponding to a circulation Γ_o must be taken into account in hydraulic-turbine calculations.

REFERENCES

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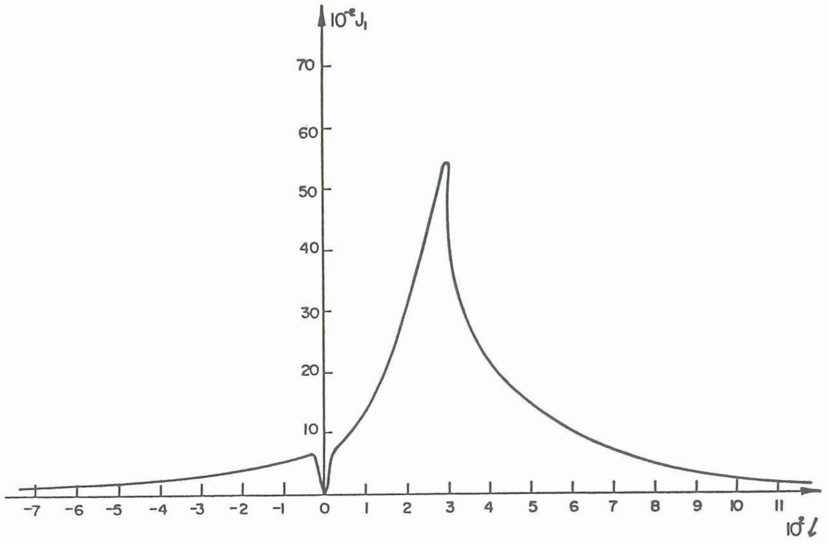

Fig. 1 - Integral J_1 for $n_1 = 6$

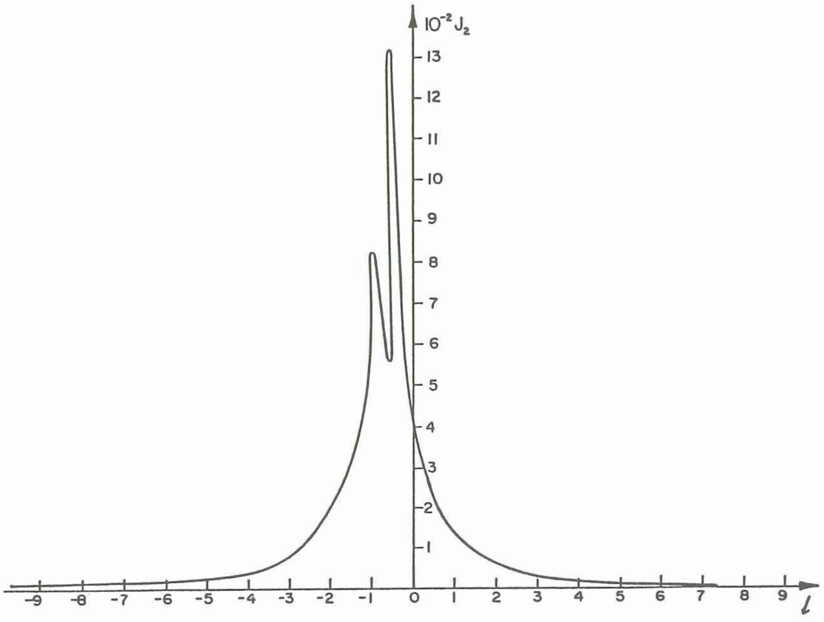

Fig. 2 - Integral J_2 for $n_1 = 6$

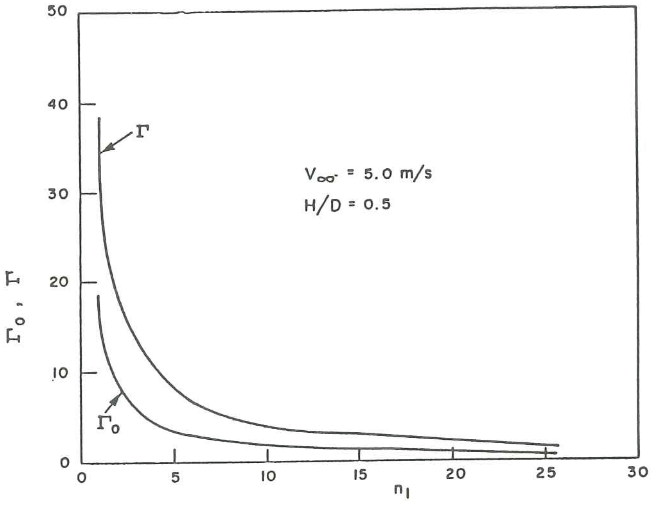

Fig. 3 - Circulations Γ_o and Γ vs n_1

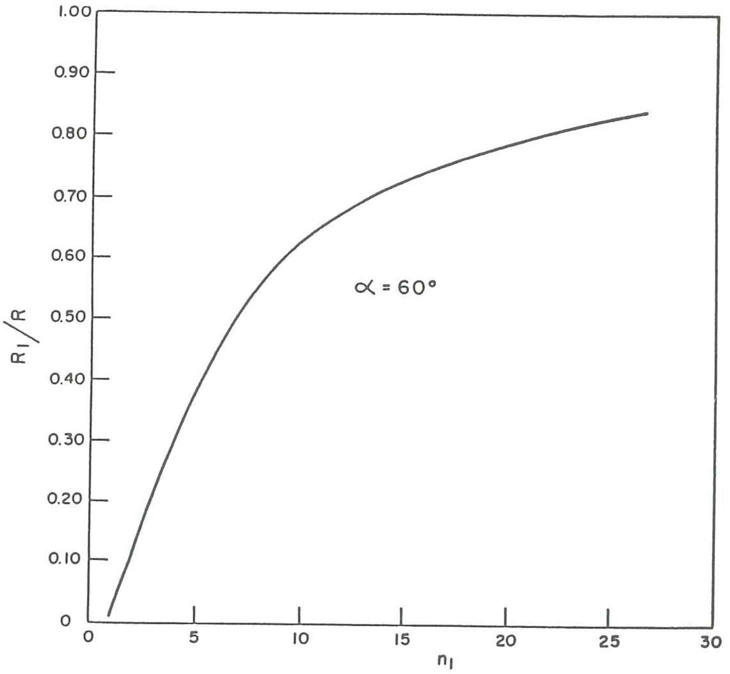

Fig. 4 - R_1/R ratio vs n_1