

DOES THE QUADRATIC EQUATION HAVE GREEK ROOTS?
A STUDY OF "GEOMETRIC ALGEBRA", "APPLICATION
OF AREAS", AND RELATED PROBLEMS

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"Pe unde iese cuvântul iese și sufletul."

Romanian proverb¹

"[L'Abbé de Condillac] ... établit que nous ne pensons q'avec le secours des mots; que les langues sont de véritables méthodes analytiques; que l'algèbre la plus simple, la plus exacte et la mieux adaptée à son objet de toutes les manières de s'énoncer est à-la-fois une langue et une méthode analytique; enfin que l'art de raisonner se réduit à une langue bien faite ... Le mot doit faire naître l'idée; l'idée doit peindre le fait: ce sont trois empreintes d'un meme cachet; et comme ce sont les mots qui conservent les idées et qui les transmettent, il en résulte qu'on ne peut perfectionner le langage sans perfectionner la science, ni la science sans le langage, et que quelque certains que fussent les faits, quelque justes que fussent les idées qu'ils auroient fait naître, ils ne transmettroient encore que des impressions fausses, si nous n'avions pas des expressions exactes pour les rendre."

A. L. Lavoisier²

"... every language-act has a temporal determinant. No semantic form is timeless. When using a word we wake into resonance ... its entire previous history. A text is embedded in specific historical time; it has ... a diachronic structure. To read fully is to restore all that one can of the immediacies of value and intent in which speech actually occurs."

"No safety-wire in the publicly available grammar stops us from talking nonsense correctly."

"This insinuation of self into otherness is the final secret of the translator's craft."

George Steiner³

"Die Sprache ist das bildende Organ des Gedankens."

Wilhelm von Humboldt⁴

"A translator is to be like his author, it is not his business to excel him."

Dr. Johnson⁵

"Error is never so difficult to be destroyed as when it has its root in Language."

Bentham⁶

"A good interpretation of anything — a poem, a person, a history, a ritual, an institution, a society — takes us into the heart of that of which it is the interpretation. When it does not do that, but leads us instead somewhere else — into an admiration of its own elegance, of its author's cleverness, or of the beauties of Euclidean order — it may have its intrinsic charms; but it is something else than what the task at hand ... calls for."

Clifford Geertz⁷

INTRODUCTION

This essay is the intellectual progeny of some longstanding preoccupations with basic historiographic matters by one of its authors. It is, specifically, a direct outgrowth of issues dealt with in an article published in the Archive for History of Exact Sciences.⁸ That article was meant as a sweeping attack on the prevailing methodology of historians of ancient mathematics, pointing out, on the one hand, the inherent inadequacies of that methodology and, on the other hand, suggesting an alternative, more sympathetic approach to the historical study of ancient texts. Being, in part, an open critical assessment of a conventional, customary, and recognized style in the writing of the history of mathematics, it was, by its very nature, more than latently polemical. It is, indeed, arguable whether this represented a praiseworthy feature or one of the many reasons for its tattered fortuna.⁹ But, be that as it may, the bête noire of its historical wrath, one of the crucial concepts in the interpretation of Greek mathematical texts, namely, the concept of "geometric algebra," seems to go on living somehow in the works of many historians of mathematics, in spite of its many shortcomings.

These shortcomings, some of them fatal, were pointed out at great length in the article "On the Need to Rewrite the History of Greek Mathematics" and need not detain us here. What needs to be stated, however, is that the criticisms levelled against the interpretive approach embodied in the idea of "geometric algebra" focussed on fundamental historico-philosophical considerations and less on the mathematical underpinnings and consequences of the positions adopted by "geometrical algebraists." Moreover it was

repeatedly stated in the course of the argument that mathematically there was nothing wrong with the reasonings of the adherents of "geometrical algebra;" what was wrong was endowing those reasonings with historical value. It is, therefore, quite possible that for the mathematically minded historians, who "assume tacitly or explicitly that mathematical entities reside in the world of Platonic ideas where they wait patiently to be discovered by the genius of the working mathematician,"¹⁰ the very fact that there was nothing wrong with the mathematics of "geometric algebra" was enough of an enticement to see "geometric algebra" as vindicated, in spite of the "irrelevant" historical, philosophical, and linguistic arguments to the contrary. If the "mathematical historians" swear by the eleventh commandment (that "mathematical equivalence is ... historical equivalence"¹¹) and if non-mathematical arguments, be they historical, philosophical, linguistic or whatever, are largely irrelevant then, indeed "geometrical algebra" is godly and immortal.

But, alas, it seems that even this concession to the practitioners of "geometric algebra" is an unnecessary one. For as the present work will convincingly show, the mathematical correspondence that has been concocted in order to demonstrate the equivalence of certain propositions in Greek geometry with other more familiar results from elementary algebra is, in fact, very weak in several essential areas. Thus, while the present work represents a continuation of the line of argument and the entire foray of "On the Need to Rewrite the History of Greek Mathematics," and while it adopts in toto its conclusions, it nevertheless concentrates directly on the mathematical content of "geometric algebra." It takes the mathematical arguments of the proponents of "geometric algebra" very seriously and at face value, drawing the ultimate implications from these arguments. It follows the mathematical arguments wherever they lead. It adopts the devil's stance (dangerous stance, as any true believer knows) in order to defeat the devil. It plays the algebraic game in order to show its unacceptability. It aims at undermining the position of the geometrical algebraists from within. It contains a two-pronged attack on their approach and results based, on the one hand, on sapping the "arithmetic" foundations of "geometric algebra" and, on the other hand, on adducing substantial and weighty evidence indicating that its alleged purpose, i.e., solving equations, could NOT have been a matter of immediate concern (if any) to Greek mathematicians of the classical age. In sum, tactically donning the hat of the geometrical algebraists, it espouses their mathematical cause and pursues their line of argument to its bitter end, creeping, as it were, under their skin, in order to show the unwanted and ludicrous, but necessary and incriminating, extreme consequences of their views; in taking up the cudgels for "geometric algebra," it strives to

display its ahistoricity as an interpretive device for Greek mathematics. On the positive side, it advances throughout an alternative interpretation, one that does no violence to the texts and their overwhelming geometric character.

The two-pronged attack on "geometric algebra" that is presented here involves a detailed examination of a good deal of mathematics, most of which concerns various results that are found in the Elements of Euclid. For this reason, a prior knowledge of the Elements, while certainly not indispensable, will prove to be very beneficial for those readers who wish to grasp the full force of the arguments presented here, many of which are fairly technical. Still, we have attempted here to make this study accessible to as large an audience as possible, and every effort has been made to keep the presentation as self-contained and self-explanatory as possible. Of course the drawback to this approach is that it adds considerable length to an already long article. This being the case, we would recommend that those readers who are thoroughly versed in their Euclid should skim over Section III, No. 4 and 5, and proceed as quickly as possible to Section III, No. 6 wherein the heart of our argument commences.

I

1. Before addressing the issue of "geometric algebra" directly, a few general remarks concerning the nature of algebra itself, as well as its relationship to arithmetic procedures, need to be made. In general, the existence of a coherent system of arithmetical operations is a necessary (though not sufficient) precondition for the existence of any system of algebra. For fundamental to the modern post-Viêtan algebraic enterprise, used in practice by the proponents of "geometric algebra," is the abstract treatment of number, wherein various arithmetic properties and arithmetical relationships between numbers are extracted, generalized, and thereby exploited via a system of symbolic manipulation. But under any suitable, historically reasonable definition of algebra, ancient Babylonian and classical Greek mathematical texts are not algebraic in character. In the Babylonian case they are arithmetical, while in the Greek they are geometrical. Both differ, then, from the algebraic mode of thinking. It is stacking the cards illegitimately and engaging in what amounts to a petitio principii to define algebra in an ad hoc manner as the type of reasoning embodied in Babylonian and Greek mathematical texts.¹² Not only is this not enlightening, as well as historically unacceptable and philosophically indefensible, but it assumes precisely that which needs proving: an underlying algebraic substructure bolstering ancient mathematical texts. And yet, this is exactly what is involved in

the traditional interpretation of ancient mathematics by historians of mathematics.

Let us see what this means by way of an example drawn from the mathematical cuneiform texts published by Otto Neugebauer, namely BM 13901, which reads in translation:

I have subtracted the [side] of the square from the area, and 14, 30 is it.¹³

Van der Waerden, in response to Unguru's criticism of his position, having defined algebra as "the art of handling algebraic expressions like $(a+b)^2$ and of solving equations like $x^2 + ax = b$,"¹⁴ has the following to say about our cuneiform text:

The statement of the problem is completely clear: It is not necessary to translate it into modern symbolism. If we do translate it, we obtain the equation

$$x^2 - x = 870.$$

The solution given in the text reads:

Take 1, the coefficient (of the unknown side). Divide 1 into two equal parts: 0; 30 times 0; 30 is 0; 15. Add this to 14, 30 and (the result) 14, 30; 15 has 29; 30 as a square root. Add the 0; 30 which you have multiplied by itself to 29; 30, and 30 is the (side of the) square.

This is the same method of solution we learn at school. According to our definition, this is algebra.¹⁵

What is one to say about this "straightforward" interpretation, and is it not rather damaging to our approach? Before attempting to answer this question, we must say a few words about van der Waerden's historical scholarship in this case. Briefly, van der Waerden introduces his own modifications in the cited text without calling this to the attention of the reader and these modifications, needless to say, are all supportive of his interpretive bias. Thus van der Waerden's translation of Neugebauer's rendering of the solution contains the former's editorial improvements in the same kind of parenthesis that the latter uses for his textual emendations, thus preventing the reader from realising that the original cuneiform has been improved twice. Since, allegedly, van der Waerden is quoting Neugebauer, such a procedure is inadmissible.¹⁶ Also, we disagree with van der Waerden's claim that "the method" described in the cuneiform text is the method "we learn at school."

In our schools, algebra is not taught by recipes. They teach chefs de cuisine and simple cooks by this method, not students in secondary schools; the latter are taught first the method of solution of a general quadratic and then one exemplifies the method by means of specific equations. Once the general method is understood (or the quadratic formula available), there is no need anymore for a whole series of particular, specific equations.

Let us now return to the question of van der Waerden's interpretation of our cuneiform text. Does it fit? And if it does, is it the only possible interpretation? By transcribing the text as $x^2 - x = 870$, van der Waerden shows that the steps followed by the scribe in the solution of the problem fit exactly the quadratic formula (without the second, negative solution, we might add):

$$\frac{1}{2} + \sqrt{\frac{1}{4} + 870} = \frac{1}{2} + \frac{59}{2} = 30$$

But does this exact fit prove that this is how the scribe proceeded? Not necessarily. There are other possible interpretations that fit equally well and that are, at the same time, more in tune with the character of Babylonian mathematics. Here is one such interpretation.

The fascination of the Babylonian mathematician (the scribe or the originator of the cuneiform mathematical tablets) with numbers and numerical operations is well known.¹⁷ The various "table texts" discussed by Neugebauer are a case in point. There are multiplication tables, tables of reciprocals, tables of "Pythagorean numbers," tables of squares and square roots, cubes and cubic roots, of sums of squares and cubes, exponential tables, etc.¹⁸ It would not, therefore, be out of character to assume that the Babylonian mathematician knew, as a result of trial and error, how to square a sum or a difference of specific numbers, say (written anachronistically), that

$$(5 \pm 3)^2 = 5^2 \pm 2 \cdot 5 \cdot 3 + 3^2 = \begin{matrix} 64 \\ 4 \end{matrix}$$

Since in the properly mathematical texts the overwhelming majority of problems are solved by starting with the known answer,¹⁹ the mere knowledge of how "to complete the square" is enough to understand fully, step by step, the scribe's procedure in the solution of BM 13901 above. Thus, the sequence of steps described in the cuneiform text fits exactly the following order of succession, the scribe having started with the knowledge that $30^2 - 30 = 14,30$:

$$30^2 - 2 \cdot 30 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 = 14,30 + \left(\frac{1}{2}\right)^2$$

$$\left(30 - \frac{1}{2}\right)^2 = 14,30;15,$$

and, since the numbers are "rigged," the scribe knows that,

$$\left(30 - \frac{1}{2}\right)^2 = (29; 30)^2$$

$$30 - \frac{1}{2} = 29; 30$$

$$30 = 29; 30 + 0; 30$$

□

Now it should be clear that this reconstruction of ours, which is entirely consistent with the scribe's procedure, is to be preferred to van der Waerden's, which assumes, against the textual evidence, the availability of the quadratic formula to the Babylonian scribe.

Speaking of his definition of algebra (quoted above),²⁰ van der Waerden says: "If this definition is applied to any [!] Babylonian or Arab text it is unimportant what symbolism the text uses."²¹ We respectfully dissent. "What symbolism the text uses" is very important indeed. It is crucial for a historical interpretation to remain faithful to whatever symbolism the text uses, and if the text fails to use any symbolism whatever, it is crucial not to introduce into it such foreign symbolism as might betray (and, as a rule, does betray) the idiosyncratic train of thought displayed in the text. To be sure, it is possible to translate existing formulas into words; no question about it. It is also possible to translate words into formulas, if there exists a formulaic language together with the rules of translation into it, which is available to and mastered by the translator. But is it possible to translate specific numbers into non-existing formulas? Assuming that there was indeed an oral tradition that accompanied Babylonian mathematics (as both Neugebauer and van der Waerden insist), could the Babylonians translate a rhetorical statement from that tradition into a formulaic (symbolic) expression? By the extant evidence, the answer is clearly, no!

For otherwise the form of the extant Babylonian mathematical texts would have been other than their actual form, where one and the same type of problem is repeated many, many times on the same tablet, only the specific, particular numerical data differing between neighboring problems. This repetition makes sense only if the recipe for the solution of a particular kind of problem had to

be learned and grasped by prolonged practice and reiteration of the steps involved in that recipe. If the oral tradition had the benefit of formulaic expression (to stretch things to their limit), it would have been a trivial matter to put such an ability into writing. What one can say orally, one can ipso facto say in writing. Alas, the written texts of Babylonian mathematics contain no manipulation of symbols; moreover, what they do contain (the form of the actual problems, the particular recipes) is incomprehensible in the presence of symbols, formulas, and general procedures of solution. To sum up, then, there are no "algebraic expressions like $(a+b)^2$ " in Babylonian texts simply because there are no algebraic expressions; and, it really follows from this, there are no equations either. Consequently, "the art of solving equations like $x^2 + ax = b$ " is a non-existent art in Babylonian texts. It comes into being only after the specific Babylonian examples have first been transcribed into algebraic language. Furthermore, that van der Waerden's definition of algebra "... is in full accordance with standard usage from 800 A.D. to the present day"²² is irrelevant even if true. For what is at issue here has nothing to do with the state of Islamic mathematics in 800 A.D. or with later developments; rather it is simply a matter of assessing the "state of the art" in Babylonian times and during classical Greek civilization, and, by any reasonable standard, there was no algebra in either of these historical periods.

The conclusion, then, seems inescapable. Van der Waerden's claim that BM 13901 is "dealing with the solution of quadratic equations"²³ is, strictly speaking, incorrect. The text in question enunciates a specific, numerical problem, and the solution-recipe involves arithmetical operations performed on the given, specific numbers. It is true that, when translated into modern symbolism, the problem leads to a quadratic equation and that the solution-recipe fits the operations performed in the quadratic formula. But this is just one possible approach to and interpretation of the text and, from what was said above, this is by far the least acceptable interpretation. Be that as it may, however, there is no equation in the text and there is no formula. These materialize only as a result of "[t]he hermeneutic motion, the act of elicitation and appropriative transfer of meaning ..." ²⁴ of the mathematician-interpreter.

Now it is clear that the Neugebauer-van der Waerden algebraic interpretation of mathematical cuneiform texts is just one possible interpretation. In principle, this should be obvious for any historical reconstruction of the past. What the historian establishes never transcends the realm of the possible. In most non-trivial cases, what the historian of ideas can claim for his interpretation of past events is that it is more likely than another possible

interpretation, never that it is the only possible interpretation. This is the very nature of what has been called "deductive reconstruction." Such reconstructions are always tentative, hypothetical, uncertain, for it is at all times conceivable that things might have been different than they are taken to be in a peculiar reconstruction. History and certainty rarely, if ever, cohabit.

The literature of the history of mathematics contains indeed interpretations of Babylonian mathematics that differ in varying degrees from Neugebauer's theory, which van der Waerden has appropriated.²⁵ To be sure, Neugebauer's interpretation is the most popular and the best known for clearly understandable reasons: Neugebauer has been involved for most of his professional life with editing and commenting upon mathematical and astronomical cuneiform texts, and his studies are among the most thorough, penetrating, and competent technical discussions of the Babylonian materials, carrying great appeal with mathematicians and mathematically-minded historians. Coupled with this is the fact that he has trained a number of very able scholars, whose approach to history is largely that of their master. But, by no means, has his work remained uncriticized. It is true that most of those who have taken issue with Neugebauer's hermeneutics have advanced algebraic interpretations of their own. Nevertheless, the significant thing is that there are alternatives and that recently the need for a non-algebraic, historically more sensitive approach to the texts has come to the fore.

Among those calling attention to this fact are Michael S. Mahoney and Árpád Szabó. The former, in an essay review of the 1969 reprint of Neugebauer's Vorgriechische Mathematik,²⁶ points out that,

It would be best ... to wield Ockham's razor when dealing with Babylonian mathematics and not to assign to the Babylonians any concept, or form of mathematical thought, for which there is no explicit documentation, nor even need ... If ... the Babylonians did mathematics, and even if they did it remarkably well, there is absolutely no evidence that they thought about mathematics.... That is why one objects to the use of [modern algebraic] language and ... concepts in interpreting Babylonian mathematics.²⁷

Á. Szabó too has taken strong umbrage with Neugebauer's historical methodology. In an article written for a Festschrift in honor of Willy Hartner (Prismata), a preprint of which S. Unguru was fortunate to receive,²⁸ Szabó says, among other things:

... so findet man ... in den mathematischen Keilschrifttexten immer wieder Beispiele mit bestimmten Zahlen. Die konkret-bestimmten Zahlen waren überhaupt die Wegweiser, die für Neugebauer das Verstehen der babylonischen Aufgaben ermöglichten. Ebenso ermöglichen diese Zahlen in vielen Fällen das Vermuten jenes antiken Rechenverfahrens, das beim Lösen der Aufgaben befolgt wurde. Aber redet man nach einer solchen Rekonstruktion von irgendeinem 'algebraischen Formalismus des Systems', so verzichtet man von vornherein darauf, die antike Vorlage und die moderne Interpretation sauber auseinanderzuhalten. Den die Formeln sind nicht unmittelbar im Text selbst gegeben. Auch wenn sie auf Grund der antiken Vorlage richtig aufgestellt wurden, sie sind keine antiken Formeln. Denn zum Aufschreiben dieser angeblich alten Formeln mit modernen Symbolen unerlässlich notwendig sind seitens des Historikers auch solche Kenntnisse, die nachweislich aus viel späteren Zeiten entstammen.²⁹

At this point, we might add that our interpretation of BM 13901, which we consider a historically more sensitive alternative to that of van der Waerden, applies in toto to all those cases in Greek mensurational texts (like Heron of Alexandria's Geometrica) and also in Diophantus's Arithmetica where so-called "quadratic equations" occur. A case in point is the following from Heron's Geometrica:

Given the sum of the diameter, perimeter and area of a circle, to find each of them separately. It is to be done thus: Let the given sum be 212. Multiply this by 154; the result is 32648. To this add 841, making 33489, whose square root is 183. From this take away 29, leaving 154, whose eleventh part is 14; this will be the diameter of the circle. If you wish to find the circumference, take 29 from 183, leaving 154; double this, making 308, and take the seventh part which is 44; this will be the perimeter. To find the area. It is done thus: Multiply the diameter, 14, by the perimeter, 44, making 616; take the fourth part of this, which is 154; this will be the area of the circle. The sum of the three numbers is 212.³⁰

Assuming again that Heron starts with the known answers, it is not necessary to suppose (as Ivor Thomas claims) that Heron deals with a quadratic equation by means of an algebraic formula, but rather that he plays with specific numbers, completing a given concrete numerical relationship to a perfect square. Thus, written symbolically, D being the diameter and $\frac{22}{7}$ the standard Archimedean value for Π , the constant ratio of the circumference to the diameter, what Heron says is:

$$D + \frac{22}{7} D + \frac{11}{14} D^2 = 212$$

Actually, since Heron knows the answer ($D = 14$), what he says is:

$$14 + \frac{22}{7} \cdot 14 + \frac{11}{14} \cdot 14^2 = 212 \quad (\text{I})$$

His solution-procedure, amounting to the completion to a perfect square of the given numerical relationship, follows:

$$\frac{11}{14} \cdot 14^2 + \frac{29}{7} \cdot 14 = 212$$

$$11^2 \cdot 14^2 + 2 \cdot 11 \cdot 14 \cdot 29 = 212 \cdot 154$$

$$11^2 \cdot 14^2 + 2 \cdot 11 \cdot 14 \cdot 29 + 29^2 = 32648 + 29^2$$

$$(11 \cdot 14 + 29)^2 = 33489$$

$$11 \cdot 14 + 29 = 183$$

$$11 \cdot 14 = 183 - 29$$

$$11 \cdot 14 = 154$$

$$11 = \frac{154}{14},$$

and this is the diameter. The perimeter and the area are obtained from the numerical relationship (I), q.e.d.

The same thing is true about the "quadratic equation" appearing in Diophantus IV.39,³¹ although Diophantus clearly represents a special and unique case, in the sense that what he is doing is definitely something unlike anything else that went on before him in Greek mathematics, namely, in Nesselmann's rather approximative trichotomous classification, syncopated algebra.³² Still, it is not necessary to agree with Ivor Thomas that "Diophantus had a perfectly general formula for solving the equation $ax^2 = bx + c$ [and] ... $ax^2 + bx = c$ and ... $ax^2 + c = bx$."³³

It should be clear by now that we believe that there are important distinctions between algebra and the concrete arithmetical relationships appearing in Babylonian and some Greek materials. For there is a vast mathematical gap involved between having a general knowledge of concrete number facts on the one hand, and

being able to abstract that knowledge and manipulate it symbolically without any reference to the concrete, on the other. Ignoring these distinctions, representative as they are of a wide gulf in mathematical outlook and technique, has been one of the main ways that confusion has arisen over the use of the term "algebra." It follows, then, that arithmetic precedes algebra, i.e., the existence of a coherent arithmetic system is required in order to have an algebraic superstructure, and without a firm arithmetical foundation, the attempt to do algebra (or, in our case, to find algebra) collapses like a house of cards. It is our contention that this is exactly the predicament of "geometric algebra," the arithmetical foundations of which turn out, at close scrutiny, to be rather shaky. To show this convincingly will require an extended discussion of the various operations that comprise the "geometric arithmetic," together with an assessment of their proper place in what we feel is a viable interpretation of classical Greek mathematics. What will emerge very clearly from this discussion is the fact that there are insuperable difficulties inherent in this, the very foundation of "geometrical algebra," and that these difficulties peremptorily show that the foundation is much too weak to support the weight of the required algebraic superstructure.

2. A noticeable feature that one encounters when reading Euclid's Elements is the widespread use of operations that we are apt to associate with arithmetic rather than geometry. When one is confronted with geometric versions of addition, subtraction, multiplication, and division (ratio formation),³⁴ there is an inevitable tendency to think of these in terms of their more familiar, modern counterparts. Moreover, this tendency will be all the stronger once one is convinced that Euclid Book II is "nothing but geometric formulations of algebraic rules."³⁵ For the existence of these operations only seems to confirm the position that there is a "geometric algebra" underlying Greek mathematics, an "algebra" that is largely hidden from view, but which clearly surfaces in Book II, in the propositions concerning "application of areas" of Book VI, and (to a more limited extent) in Books X and XIII. Those who espouse the position that there was such a "geometric algebra" have little difficulty in dealing with these "arithmetic" operations in Greek geometry, because, for them, they are, for all intents and purposes, the same operations we would employ today, were we forced to express our algebra entirely in geometrical terminology. They would argue that there are no significant distinctions (ontological and of other kinds) between the Greek operations and their modern counterparts, and that, in most instances, nothing is lost by transcribing Greek mathematics into modern notation and performing modern operations similar to "known" Greek techniques.³⁶

The practitioners of "geometric algebra" (none of whom were ancient Greeks) have, however, not been entirely content with the "arithmetical" operations that explicitly appear in the Elements. Not that there is anything wrong with them; it is just that they are too limited for the purposes required. What is needed in order to have an actual (as opposed to a mythical) "geometric algebra," is an arithmetic capable of handling arbitrary magnitudes, and so they invented, what we shall call, the "geometric arithmetic."³⁷ Using this, one is able to perform arithmetic operations in the general realm of magnitude (μέγεθος) rather than in the more restricted arena of (natural) number (ἀριθμός) or, at most, in that of ratios of numbers. On the other hand, without such a "geometric arithmetic" there is no longer a foundation on which to build a "geometric algebra," and the entire "algebraic" edifice crumbles and collapses under its own weight.

3. The operations explicitly referred to in the Elements have a good deal in common with those of the "geometric arithmetic," but the latter has two decided advantages. In the first place it presents a more unified, and coherent, system of operations (precisely because it so closely resembles the operations underlying modern-day algebra). Thus in Book V of the Elements, there is no simple inverse relation between multiplication and ratio-formation, whereas in "geometric arithmetic," ratio-formation is just one form of division, and division is indeed inversely related to the multiplication operation. The second advantage that the "geometric arithmetic" presents is, as we have already indicated, that it provides the necessary foundation for the manufacture of a "geometric algebra."

There are, however, drawbacks to the system of "geometric arithmetic," and one of the main theses of this paper is concerned with showing that these drawbacks easily outweigh the "advantages" cited above. Initially our attention will be focused on the "geometric arithmetic" itself, elucidating its structure and the assumptions underlying its use. This investigation will have little difficulty in revealing the seam along which the operations of "geometric arithmetic" have been grafted onto the operations explicitly enunciated in the Elements. Thus the stitching, the gluing together of the two operations of "multiplication" (that found in the Elements and the generalized form found in the "geometric arithmetic") turns out to have been most untidily done, and even a superficial investigation of its character leads to the preliminary conclusion that the surgery should probably be regarded as a failure. But, of course, the only real test of a surgical intervention is to see what happens once the patient has left the operating table and is out on his own. Thus the crucial test for

"geometric arithmetic" consists in applying it, in conjunction with the established techniques of "geometrical algebra" (e.g., Book II), to solve some of the important algebraic problems that supposedly engaged the interests of the finest mathematicians of antiquity. The results obtained by pursuing this critical test (critical not just for "geometric arithmetic," but for the entire edifice of "geometrical algebra") offer a resounding confirmation of our preliminary conclusion. For the "natural" algebraic solutions that one obtains by pursuing the techniques of "geometric arithmetic" to their logical end, lead to the creation of a "geometrized" algebra, utterly unlike anything known in the extant corpus of Greek mathematics.

Let us reiterate, then, that the system of operations we are about to consider ("geometric arithmetic") does not explicitly appear as such in any of the Greek texts. We emphasize system here, because many of the isolated operations of "geometric arithmetic" are indeed performed in the Elements, but they are never organized and presented as a unified and coherent network of "arithmetic" for manipulating general magnitudes. Thus in arguing against the existence in Greek mathematics of a generalized arithmetic replete with the unconstricted operations of addition, subtraction, multiplication, and division, we are only making the assertion that Greek mathematics is more or less what it appears to be. It is our contention that one obtains a much truer picture of the nature of Greek mathematical activity by taking the texts at pretty near face value, rather than by seeking to imbue them with a hidden motivation and methodology. The fact that there is no mention of algebraic equations anywhere in the Greek mathematical texts under discussion is the strongest possible a priori evidence that solving such equations was not a matter of central (or any) importance for the Greeks. This is not just a matter of language or the lack of a suitable symbolism, rather it is a reflection at the very deepest level of intrinsically different intentionalities that distinguish Greek mathematical activity from the modern.³⁸

Concerning the interpretation of the operations of "geometric arithmetic," one is basically faced with choosing between the following two alternatives: One possibility is to accept the position of the "geometrical algebraists" that "geometric arithmetic" was indeed an integral part of Greek mathematics, forming the necessary foundation for the creation of a Greek "geometric algebra." Accordingly "geometric arithmetic" is part and parcel of an underlying algebra that was not fully discovered until the 19th century, at which time its existence was discerned and its body restored by historians of mathematics who used it to legitimize their interpretation of much of Greek mathematics as algebra in geometric dress. The other option, the one we will argue for here, suggests that the

"geometric arithmetic" is really a hybrid creature which can be dissected along the following lines. First it incorporates the "arithmetic" operations of addition, subtraction, and ratio-formation as they are actually found in the Elements. To these, however, it adds a whole arsenal of geometric operations which are grafted onto the usual "arithmetical" operations. For the "geometrical algebraists" (who have, in fact, invented this system themselves), the result is an "arithmetic" that generalizes the "arithmetic" operations explicitly delineated in the Elements; but for us, this is accomplished only at the cost of badly blurring the distinction between certain operations which are geometric, (e.g., rectangle formation) and others which are "arithmetic" (e.g., formation of a ratio). What is even more serious, the operations of "geometric arithmetic" fail to accomplish their intended purpose. For, as we shall show in the course of our analysis, they fail to generalize the actual Euclidean "arithmetic" operations in a consistent manner. In what follows, we will find it convenient to take T. L. Heath as our main authority on matters concerning the "geometric arithmetic," since he is a prime spokesman for the, by now, conventional view that there was such an arithmetic which served as the foundation for the "geometric algebra," and also because he is a scholar with universally recognized credentials.

4. A natural point of departure, for our purposes, will be to consider initially the most elementary "arithmetic" operations, addition and subtraction, as they actually occur in the Elements. Our first observation is that these operations are never defined; neither is there any explicit indication as to which figures may be added to, or subtracted from, which nor is there any explanation of the manner by which these operations are to be carried out. The only explicit statement concerning these operations, and an important indicator of the broad context in which the Greeks viewed them as being applicable, occurs in the Common Notions (Κοινὰ ἔννοια):

C.N. 2: If equals be added to equals, the wholes are equal.

C.N. 3: If equals be subtracted from equals, the remainders are equal.³⁹

Here it is important to notice that addition and subtraction hold a special place in the Euclidean tradition, owing to the fact that they are the only arithmetic operations mentioned in the Common Notions. This suggests that addition and subtraction are unique insofar as their range of applicability is concerned. For, as we shall see later, the fact that there is no corresponding statement in the Common Notions concerning multiplication, viz., "If equals be multiplied by equals, the products are equal," is difficult to explain unless one accepts the view that the multiplication

operation is more restrictive and does not apply with the same generality as do addition and subtraction. This is the position we shall take up and defend later in this paper. The implications this has for "geometric algebra" are, of course, devastating, for without a generalized multiplication operation, there is no arithmetic substructure upon which to build an algebraic system worthy of the name.

Because there are no specific statements made in the Elements that would shed light on the nature of the operations of addition and subtraction, we must learn about this matter from what the Greeks do rather than from what they say. What we find in Greek practice confirms the view that addition and subtraction are applicable in the broadest possible context, i.e., to arbitrary magnitudes, but with the important proviso that the magnitudes in question be homogeneous, i.e., of the same kind. Again, there is no explicit statement in Euclid to this effect,⁴⁰ nor is there any indication as to which magnitudes are of which kind. It is not even clear how many different types of magnitudes were recognized by Greek geometers, nor is it clear that these matters were ever given much thought. There is good reason to believe, however, that the homogeneity relation between magnitudes was primarily an intuitive idea so far as Greek geometry was concerned, whereas in philosophic circles (which were, of course, very closely tied to the mathematical) these matters were hotly debated.⁴¹ Taking these things into consideration, the best we can hope to do is learn what we can from the Greek practice involving the manipulation of different kinds of magnitude. It turns out that by doing so, several important generalizations can be made.

5. When it comes to studying the exact manner in which the Greeks handled different kinds of magnitude, it is just as important to observe what they did not do as it is to observe what they did. Thus one will not find, for example, a line added to a rectangle anywhere in Euclid, since magnitudes represented by figures of different dimension are not homogeneous.⁴² Nor will one find an angle subtracted from a magnitude represented by a plane figure, say a square, this in spite of the fact that "... Euclid certainly regarded angles as magnitudes."⁴³ On the other hand, rectilinear angles did represent magnitudes of like kind, and therefore could be added or subtracted as in I.17:

In any triangle two angles taken together [i.e., added] in any manner are less than two right angles.⁴⁴

We also know that rectilinear and even curvilinear plane figures were sometimes added, e.g., when Hippocrates shows that squaring

the circle is equivalent to squaring a certain lune, he does so by adding a hexagon to both sides of a certain equality of geometric figures, thereby obtaining the very pretty result that a triangle plus a hexagon are equal to the aforesaid lune plus a circle.⁴⁵

Proposition I.47, the so-called "Pythagorean Theorem," gives an excellent illustration of the broad applicability of Common Notion 2 (the property that says equality is preserved when equals are added to equals). The proof makes use of C.N. 2 first for the addition of angles, and then for the addition of plane areas. The motivation for the argument comes from the visual appeal of the "windmill" figure, while the proof itself relies on the idea that two-dimensional plane figures are homogeneous, and hence can be added.⁴⁶ An illustration of subtraction in conjunction with C.N. 3 can be found in the proof of II.11, which we will examine in detail when we discuss examples of the alleged Greek solution of quadratic equations.⁴⁷

These, then, are some examples illustrating the manner in which the Greeks added and subtracted geometric figures. Clearly these operations are not nearly as general as the modern operations on general magnitudes, whereby the size of a geometric figure can be thought of as a positive real number completely independent of the figure that happens to represent it. Once magnitude is dissociated from geometry, it has the freedom from ontological commitments that makes it possible to develop arithmetical operations and eventually symbolic manipulations which are the very hallmark of an algebraic system.⁴⁸ But first magnitude must become number. What we have seen is that, although addition and subtraction are employed for general magnitudes in the Euclidean tradition, the dependence of these operations on a geometric formulation imposes a limitation that makes these operations qualitatively different from their modern counterparts. The modern notion of real number transcends this limitation, making it possible to equate and compare figures of differing dimensions, equating these in turn with angles or anything whatsoever capable of being measured. When number reigns supreme, everything can be related numerically to anything else. This the Greek could not do.

6. With this as background, we shall now consider what Heath has to say concerning the role these fundamental operations (addition and subtraction) play in the "geometric arithmetic":

The addition and subtraction of quantities represented in the geometrical algebra by lines is of course effected by producing the line to the required extent or cutting off a portion of it.⁴⁹

Thus the prototypic representation of a one-dimensional magnitude is a straight line of appropriate length, whereupon addition and subtraction are performed in the obvious manner. For two-dimensional magnitudes the prototypic figure is the rectangle, which gives a geometric representation for the product of two magnitudes:

The addition and subtraction of products is, in the geometric algebra, the addition and subtraction of rectangles or squares; the sum or difference can be transformed into a single rectangle by means of the application of areas to any line of given length, corresponding to the algebraical process of finding a common measure.⁵⁰

The "application of areas" (παραβολή των χωρίων) turns out to be the key ingredient, in fact, the very heart of the "geometric algebra." Thus in debunking the latter idea, it is imperative that we give a good deal of considered attention to the former. Whatever the status of "geometric algebra" might be, there is no question but that the "application of areas" played an important role in Greek mathematics. This will be made abundantly clear in Section III below, wherein we study its alleged use as a means for obtaining solutions to second degree equations. But, for now, we must postpone detailed discussion of this important topic, and merely indicate the manner in which it applies to the immediate problem of adding or subtracting rectangles.

7. The following two propositions are fundamental to the "application of areas," and, furthermore, form a key cornerstone in the "geometric arithmetic":

I.44: To a given straight line to apply, in a given rectilinear angle, a parallelogram equal to a given triangle.⁵¹

I.45: To construct, in a given rectilinear angle, a parallelogram equal to a given rectilinear figure.⁵²

Using these results, it is an easy matter to prove the following corollary, which is, surprisingly enough, conspicuously absent from the Elements. We shall call it I.45A:

I.45A: To a given straight line to apply, in a given rectilinear angle, a parallelogram equal to a given rectilinear figure.

The following argument gives a simple proof for this important result. We are given a line AB, and angle R, and a rectilinear

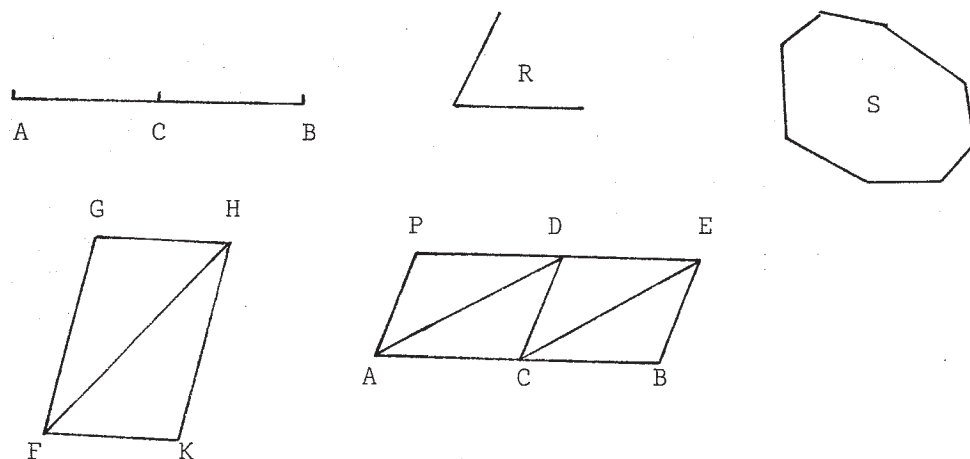


Fig. I.1

figure S, and we wish to construct a parallelogram with base equal to AB, base angle equal to R, and area equal to S. First apply I.45 to transform the figure S into the parallelogram FGHK equal in area with S (the angle GFK being arbitrary). Next, we use I.10⁵³ to bisect AB at C. By I.34,⁵⁴ the diagonal of the parallelogram FGHK divides the figure into two equal triangles. Using I.44, we can apply to AC a parallelogram equal to triangle FGH in the angle R. On the remaining segment CB we can apply another parallelogram in the same angle and equal to triangle FHK (which equals FGH, by I.34). Since the applied parallelograms are equal, so are the four triangles obtained by joining their diagonals. Thus triangle ADC equals triangle CEB and they lie on equal bases AC and CB. Hence, by I.40,⁵⁵ the triangles ADC and CEB are in the same parallels, proving that ABEP is the desired parallelogram.⁵⁶

There can be little doubt that the geometers of Euclid's day, and probably of earlier times, were well aware of the full power of what we have called I.45A, even though the explicit statement and proof of I.45 that appears in the *Elements* says nothing about applying the given rectilineal figure to a given line. However, this seems to have been nothing more than a minor slip on the part of the author, for when I.45 is eventually required in the proof of Proposition VI.25,⁵⁷ the full force of I.45A is, as a matter of fact, utilized. Now Heath makes extensive use of the full force of I.45A (but without ever saying so), making it into a crucial tool for the operations of the "geometric arithmetic." Because of this, we will find it convenient in subsequent discussions to refer to this result simply by number. Bearing this in mind, let us now consider precisely what role Proposition I.45A plays in the context of the "geometric arithmetic."

8. If we restrict I.45A to the case where the given rectilinear figure is a rectangle and the given angle a right angle, it can readily be seen that this is exactly what is needed in order to add and subtract rectangles just as Heath indicated above. For if we are given two rectangles A and B, I.45A enables us to transform B to a rectangle B', where B' is equal to B and has the same height as A. Then $A + B = A + B' = C$, where C is formed by adding the bases of A and B' (an operation that Heath has already defined, see #5 above). To subtract B from A we simply follow the same procedure, only this time, subtracting the base of B' from the base of A.

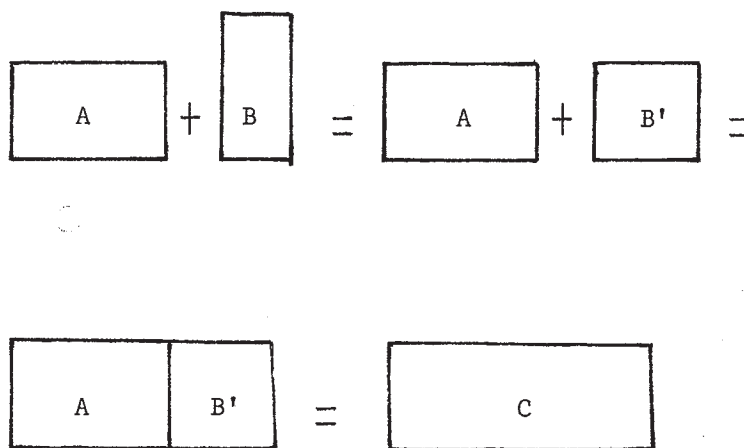


Fig. I.2

The key step in adding A to B is the transformation of B to B' using I.45A. In the language of "geometric arithmetic," this procedure is one form of "division":

The division of a product of two quantities by a third is represented in the geometrical algebra [or, as we would prefer to say, "geometric arithmetic"] by the finding of a rectangle with one side of a given length and equal to a given rectangle or square. This is the problem of application of areas solved in I.44, 45 [actually, I.45A].⁵⁸

The use of "application of areas" in order to add and subtract rectangles plays a key role in the arguments appearing in Book X. One finds "additions" of rectangles, for example, in Propositions X.23, 25, 41, 47, etc., while X.38, 75, 78, etc. utilize subtraction, and X.60-65 invoke both operations. Thus there is no question about the significance of these techniques. The issue,

rather, concerns whether or not the Greeks employed these techniques as part of a systematic arithmetic for general magnitudes. In this regard, it is interesting to look at Heath's remarks following Proposition I.44:

This proposition will always remain one of the most impressive in all geometry when account is taken (1) of the great importance of the result obtained, the transformation of a parallelogram of any shape into another with the same angle and of equal area but with one side of any given length, e.g. a unit length, and (2) of the simplicity of the means employed...⁵⁹

Later, in Section III, we will discuss I.44 in detail as part of a survey of the method of "application of areas," but here we would be remiss not to mention the fact that Heath seems to misrepresent intentionally Euclid in order to bolster his view that I.44 and I.45 are two key ingredients in the machinery of the "geometric arithmetic." Proposition I.44 does not deal with the "transformation of a parallelogram;" the figure undergoing a "transformation" is a triangle. Furthermore, the proposition says nothing about the transformation of a figure of "any shape into another with the same angle ..." — the given angle of the resulting parallelogram is, in fact, completely arbitrary.

Finally, there is the suggestion that one side of the constructed parallelogram might be a "unit length." This remark appears to suggest (is there any other feasible interpretation for it?) a possibility pregnant with implications for Greek geometry, namely, the utilization of I.44 as a mechanism for the determination of plane areas! For if a plane figure can be transformed into a rectangle (which happens to be the "[m]ost important of all ... parallelograms ... [as it] corresponds to the product of two magnitudes in algebra ..."),⁶⁰ then the "great importance" of I.44 is (according to Heath) due to the fact that this rectangle can be transformed "into another with the same angle and of equal area but with one side of any given length, e.g. a unit length;" the upshot of all this being that, since the newly formed rectangle represents a product with one side of unit length, the other side "measures" the area of the original plane figure. Thus it seems impossible to escape the conclusion that Heath is alluding here to a procedure for measuring the area of plane figures. Furthermore, by utilizing the above transformation, the problem of determining the size of a two-dimensional figure is reduced to a one-dimensional problem, namely "measuring" the line that forms the other side of the newly formed rectangle. This is an attractive idea, especially when taken in conjunction with a similar interpretation for Euclid's approach in Book X of the Elements.⁶¹ Alas, it has

the unfortunate drawback that it never seems to appear anywhere in the extant corpus of Greek mathematics. Considering the tight strictures that the Greek concept of magnitude imposes on doing "arithmetic," however, (i.e., the strict adherence to the principle that only homogeneous magnitudes can be combined, etc.), it should not be at all surprising that the Greeks themselves "overlooked" the "great importance" of I.44. For had they employed this proposition in the manner Heath suggests, the integrity of the principle that magnitudes of different dimension are distinct would have been weakened, thereby opening the way for the formulation of a conception of magnitude not just as size, but as generalized number, completely independent of any particular geometric representation. As we shall soon see, the evidence that the Greeks ever reached such a modern conception of magnitude is, to put it mildly, extremely weak.

The above mistakes that appear in Heath's commentary are, taken by themselves, hardly worth pointing out (except for, perhaps, the last one); taken together, however, they are very significant, as they reveal important interpretive biases that have apparently led Sir Thomas to twist the evidence a bit. If we turn to Heath's remarks following I.45, we find (curiously enough) the same combination of bold assertion and misrepresentation of fact:

We have now learnt how to represent any rectilinear area, which can of course be resolved into triangles, by a single parallelogram having one side equal to any given straight line [our emphasis] and one angle equal to any given rectilinear angle. Most important of all such parallelograms is the rectangle, which is one of the simplest forms in which an area can be shown. Since a rectangle corresponds to the product of two magnitudes in algebra, we see ... [among other things that such a representation] enables us to add or subtract any rectilinear areas and to represent the sum or difference by one rectangle with one side of any given length, the process being the equivalent of obtaining a common factor.⁶²

We have emphasized the phrase "having one side equal to any given straight line" in order to call attention to the fact that, again, this is not a faithful rendering of Euclid. There is no mention in I.45 of a given line forming one side of the parallelogram; Heath is, once again, alluding to what we called Proposition I.45A, and not I.45. It is certainly surprising that Heath never once alludes to the fact that the key Proposition I.45A, which is used implicitly in the proof of VI.25 and throughout Book X, is neither enunciated nor proved anywhere in the Elements!

We have not called attention to these "sins" of omission and commission in Heath's commentary merely for the sake of nit-picking. Rather our purpose in bringing this up is to illustrate the sometimes rather subtle interpretive bias that colors Heath's otherwise admirable commentary on the Elements. For it seems to us that there is something more than mere accident behind Heath's uncharacteristic carelessness in discussing I.44 and 45, especially considering the "great importance" he attaches to these results. Why is it, we must ask, that this great scholar, who normally devotes meticulous attention to even the slightest details, is so sloppy when it comes to these important matters to which, moreover, he pays such short shrift?

Our impression is that these are not just haphazard oversights, but rather clear reflections of the interpretive bias that colors Heath's otherwise illuminating commentary. For the source of these errors is intimately connected with Heath's desire to extract the essential significance of I.44, 45 for Greek mathematics, which, for him, consists in the possibility of incorporating these results into a unified system of generalized arithmetic. According to this view, Propositions I.44, 45 have as their only raison d'être, the addition of a needed weapon to the arsenal of operations that comprise the "geometric arithmetic."

Now, by these remarks, we do not mean to imply that a commentator could or should be without interpretive biases. If that were the case there would be no point in writing a commentary in the first place. Neither do we mean to imply that there is anything secretive or mysterious about Heath's views concerning the nature of the mathematics in the Elements. These are plainly written and readily apparent to any careful reader. The point is that before we accept Heath's views lock, stock, and barrel, we need to examine precisely what the Euclidean text says. In the course of our analysis, we shall see to what extent Heath's views are justified. In the meantime, it is a healthy activity to expunge from Heath's notes those remarks which clearly run counter to the letter and the spirit of Greek mathematics. Certainly one would not want to do without Heath's invaluable commentary, but it is important that we separate the wheat from the chaff, so to speak, by rendering unto Euclid what is Euclid's, while rendering unto Sir Thomas what is Sir Thomas'.

II

1. In section I we discussed the operations of addition and subtraction, both as they appear in the Elements and as important components of the "geometric arithmetic." This discussion provides us with the necessary background information that sets the stage for much of the analysis that will now follow. One of the key ideas we discussed (and that bears repeating, as it directly affects the main issue in this section) concerns the fact that, throughout classical Greek mathematics, there is a strict adherence to the principle that only magnitudes of like species can be added or subtracted. In particular, this means that there was no generalized concept of number underlying Greek magnitude, and, hence, no idea of combining magnitudes of different dimensions. We also saw that, except for the somewhat subtle matter of homogeneity, addition and subtraction were perfectly general operations that could be applied to arbitrary magnitudes. It was this generality of application that, we conjectured, accounts for the fact that addition and subtraction are the only "arithmetic" operations appearing in the Common Notions.

Our discussion of addition and subtraction also indicated how these operations have been especially adapted to, and thereby incorporated into, the "geometric arithmetic." Of particular interest in this regard is the use of "application of areas" as a means of making the addition and subtraction of two dimensional figures truly operational. As we noted earlier, this technique is an important device in the arguments of Book X.

This is the background the reader should keep in mind when considering the arguments that follow. It will be observed, however, that nothing we have said so far, pertaining to addition and subtraction, is terribly damaging to the stronghold of "geometric arithmetic," and, as a matter of fact, this powerful interpretive bastion is not terribly vulnerable along this particular front, the reason being that addition and subtraction are indeed general operations. The weak link in the "geometric arithmetic" turns out to be the multiplication operation, which represents the product of two magnitudes (given as lines) by the formation of the rectangle with the given lines as sides. As we shall argue momentarily, this operation, unlike addition and subtraction, does not fit nearly so nicely into the framework of "geometric arithmetic." For, as we shall see, the operations of rectangle formation and ordinary multiplication, as explicitly performed throughout the Elements, are in fact incompatible with one another, i.e., rectangle formation cannot be "generalized multiplication" without producing inconsistency in the system of operations that we know the Greeks utilized. Before coming to this, however, we must

first consider the evidence from the other side, beginning again with Heath's views on this subject.

There is no direct textual evidence that the Greeks ever viewed the formation of a rectangle on two given sides as multiplication, and Heath appears to rest his case that they did view it in this manner on the "plausibility" of extending the rectangular representation of products in the Pythagorean $\Psi\acute{\epsilon}\phi\omicron\iota$ ("pebble") arithmetic to arbitrary magnitudes. In the "pebble" arithmetic:

A "plane number" is ... described as a number obtained by multiplying two numbers together, which two numbers are sometimes spoken of as "sides," sometimes as the "length" and "breadth" respectively, of the number which is their product.

The product of two numbers was thus represented geometrically by the rectangle contained by the straight lines representing the two numbers respectively. It only needed the discovery of incommensurable or irrational straight lines in order to represent geometrically by a rectangle the product of any two quantities whatever, rational or irrational ...⁶³

Heath's argument, then, rests on the assumption that since, in the Pythagorean $\Psi\acute{\epsilon}\phi\omicron\iota$ -arithmetic, the product of two numbers was represented by a rectangle, it was natural for the later Greeks, who had adopted much of the Pythagorean number theory already, to use this geometrical formulation as the definition of multiplication for arbitrary magnitudes. Thus Heath is drawing on the fact that the terms 'square', 'plane', and solid' number, etc., defined in Euclid Book VII are believed to have evolved from the Pythagorean $\Psi\acute{\epsilon}\phi\omicron\iota$ -arithmetic. But this is really a gratuitous argument. For what one finds throughout the "arithmetical books" of the Elements, i.e. Books VII-IX, is that, even though abundant use is made of the terms 'plane-', 'square-', etc. numbers, yet never are these represented by rectangles or squares but always by lines.⁶⁴ These Pythagorean terms, as they appear in Euclid, are merely designations, a scheme for the purpose of classifying numbers; they serve as standard categories for differentiating various species of number via natural geometrical analogues. There is, however, no wholesale adoption of the Pythagorean "pebble" arithmetic in the Elements, much less an extension of it to general magnitudes. Rather what we find is that the Pythagorean categories are employed in Euclid as an analogy for the classification, and not the geometrical representation, of numbers.

Is there any other evidence, then, that might be brought against our thesis that rectangle formation is not generalized

multiplication in Greek mathematics? Certainly many authorities would point to Book X in this regard,⁶⁵ but as we have already indicated (cf., n. 61), Book X should be regarded as a qualitative treatment of commensurability relations between geometric figures, wherein rectangle formation as a geometric operation plays a key role. Moreover, there is a strict adherence throughout Book X to the integrity of the homogeneity relation underlying the Greek "arithmetic" dealing with magnitudes. One other potential counter-argument we would like to anticipate stems from Euclid VI.16:

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional.⁶⁶

Certainly this has a familiar algebraic ring to it, as it seems to say that, $a/b = c/d$ if and only if $ad = bc$. So it is rather surprising, at first, to find that Heath, who generally makes a practice of transcribing the results of Greek proportion theory into modern notation, says nothing in his commentary that even hints at the obvious parallel between VI.16 and the arithmetic property that says "in fractions, the product of the means equals the product of the extremes." Could it be that Sir Thomas has entirely forgotten that, in his own words:

The equivalent of multiplication is the construction of the rectangle of which the given lines are adjacent sides. The equivalent of the division of one quantity represented by a line by another quantity represented by a line is simply the statement of a ratio between lines on the principles of Books V. and VI.⁶⁷

If we examine the situation more closely, however, it turns out that there are very good reasons both for what Heath says about VI. 16, and for what he does not say. To see this clearly, we must turn to his remarks made at the very beginning of Book VI:

The theory of proportions has been established in Book V. in a perfectly general form applicable to all kinds of magnitudes (although the representation of magnitudes by straight lines gives it a geometrical appearance); it is now necessary to apply the theory to the particular case of geometrical investigation.⁶⁸

Here Heath has aptly characterized not only the underlying structure, but also the motivation behind the layout of Books V

and VI. Book V presents a general theory of proportion that deals with the theoretical manipulation, via ratio and proportion, of any homogeneous entities whatever that happen to possess magnitude. Using this theory, one no longer had (closely paraphrasing Aristotle) to develop separate arguments for numbers, lines, solids, and times, as it was now possible to consider the entire genus of magnitude at once and prove pertinent conclusions for all kinds of magnitude by one demonstration.⁶⁹ Book VI, on the other hand, shows how the general theory contained in Book V can be applied to the situation of paramount interest, namely the study of geometric figures. It is, therefore, highly significant that VI.16 occurs where it does, and not in Book V, as this is a strong indicator that this proposition was primarily a geometric result and not a theoretical relationship on a par with the other results obtained in Book V. Heath's remarks following VI.16 are, in fact, fully consistent with this interpretation. He observes, for example, that VI.16 is actually only a particular case of VI.14:

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.⁷⁰

The proof of VI.16 amounts to nothing more than a trivial two-fold application of VI.14, which is itself proved by way of a straightforward application of the geometric fact contained in VI.1 that "... parallelograms which are under the same height are to one another as their bases."⁷¹ The above reasons seem to us to constitute a sufficiently compelling answer to those who would maintain that VI.16 is nothing more than the Greek formulation of the property that multiplication and division are inversely related. The fact that this proposition occurs in Book VI and not in Book V, emphatically suggests that its motivation has more to do with its geometric content than with its alleged algebraic utility.

2. We must now turn to the evidence against the view that rectangle formation meant, for the Greeks, generalized multiplication, by first considering Greek multiplication as it actually appears in the Elements. In Definition VII.15, we have an explicit statement of what it means to multiply numbers:

A number is said to multiply [πολλαπλασιάζειν] a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.⁷²

Thus multiplication of numbers is the familiar operation of repeated addition, an operation which, when applied to one-dimensional magnitudes, produces not a two-dimensional but another one-dimensional magnitude. But multiplication is not confined to numbers in Greek mathematics, as can be seen from a cursory inspection of Book V, wherein it serves as the very backbone of the general theory of proportion. Thus we encounter the term multiple [πολλαπλάσιος] in Definition V.2,⁷³ and we see it actually used as a criterion for homogeneity in Definitions V.3 and 4:

- V.3: A ratio is a sort of relation in respect of size [πηλικότης] between two magnitudes of the same kind.⁷⁴
 V.4: Magnitudes are said to have a ratio to one another which are capable, when multiplied [πολλαπλασιαζόμενα, from πολλαπλασιάζειν], of exceeding one another.⁷⁵

Here, and throughout Book V, multiplication means one thing and one thing only — repeated addition. Furthermore, Definitions V.3 and 4 together imply that two magnitudes are homogeneous if and only if some multiple of one exceeds the other, i.e. if and only if the magnitudes have a ratio with one another.

We are now in a position to see the essential interplay between the operations of addition, subtraction, multiplication, and ratio formation as they actually occur in the Elements of Greek mathematics. The key feature upon which we wish to focus our attention concerns the fact that these operations are only applied to homogeneous magnitudes, i.e., magnitudes of the same dimension. In this sense these operations form a coherent system of "arithmetic," but one with idiosyncrasies all its own. For example, there is an asymmetric quality about Greek multiplication that differentiates it from the other operations. Whereas addition, subtraction, and ratio-formation are defined for arbitrary pairs of homogeneous magnitudes, multiplication requires that one of these magnitudes be a number.⁷⁶ The interplay between ratio-formation and the other operations is also highly idiosyncratic. The ratio between two magnitudes of the same kind can equal a number, a ratio of numbers, or, when the magnitudes are incommensurable, their ratio may be one of several types as, for example, in Book X. There is interplay between addition and ratio formation in V.18⁷⁷ (anachronistically, $a:b = c:d$ implies $(a+b):b = (c+d):d$) and between subtraction and ratio formation in V.19⁷⁸ ($a:b = c:d$ implies $(a-c):(b-d) = a:b$). These properties have a significance in Greek ratio and proportion theory far beyond the mere fact that they happen to be, for us, valid rules that apply to fractions. They also happen to be, for us, utterly trivial. One way we might verify both V.18 and 19 is by multiplying means and extremes. Thus, for V.18,

$$(a+b):b = (c+d):d \Leftrightarrow (a+b)d = b(c+d)$$

$$ad + bd = bc + bd$$

$$ad = bc$$

But $ad = bc \Leftrightarrow a:b = c:d$. Hence $a:b = c:d$ is actually equivalent to $(a+b):b = (c+d):d$. Yet this argument cannot be applied using the Greek methods, as there is no general principle in Greek mathematics that in any way resembles $a:b = c:d$ if and only if $ad = bc$ (VI.16 applies only to lines).⁷⁹

Another curious quality of ratios is that, unlike the other operations, ratio-formation produces a quantity without extension. The ratio between two three-dimensional magnitudes is not a three-dimensional magnitude, nor is it a two- or even one-dimensional magnitude; it is a pure, dimensionless quantity residing in a realm of magnitude without extension, a veritable scalar quantity. This gives us an important clue (which we will now pursue) to how the Greeks themselves viewed the essential interplay between ratio-formation, on the one hand, and generalized "multiplication" on the other.

We have seen that, in the Greek "arithmetical" system, ratio is defined for any pair of homogeneous magnitudes, and that ratios themselves are magnitudes without dimension. Multiplication, on the other hand, is confined, in the Elements at least, to the situation where at least one of the factors is a number. We have also seen that the attempt to generalize this multiplication operation, by introducing rectangle formation as an arithmetic operation, has been accomplished only by ignoring substantial textual evidence to the contrary. The question thus arises: Was there ever a generalized "multiplication" operation in Greek geometry that served as the inverse of Greek style ratio-formation?

One does not need to look very far before answering this question, as there is indeed a well-known Greek technique, which effectively generalizes the operation of multiplication as repeated addition, and which also comes one step closer to being a true inverse for ratio-formation, namely the technique of solving for the fourth proportional.⁸⁰ Thus if a , b , and c are arbitrary lines, then applying VI.12, one can find x such that $a:b = c:x$, i.e., $b:a = x:c$. It will readily be seen that, in the case where $b:a$ is a number m (i.e., b is a multiple of a), the line x obtained via VI.12 is precisely the same as the line obtained by multiplying c times m . Now it must be recognized that this procedure is still a far cry from being a true inverse to ratio-formation, as, for one thing, it is only explicitly worked out for

lines in the Elements. If a , b , and c were arbitrary curvilinear plane figures, for example, the prospects for obtaining x such that $a:b = c:x$ would be slim indeed.

The key point we wish to emphasize, however, is not this. Rather it concerns the fact that, once we view the technique of solving for the fourth proportional as a kind of generalized multiplication, there is still a completely coherent system of operations at work here that, nevertheless, preserves the homogeneity of the magnitudes involved. One can compare the relative sizes of two homogeneous magnitudes by forming their ratio, and one can "multiply" a given magnitude (assuming the fourth proportional can be found) by a ratio, thereby realizing another homogeneous magnitude, bearing a prescribed ratio to the original. Thus all of the various "arithmetical operations" come together to form a coherent network for the manipulation of general, but homogeneous, magnitudes, and if this system lacks the absolute freedom that we, who have had the benefit of an algebraic heritage placed at our disposal, so easily take for granted, it still presents a neat, dovetailed system of operations, altogether suited to the Greek view of magnitude as a genus wherein each species must be treated separately.

3. A corollary to the above is that addition, subtraction, and multiplication (viewed as repeated addition) all preserve dimension, and it is absolutely essential that they do so. For, as we have seen, addition, subtraction, and ratio formation all require that the dimensions of the magnitudes involved be equal. It follows that the introduction by Heath and others of the operation of rectangle formation as generalized multiplication represents a radical break with the intrinsic principles underlying the operations explicitly performed in the Elements. The representation of products via rectangle formation, which is the very cornerstone of "geometric arithmetic," overlooks precisely the fundamental tenet of homogeneity that governs the entire Greek treatment of magnitude. This situation, once its implications are realized, presents a real dilemma for the "geometric arithmetic," even though the practitioners of "geometric algebra" have either underestimated its significance or else overlooked it completely. Thus having opted for the view that rectangle formation was, for the Greeks, generalized multiplication, they have conveniently overlooked the difficulty that arises here due to the peculiar restrictions that the Greek concept of ratio places on the magnitudes involved. For if rectangle formation is supposed to be an extension of the known technique for multiplying numbers, this new technique ought to yield the same answers as the old one, which it clearly does not, if one adheres to the principles employed in

Greek geometry. For instance, multiplication of four times three, using the two definitions available for the purpose, produces, on the one hand, a rectangle of twelve square units, on the other, a line of twelve units. But, being magnitudes of different dimension, these two products (which are presumably equal to one another) have no ratio to one another. As a matter of fact, according to the rectangle definition, the product, twelve, cannot be compared with either of its factors, four and three!

Yet any attempt to bypass this first horn of the dilemma brings us face to face with the second. Seeing that it is patent nonsense to have a number which has no ratio to any of its factors, one might seek to get around this by simply transforming one number to another "equal" number "reexpressed" by using the appropriate dimension. With this approach, there is never any difficulty in "dividing" the line of length four into the rectangle with sides of length four and three, as one can transform the line into a rectangle (4 units by 1 unit) and then form the ratio between the two rectangles to obtain the answer, i.e., the ratio 3:1 representing the number 3. But it should be apparent that this approach amounts to nothing less than the abandonment of dimension (and hence the principle of homogeneity) altogether, a mistake which Heath, van der Waerden, et al. (at least sometimes) carefully avoid. Yet it is also apparent that if one wishes to cling to the view that rectangle formation represents a generalization of ordinary Greek multiplication, then these two definitions can be reconciled in no other way. If four times three equals, on the one hand, a rectangle, and, on the other hand, a line, then clearly if they are to be equal, one must insist that magnitude is independent of dimension.

Now the stance that Heath and others have adopted towards this dilemma does nothing, in truth, to remedy the situation. According to Heath, as we saw,

The equivalent of the division of one quantity represented by a line by another quantity represented by a line is simply the statement of a ratio between lines on the principles of Books V and VI. The division of a product of two quantities by a third is represented in the geometrical algebra by the finding of a rectangle with one side of a given length and equal to a given rectangle or square. This is the problem of application of areas solved in I.44, 45.⁸¹

One would assume that these operations generalize to three dimensions, even though Heath does not say so explicitly. If this were so, division of magnitudes of the same dimension, whether lines, rectangles, or rectangular solids, would be accomplished

via ratio-formation, whereas division of a three-dimensional magnitude by an one- or two-dimensional magnitude would involve an application of volumes. The trickier of the two latter cases involves the construction of a rectangular solid on a given base and equal to a given rectangular solid. But this is really straightforward. For if, when written symbolically, ab and xyz are the given base and solid respectively, then applying our I.45A to the rectangular area xy , we can construct $ap = xy$, and reapplying I.45A to pz we have $bq = pz$. Thus, $xyz = apz = abq$, q.e.d. as desired! This, needless to say, never appears in Greek mathematics.

What we have, then, is a bifurcated definition of division which utilizes ratio-formation when the dimensions are equal, and an appropriate "application of areas" when they are not; or, for that matter, even "application of volumes" when needed! This might, at first, seem to solve the dilemma posed by the two forms of multiplication (i.e. the one found in Greek mathematics and the other in "geometric arithmetic"): To divide the rectangle of area twelve by the line of length four, one simply applies a new rectangle, equal to the old one, to the line of length four, producing another line (the other side of the newly formed rectangle) of length three.

But it should be clear that this technique really does nothing to resolve the issue we have raised here, since there still remains a rift in the operation of division. This rift only serves to camouflage the same basic difficulty that one faces all along when one attempts to use rectangle formation as a definition for generalized multiplication, namely, the homogeneity relation between magnitudes gets lost in the shuffle. For example, take an arbitrary rectangle measuring twelve square units and "divide" it by a line segment of length twelve. This could be done, from what was said above, by constructing a twelve unit by one unit rectangle. Now if rectangle formation actually extends the usual Greek operation for numbers, it follows that $12 \div 1 = 12$, i.e., the rectangle and the line are both equal. But if they are equal, they must have a ratio which clearly they do not unless the line can be transformed to the rectangle and vice-versa, i.e. unless magnitude is independent of dimension. So, again, we are led to the inevitable conclusion that viewing rectangle formation as generalized multiplication requires that we abandon the homogeneity relation underlying the Greek theory of magnitude, and, in particular, that we view Greek magnitude as being independent of dimension.

If we stop to consider what this bifurcated form of division amounts to, we soon realize that the roots of its split-appearance

are already present in the conflicting definitions for multiplication that we have considered above. On the one hand, the Greek definitions for multiplication and ratio-formation go hand in hand to form, along with addition and subtraction, a reasonably integrated system of "arithmetical" operations that preserves the integrity of the homogeneity relation between magnitudes; whereas, on the other hand, rectangle formation and "application of areas" are natural, inverse, geometric operations. However the attempt to put the arithmetical and geometrical operations together to form an arithmetic for general magnitudes creates a hybrid creature that plays havoc with the central assumptions of Greek geometry.

4. It is our contention that the dilemma posed by the incompatibility between rectangle formation as generalized multiplication and the homogeneity relation underlying Greek magnitude can be resolved in but one reasonable way, namely by taking rectangle formation at face value, precisely the way we find it throughout Greek geometry, and viewing it as a geometric operation and not as part of a system of generalized arithmetic. Thus, while recognizing the operational character of this construction, as well as the close analogy it shares with the geometric properties of numbers, we cannot accept that this construction was part of an "algebraicized" geometry which sought to extend the ordinary "arithmetic" operations found in the Elements. These two systems (geometrical and "arithmetical") cannot be fused into one without altering altogether the character of Greek geometry, since the "arithmetic" system found in the Elements cannot withstand the type of surgery required to support an algebra for general magnitudes. It follows that rectangle formation, "application of areas," etc. should be viewed as geometric operations and not as part of an integrated, generalized system of "arithmetic." Thus the only "division" in Greek mathematics is ratio-formation, and we can quite confidently assert that no Greek would have confused this operation with the application of a rectangle to a given line by thinking of the two as but different forms of a more general "division" operation. In regard to rectangle formation, we are inclined to view the approach of Dijksterhuis as speaking sympathetically to the issue at hand.⁸² For, by adopting a special notation for various geometric operations, e.g.,

$\Pi(a,b)$ = parallelogram with sides a, b .

$O(a,b)$ = rectangle with sides a, b .

$T(a)$ = square of side a ,

he, at once, calls attention to the importance and operational character of these constructions, while at the same time emphasizing that they are geometric operations, and not part of a "geometric arithmetic," and, as such, should not be confused with the ordinary operations of Greek "arithmetic" from which, as we have seen, they differ fundamentally.

The alternative position concerning "geometric arithmetic" is motivated by the belief that the study of magnitude for its own sake, i.e. magnitude divorced from geometry, held a central place in Greek mathematics. The proponents of this view argue that, to accomplish this, the Greeks found it convenient to associate certain geometric operations (e.g. rectangle formation and "application of areas") with other arithmetical operations (e.g. multiplication and division) in order to establish the necessary foundation for the study of various numerical relationships between certain magnitudes, or classes of magnitudes.

But, given what has been said above, it should by now be clear that this position can no longer be accepted. Not only is its motivation wrong, but the very argument used to support it is both methodologically inappropriate and actually inapplicable. The approach of van der Waerden and Freudenthal illustrates what we mean very clearly.⁸³ For they have adopted a viewpoint regarding these matters altogether consonant with the spirit of modern-day mathematics, which teaches us that there is no essential (mathematical!) difference between the notions of rectangle formation and multiplication of magnitudes, so long as we can exhibit an isomorphism between their mathematical structures. Forming the square on a and writing a^2 are, accordingly, only different names for what is the same mathematical operation, once we have stripped away superfluous notation. It is, therefore, altogether fitting that Freudenthal should adopt as his motto, "But what is in a name?",⁸⁴ since for him, there is absolutely no loss of information in transcribing a geometric operation into an algebraic one, once the alleged isomorphism has been identified.

The objections to the use of this approach as a methodological principle in the study of history of mathematics have been thoroughly covered in Unguru's "On the Need to Rewrite the History of Greek mathematics" and therefore, need not detain us here.⁸⁵ What does need pointing out is the fact that what the proponents of "geometric algebra" have taken as a virtual mathematical verity (namely the alleged isomorphism between the structures of "geometric algebra" and modern day elementary algebra) is, in fact, nothing more than a superficial similarity that conveys nothing significant about the fundamental character and assumptions of

Greek geometry. For, as we have already seen, the attempt to integrate the fundamental operations of "geometric arithmetic" into Greek geometry (i.e., the attempt to view these operations as already embedded within the Greek system) leads to immediate contradiction and confusion. The fact is (and this will become more and more clear in the course of our analysis) that the alleged mathematical correspondence between portions of Greek geometry and elementary, modern-day algebra is, in reality, very weak. It is not enough to point out, as do van der Waerden and Freudenthal, that there is a "resemblance" between some of the propositions in Books II, X, and XIII and various familiar algebraic identities, in order to conclude that the outwardly geometric Greek operations were actually algebraically motivated. One must also show that such an interpretation is consistent with what we find in the actual performance practice of Greek mathematicians, and here, in this all important respect, the "isomorphism" breaks down.

Thus the argument of the "geometrical algebraists" is not only methodologically misguided, it is also actually inapplicable. Moreover, it imputes to Greek mathematics a hidden motivation that is altogether misleading. For the fundamental assumption underlying the theory of "geometric algebra" is that the study of magnitude for its own sake, i.e., as number, was of fundamental importance for Greek mathematics; on no other assumption can the notion of an algebra (whether "geometric" or not) be made intelligible. The fact that van der Waerden et al. completely downplay this foundational principle is further testimony to their unwillingness to assess accurately the implications of their position that Greek mathematics was algebraically motivated. For if arbitrary magnitudes (represented by lines) could be added, subtracted, multiplied, and divided; if they could be manipulated as quantities satisfying all the familiar properties of elementary algebra (associativity, distributivity, etc.), and treated as constants and unknowns in algebraic equations; if all that is true, then certainly magnitude, for the Greeks, was number. And yet the proponents of "geometric algebra" seem to be most reluctant to follow their position to this, its inevitable conclusion, and for a very good reason — there is not a shred of evidence that would support such a claim. The attempt to see algebra lurking behind the propositions of Greek geometry can only be done at the cost of overlooking the intrinsic geometrical setting that persistently motivates Greek mathematics. Searching for its motivation by way of a hidden algebra based on geometric operations, whose "arithmetic content" is never explicitly stated, amounts, in our opinion, to an inversion of means and ends. For while the practitioners of "geometrical algebra" hold to the view that the geometrical form (in, for example, the "application of areas") is incidental to the true algebraic content, it is our contention that "arithmetic"

operations and quasi-"algebraic" relations were never an end unto themselves, but were always used as a means for the solution of problems that were grounded primarily, if not exclusively, in the rich soil of Greek geometry.

5. We have seen so far that the attempt to understand Greek mathematics as "geometric algebra" requires first of all an appropriate system of arithmetical operations, i.e., a "geometric arithmetic," and, secondly, something even more fundamental, an appropriate geometric quantity, upon which the arithmetical system can operate. Both of these are essential ingredients without which the possibility of having a "geometric algebra" is simply unthinkable. The appropriate geometric quantity for "geometric arithmetic" is the Greek notion of magnitude, and it is for this reason that the acceptance of "geometric algebra" necessarily entails the view that, for the Greeks, magnitude was number.

Now, as we have already remarked, few writers have taken it upon themselves to address this issue head-on. There have been several, however, who have found the Greek notion of ratio, rather than the concept of magnitude itself, amenable to a treatment similar to the modern approach toward the positive real numbers. In particular, Definition V.5, which gives a criterion for the proportionality of magnitudes, seems to these writers to resemble closely familiar properties of the real number system. Here is Heath's summary of the views of one such author:

Max Simon remarks ..., after Zeuthen, that Euclid's definition of equal ratios is word for word the same as Weierstrass' definition of equal numbers. So far from agreeing in the usual view that the Greeks saw in the irrational no number, Simon thinks it is clear from Eucl. V. that they possessed a notion of number in all its generality as clearly defined as, nay almost identical with, Weierstrass' conception of it.⁸⁶

If Heath never explicitly endorses this view, he certainly sounds a sympathetic note, by continuing:

Certain it is that there is an exact correspondence, almost coincidence, between Euclid's definition of equal ratios and the modern theory of irrationals due to Dedekind.⁸⁷

And, after explaining this correspondence, he concludes that:

... Euclid's definition divides all rational numbers into two coextensive classes, and therefore defines equal ratios

in a manner exactly corresponding to Dedekind's theory.⁸⁸

Now, what Heath says here is, strictly speaking, correct; there is, interestingly enough, a very close, mathematical correspondence between Definition V.5 and the Dedekind Cut. However the inference that there is something more than mere happenstance at work here is misleading. For despite the formal correspondence discernible between the two, they are, from the vantage point of the history of ideas, totally unrelated. The overriding distinction between the two ideas has to do with their motivation or intentionality. Dedekind's method amounts to showing how it is possible to construct the real number continuum out of the rational numbers, by considering all possible partitions of the latter as a linearly ordered set. When each of these partitions or cuts is assigned a real number, it can be proven that all the "holes" are filled in, or, to speak in more mathematical terms, that, in this larger system, all Cauchy sequences converge.⁸⁹ But, to state the obvious, Definition V.5 has an entirely different motivation from this. There is no question of constructing something out of something else, because the ratios themselves, which are the object of the definition, are already given, they already exist. Nor is there any interest in showing that the collection of all ratios forms a complete space (i.e. showing that all Cauchy sequences converge). What the Greek mind is interested in is defining a criterion of equality of ratios, the net effect of which is to make it possible to order the collection of ratios so as to ascertain whether a given ratio is equal to, less than, or greater than, another given ratio.⁹⁰ If there is more to the theory than this, then we ask, (and, in spite of the strangeness and the apparent ahistoricity of this question, it surely is apposite when addressed to geometrical algebraists) where in Greek mathematics is there any evidence indicating that Definition V.5 was ever used in order to argue that a certain Cauchy sequence of ratios must converge to a definite limit ratio? ...

Now according to Heath, there is another important property of ratios, which along with the idea of the Dedekind Cut, makes ratio, rather than magnitude, more suitable for comparison with the modern notion of (positive) real number:

We have already in Books I and II made acquaintance with one important part of what has been well called geometrical algebra, the method, namely, of application of areas. We have seen that this method, working by the representation of products of two quantities as rectangles, enables us to solve some particular quadratic equations. But the limitations of such a method are obvious. So long as general quantities are represented by straight lines only, we cannot, if our

geometry is plane, deal with products of more than two such quantities; and, even by the use of three dimensions, we cannot work with products of more than three quantities, since no geometrical meaning could be attached to such a product. This limitation disappears so soon as we can represent any general quantity, corresponding to what we denote by a letter in algebra, by a ratio; and this we can do because, on the general theory of proportion established in Book V, a ratio may be a ratio of two incommensurable quantities as well as of commensurables. Ratios can be compounded ad infinitum, and the division of one ratio by another is equally easy, since it is the same thing as compounding the first ratio with the inverse of the second. Thus e.g. it is seen at once that the coefficients in a quadratic of the most general form can be represented by ratios between straight lines, and the solution by means of Books I and II of problems corresponding to quadratic equations with particular coefficients can now be extended to cover any quadratic with real roots.⁹¹

Heath's claim, regarding the suitability of utilizing ratios as coefficients for quadratic equations, will be criticized later in Section III. Here we wish to focus on his contention that the Greeks used ratios in order to get around the problem of the increase in dimension that accompanies the multiplication of magnitudes. Heath seems to suggest that the Greeks could simply replace their magnitudes by ratios, and rectangle formation by the operation of compounding ratios. His assertion is that these entities and operations were, for the Greeks, mathematically equivalent. All we can say is that, so far as we are aware, there is not a single instance in Greek mathematics wherein magnitudes and rectangle formation are replaced by ratios and the operation of compounding them in order to obtain otherwise impossible results. Elsewhere in his commentary, Heath reverts to his usual, sober outlook, and frankly admits that the operation of multiplying ratios is unknown in Greek geometry.⁹² How he can then claim that compounding ratios is the same as forming the rectangle on two lines is a mystery to us. Moreover, were this the case, one wonders why the Greeks should ever have bothered with their "geometric algebra" at all — it would have been so much easier simply to have used ratios! It is our impression that Heath's ideas here are nothing more than an ex post facto reconstruction with no historical basis whatsoever.

In our opinion, the proper perspective on Greek magnitude and ratio comes from viewing them as grounded in nothing more than the notion of size ($\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\varsigma$). Heath has aptly suggested that the relationship between magnitude, $\mu\acute{\epsilon}\gamma\epsilon\theta\omicron\varsigma$, and size, $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\varsigma$, can

be understood by thinking of size as an attribute possessed by magnitude.⁹³ Ratio, on the other hand, is, to paraphrase Euclid, a sort of relation between the sizes of two magnitudes of the same kind.⁹⁴ It is, in effect, a theoretical measuring stick that tells us something about the relative sizes of two magnitudes. Thus we have come full circle returning again to the whole crux of the matter for Greek geometry. To be algebraic, to have a "geometric algebra," requires that magnitude be viewed as number, which amounts to more than being amenable to measure, it must also be amenable to manipulation via addition, subtraction, multiplication, and division. Yet, as we have conclusively shown, it is impossible to integrate these operations into a cohesive, general arithmetic system without doing irreparable damage to the integrity of the original Greek system.

(To be continued in Volume II, 1982.)

NOTES

*The contribution and original ideas of David Rowe, a Ph.D. candidate in mathematics and a graduate student in the history of science at the University of Oklahoma, constitute a major and substantial portion of this study. Rowe's work is part of a forthcoming Master's thesis in the History of Science.

¹"The Soul and the Word leave [the body] through the same orifice."

²Traité Élémentaire de Chimie, 2 vols., vol. 1 (Paris, 1789), pp. V-VII ("Discours Préliminaire").

³After Babel: Aspects of Language and Translation (London, etc.: Oxford University Press, 1975), pp. 24, 216, 359.

⁴As quoted in ibid., p. 81.

⁵Ibid., p. 402.

⁶As quoted in C. K. Ogden and I. A. Richards, The Meaning of Meaning (New York: Harcourt, Brace and World, Inc., n.d. — first published in 1923), p. XXIV.

⁷The Interpretation of Cultures (New York: Basic Books, 1973), p. 18.

⁸S. Unguru, "On the Need to Rewrite the History of Greek Mathematics," Archive, vol. 15, no. 1 (1975), pp. 67-114.

⁹Cf. B. L. van der Waerden, "Defense of a 'Shocking' Point of View," Archive for History of Exact Sciences, vol. 15, no. 3 (1976), pp. 199-210; Hans Freudenthal, "What Is Algebra and What Has It Been in History," ibid., vol. 16, no. 3 (1977), pp. 189-200; André Weil, "Who Betrayed Euclid?", ibid., vol. 19, no. 2 (1978), pp. 91-93. A concise and drastically civilized rejoinder to the above appeared, after many exertions, in S. Unguru, "History of Ancient Mathematics: Some Reflections on the State of the Art," Isis, vol. 70, no. 254 (1979), pp. 555-565. The preceding list does not exhaust the numerous reactions to the "On the Need to Rewrite"-article. Most of these reactions, however (all of the favorable ones), came in private communications from colleagues, friends and students, either in personal letters (scripta manent) or orally (verba volant).

¹⁰"History of Ancient Mathematics," p. 555.

¹¹Ibid.

¹²While one cannot dispute what Hans Freudenthal ("What is Algebra and What has it been in History?", p. 193) says about there being no "Supreme Court to decide such questions" as "What is algebra?", it should be clear that the implications inherent in his retort to Unguru ("But What is in a name?", ibid., p. 194) lead to precisely the kind of disregard for the historicity of ideas that we find exemplified so well in van der Waerden's Science Awakening (New York: John Wiley and Sons, Inc., 1963, hereinafter referred to as SA). Thus algebra is, more or less, anything one

wants it to be. For Freudenthal, the "ability to describe relations and solving procedures, and the techniques involved in a general way, is ... such an important feature of algebraic thinking that I am willing to extend the name 'algebra' to it ..." (*ibid.*, pp. 193-94). Our opinion on this matter can be gleaned from S. Unguru, "History of Ancient Mathematics," pp. 557-561. As in that paper, here too we take algebra to be that branch of mathematics whose primary purpose is finding unknowns, i.e. solving equations. In the final analysis, the approach of Freudenthal and van der Waerden is nothing more than a convenient cover for all those who wish to transcribe ancient mathematics into modern symbolism with impunity, and it inevitably leads to a confusion between the suppositions that govern the practice of ancient mathematics and those which govern our own. Calling the Greek techniques of alternation, inversion, composition, separation, and conversion (cf. T. L. Heath, The Thirteen Books of Euclid's Elements, 3 vols. (Cambridge: At the University Press, 1908), hereinafter referred to as EE, vol. 2, pp. 114-115) "algebraic operations" (Freudenthal, *op. cit.*, p. 195) serves only to identify them with modern rules for manipulating fractions, the net result being that important distinctions that should be made between the two become lost.

¹³Otto Neugebauer, Mathematische Keilschrift-Texte (MKT), III, p. 6, in Quellen und Studien zur Geschichte der Mathematik Astronomie und Physik (Q.u.S.), vol. 3 (1937), Abteilung A (Quellen). The notation 14, 30 is Neugebauer's way of expressing numbers in the Babylonian sexagesimal system. Thus $14,30 = 14 \times 60 + 30 = 870$, whereas $14,30;15 = 14 \times 60 + 30 + 15 \times 60^{-1} = 870 \frac{1}{4}$.

¹⁴"Defence of a 'Shocking' Point of View," p. 199.

¹⁵*Ibid.*, pp. 200-201.

¹⁶Van der Waerden's emendations themselves are not without interest. For example he adds after "coefficient," "(of the unknown side)" which appears neither in the original nor in Neugebauer's German translation (cf. "Defence of a 'Shocking' Point of View," p. 201 with MKT, vol. III, p. 6, line 5 and p. 1, line 5). Furthermore, he ignores Neugebauer's distinction between two different kinds of emendation (for clarification purposes and "Ergänzung zerstörter Stellen" (MKT, vol. II, p. 7 in Q.u.S., A, vol. 3 (1935)) throughout the "quotation." Moreover, there are serious questions about Neugebauer's own translation. For example, Neugebauer translates "möglichst unbestimmt" pišitam by "coefficient" (MKT, III, p. 5), while Thureau-Dangin translates it always as "l'unité," (*ibid.*, p. 11) and points out that "la signification est très incertaine" (*ibid.*).

¹⁷See O. Neugebauer, The Exact Sciences in Antiquity (Princeton, N.J.: Princeton University Press, 1952), passim.

¹⁸Ibid., chapter II, "Babylonian Mathematics," pp. 28-53.

¹⁹O. Neugebauer, Vorgriechische Mathematik (Berlin: Springer-Verlag, 1934; reprinted Springer-Verlag, 1969), p. 33.

²⁰Cf. text to note 14 above.

²¹"Defence," p. 200, our italics.

²²Ibid.

²³Ibid.

²⁴After Babel, p. 296.

²⁵Most of these alternative interpretations are, however, algebraic. Cf. S. Gandz, "The Origin and Development of the Quadratic Equations in Babylonian, Greek and Early Arabic Algebra," Osiris, vol. 3 (1938), pp. 405-557; Kurt Vogel, Vorgriechische Mathematik, 2 vols. (Hannover: Paderborn, 1958-59); "Bemerkungen zu den quadratischen Gleichungen der babylonischen Mathematik," Osiris, vol. 1 (1936), pp. 703-717; "Zur Berechnung der quadratischen Gleichungen bei den Babyloniern," Unterrichtsblätter für Mathematik und Naturwissenschaften, vol. 39 (1933), pp. 76-81; also, Thureau-Dangin's works.

²⁶"Babylonian Algebra: Form vs. Content," Studies in History and Philosophy of Science, vol. 1 (1970-71), pp. 369-380.

²⁷Ibid., p. 377. Mahoney's essay should be read in its entirety for the cogent criticism it contains of Neugebauer's (i.e., van der Waerden's) approach.

²⁸"Zum Problem der sog. 'Geometrischen Algebra' in Euklids Elementen," completed in 1975. We had no access to the printed volume Prismata in which it appeared.

²⁹Ibid., pp. 19-20 of the manuscript.

³⁰Ivor Thomas, Selections Illustrating the History of Greek Mathematics (SGM), 2 vols. (London and Cambridge, Mass.: William Heinemann Ltd. and Harvard University Press, 1951), in vol. 2, pp. 503-505.

³¹Ibid., pp. 527-535.

³²For Jacob Klein (cf. Greek Mathematical Thought and the Origin of Algebra (Cambridge, Mass.: M.I.T. Press, 1968), pp. 127-149, *passim*), Diophantus's Arithmetica belongs to the category of theoretical logistic.

³³Op. cit., p. 533.

³⁴Addition and subtraction can be found at the very beginning of the Elements in Common Notions 2 and 3, which state that equality is preserved when equals are added (respectively subtracted) to (or from) equals. These principles can be seen in practice in the proofs of Propositions I.47 and II.11. Multiplication and ratio formation do not occur until Book V, where they are fundamental to the theory of general proportion that is developed therein. Cf. EE, vol. 2, pp. 113-114, for the use of multiple ($\pi\omicron\lambda\lambda\alpha\pi\lambda\acute{\alpha}\sigma\iota\omicron\varsigma$) and the definition of ratio ($\lambda\acute{o}\gamma\omicron\varsigma$). The concept of ratio that appears in Euclid (Definitions V.3 and 4) is for some modern tastes rather opaque, and D. H. Fowler ("Ratio in

Early Greek Mathematics," Bulletin of the American Mathematical Society, vol. 1 (1979), no. 6, pp. 807-846, on p. 812) has even argued that $\lambda\acute{o}\gamma o\varsigma$ should be taken as an undefined term in the Elements! This would appear, however, to be a substitution of modern for ancient standards of rigor, as it is not at all uncommon to find important concepts in Greek mathematics that are barely defined (or never defined at all), yet which seem to have been understood intuitively. It is a mistake to think that there is an airtight logical system in the Elements that relegates each concept to one of two categories, i.e., that regards each concept as being either defined or undefined. An even worse mistake would be to imagine that the undefined terms of Greek mathematics have the same freedom from ontological commitments as do undefined terms in modern mathematics. The fact is that there seems to be a vast grey area between the defined and undefined terms of Greek mathematics. Multiplication is a perfect example of this, as it is never explicitly defined in Book V, and yet its definition as repeated addition is altogether clear once we understand the principles behind the Greek theory of general proportion (particularly Definition V.5). An explicit definition of multiplication ($\pi\omicron\lambda\lambda\alpha\pi\lambda\alpha\sigma\iota\alpha\sigma\mu\acute{o}\varsigma$) is given in Definition VII.15, as part of the proportion theory for numbers.

³⁵B. L. van der Waerden, SA, p. 118.

³⁶A major fallacy behind the notion of "geometric algebra" concerns the question of which "algebraic operations" were actually "known" to the Greeks. If we are to accept van der Waerden's more extreme views on the subject, it would seem that the "known" operations consist of anything that one can reconstruct and render into algebraic language. Thus, after asserting that "geometric algebra" was derived from Babylonian sources, he informs us that in Greek hands this Babylonian "algebra" is "... translated into geometric terminology. But since it is indeed a translation which occurs here and the line of thought is algebraic, there is no danger of misrepresentation, if we reconvert the derivations into algebraic language and use modern notations" (SA, p. 119, our emphasis). For a detailed discussion and criticism of the methodological assumptions underlying the concept of "geometric algebra" and of the practices employed by its proponents, cf. Unguru, "On the Need to Rewrite the History of Greek Mathematics," passim.

³⁷The earliest formulation of the operations comprising the "geometric arithmetic" that we are aware of appears in H. G. Zeuthen, Die Lehre von den Kegelschnitten im Altertum, (Hildesheim: Georg Olms, 1966, being a photographic reproduction of the Copenhagen, 1886 edition), pp. 14-15.

³⁸In this regard, Jacob Klein (Greek Mathematical Thought and the Origin of Algebra, pp. 117-118) has this to say: "The difficulties in the way of an adequate understanding of the Greek doctrine of number lie above all ... in our own manner of dealing

with concepts -- in the nature of our own intentionality." By intentionality Klein means, "... the mode in which our thought, and also our words, signify or intend their objects" (p. 118). Thus he continues: "The necessity of abstaining as far as possible from the use of modern concepts in the interpretation of ancient texts is therefore generally accepted, and even stressed. It is clear, to be sure, that the feasibility of an interpretation not based on modern presuppositions must always be limited; even if we succeed in ridding ourselves completely of present-day scientific terminology, it remains immensely difficult to leave that medium of ordinary intentionality which corresponds to our mode of thinking, a mode essentially established in the last four centuries. On the other hand, the ancient mode of thinking and conceiving is, after all, not totally 'strange' or closed to us. Rather, the relation of our concepts to those of the ancients is oddly 'ruptured' -- our approach to an understanding of the world is rooted in the achievements of Greek science, but it has broken loose from the presuppositions which determined the Greek development. If we are to clarify our own conceptual presuppositions we must always keep in mind the difference in the circumstances surrounding our own science and that of the Greeks" (p. 118). Here, and throughout his book, Klein displays a rare and admirable sensitivity to issues that are vital for the interpretation of ancient mathematics. Another writer who must also be commended for his insight into the specific dangers involved in the "geometric algebra"-approach to Greek mathematics is E. J. Dijksterhuis (cf., n. 82). J. Klein also calls attention to this issue (ibid., p. 122).

³⁹EE, vol. 1, p. 155.

⁴⁰The only explicit mention of homogeneity occurs in Definition V.3: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind" (EE, vol. 2, p. 114).

⁴¹In this connection, cf. Heath's discussion on species of "lines" (EE, vol. 1, pp. 159-165) and species of "angles" (pp. 176-179).

⁴²One finds this practice employed only in the late Hellenistic period, for example in Heron (fl. 2nd half of 1st Cent. A.D.). We have encountered such an example above (see text to n. 30).

⁴³Heath's view as expressed in EE, vol. 1, p. 178.

⁴⁴Ibid., p. 281.

⁴⁵Thomas, SGM, vol. 1, pp. 248-253.

⁴⁶See below our discussion in Section III, no. 5. For a thorough treatment of I.47, cf. EE, vol. 1, pp. 349-50 followed by Heath's analysis, on pp. 350-368.

⁴⁷See below our section III, no. 5 or EE, vol. 1, pp. 402-403.

⁴⁸A first step toward this liberation of magnitude from geometry was taken by François Viète (1540-1603), in his In artem analyticen Isagoge (Introduction to the Analytical Art) which first

appeared in 1591. In Chapter II (pp. 322-324 of the appendix to Jacob Klein's Greek Mathematical Thought and the Origin of Algebra) Viète laid down his "stipulations (symbola) governing equations and proportions." These principles give a good indication of the enormous gap that exists between the "arithmetic" operations together with their concomitant properties as found in Euclid, and the modern "laws of algebra" that we are nowadays so accustomed to taking for granted. For example, the first four symbola include the Common Notions of the Elements, and many of the others are similar to results that are proven in the Elements (cf. Klein's listing in op. cit., p. 263). But, significantly, there are also several algebraic rules that are missing from Euclid like:

- (5) If equals are multiplied by equals, the products are equal.
- (6) If equals are divided by equals, the results are equal.

One can, of course, take the position that Viète was only reconstructing techniques that were latent in Greek mathematics all along. (This, in fact, seems to have been Viète's own view of the situation.) To take this position seriously, however, one must at least attempt to elucidate the precise manner in which these techniques were then utilized by Greek mathematicians, and this the practitioners of "geometric algebra" seem none too eager to do. Needless to add, we disagree with the view that Viète's techniques were actually Greek.

⁴⁹EE, vol. 1, p. 374.

⁵⁰Ibid.

⁵¹Ibid., p. 341.

⁵²Ibid., p. 345.

⁵³"To bisect a given finite straight line" (ibid., p. 267).

⁵⁴"In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas" (ibid., p. 323).

⁵⁵"Equal triangles which are on equal bases and on the same side are also in the same parallels" (ibid., p. 337).

⁵⁶The simplest way to prove I.45A is to make a slight modification in the proof of I.45. The proof in Euclid uses I.42 followed by I.44, whereas it is just as easy to use I.44 twice, which insures that the parallelogram ultimately obtained has one side equal to a given length as required in I.45A. Cf. below, section III, no. 4 or EE, vol. 1, pp. 345-346.

⁵⁷"To construct one and the same figure similar to a given rectilinear figure and equal to another given rectilinear figure" (EE, vol. 2, p. 253).

⁵⁸EE, vol. 1, pp. 374.

⁵⁹Ibid., pp. 342-43.

⁶⁰Heath's remark in EE, vol. 1, p. 347.

⁶¹In Book X a means is established for classifying various types of incommensurable magnitudes. The basic idea can be seen from Definition X.2, which introduces the notion of magnitudes (represented by lines) that are commensurable in square (ὁυνάμει σύμμετρος), i.e., lines the squares on which are commensurable with one another. Thus if a given line is taken as unit, the collection of all lines which are commensurable in square with this given line includes all of the lines commensurable in length (μήκει) with the given line as well as many others besides, e.g., the diagonal of the square on the unit, for the square on the diagonal is exactly twice the area of the unit square, and hence is commensurable with it. One sees here, then, the attempt to utilize two dimensional figures in order to make assertions about one-dimensional figures, namely lines. Moreover, there are numerous instances throughout Book X wherein "application of areas" is utilized. In fact, almost everywhere that one finds a two-dimensional figure in Book X, the proof accompanying it involves an "application of area," e.g., Propositions 20, 22, 23, 25, 26, 38, 41, 60-65, 72, 75, 78, 81, 84, 97-102, 108, 109, 111, 114. Occasionally the proofs call for an "application with defect" (never with "excess") and in these cases the "defect" is always a square. (Cf., for example, X.17, 18, 33, 34, 54, 55, 91-99.) With this information in hand, it is easier to understand Heath's misreading of I.44, as the circumstances he describes fit precisely those found in Book X, wherein I.45A is implicitly used in order to transform one rectangle into another with one side of a given length. (Only in Proposition X.38 does Heath mistakenly cite I.44 as the means for accomplishing this transformation, elsewhere he simply asserts that the application is done, without reference to the means by which it is accomplished.) But nowhere in Book X is an "application of area" performed where one side is designated to be a unit length! In fact, the entire procedure underlying Book X has nothing to do with measuring figures per se. In many situations, a line in Book X can be replaced by another line commensurable with it and the argument will be unaffected. Indeed the whole point of Book X seems to be to demonstrate the existence of certain "equivalence classes" that arise in the study of magnitude (cf. X.111, EE, vol. 3, pp. 242-243); its object is qualitative, not quantitative. There is absolutely no distinction between different magnitudes within a given "equivalence class," hence size is irrelevant. Furthermore, there is a strict adherence to the integrity of dimension throughout Book X. Although information pertaining to the square on a line is utilized to make assertions about the line itself, there is not the slightest suggestion anywhere that one can correlate magnitudes of different dimensions by using ratios or anything else.

⁶²EE, vol. 1, p. 347.

⁶³Ibid., p. 372.

⁶⁴EE, vol. 2, pp. 277-424.

⁶⁵Cf., SA, pp. 168-172.

⁶⁶EE, vol. 2, p. 221.

⁶⁷EE, vol. 1, p. 374.

⁶⁸EE, vol. 2, p. 187.

⁶⁹Cf., Aristotle, The Works of Aristotle, ed. Ross (Oxford/London: Oxford University Press/Humphrey Milford, 1937 reprint of the 1st ed. of 1928) vol. 1, Analytica Posteriora, I.5, 74^a 17-25.

⁷⁰EE, vol. 2, p. 216.

⁷¹Ibid., p. 191.

⁷²Ibid., p. 278.

⁷³Definition V.2: "The greater is a multiple of the less when it is measured by the less" (ibid., p. 113).

⁷⁴Ibid., p. 114.

⁷⁵Ibid.

⁷⁶Thus multiplication, viewed as a binary operation, does not necessarily involve homogeneous magnitudes. The homogeneity restriction arises only when multiplication is viewed as an n-fold application of addition.

⁷⁷Proposition V.18: "If magnitudes be proportional separando, they will also be proportional componendo" (ibid., p. 169).

⁷⁸Proposition V.19: "If, as a whole is to a whole, so is a part subtracted to a part subtracted, the remainder will also be to the remainder as whole to whole" (ibid., p. 174).

⁷⁹Euclid assumes, in the course of proving Prop. V.18, that given an arbitrary ratio and an arbitrary magnitude, there exists a fourth proportional. But, as Heath rightly points out, this is a logical error (ibid., pp. 169-170).

⁸⁰In this context, Euclid's remarks in the Data are pertinent. Definition II of Euclid's Data reads: "A ratio is said to be given, when a ratio of a given magnitude to a given magnitude which is the same ratio with it can be found" (The Elements of Euclid, Books I-VI, XI, XII, and the Data, ed. R. Simson (London: G. Woodfall, 20th ed., 1822), p. 359). This indicates the usual format (as well as the name) when solving for the fourth proportional, i.e., one is given a pair of homogeneous magnitudes and a third magnitude, from which one must construct a fourth magnitude such that the ratios between the given pair and the latter two magnitudes are equal.

⁸¹EE, vol. 1, p. 374.

⁸²Dijksterhuis was also one of the first to recognize the dangers inherent in the notion of "geometric algebra." The following passage, "translated" by E. M. Bruins, from the second volume of De Elementen van Euclides was written in 1930 and shows that the main features of the present situation were already recognized even fifty years ago:

The second Book of the Elements, elaborating the propositions I. 43-47, brings about the foundation of a method of research typical for Greek mathematics of which the importance can be briefly depicted indicating that it enabled the Greek mathematicians to obtain without the help of an algebra a great number of results, which in our times seem to be almost inseparably connected with an application of algebraic concepts and methods to geometry. The explanation of this theory is rendered more difficult by a danger, which always threatens everybody, who wishes to write about Greek mathematics from the modern point of view, but here in particular, namely, that one using to explain classical reasonings — for abbreviation and clarification — modern concepts and symbols, comes to ascribe to these reasonings a tenor which, historically, they did not have. This danger is threatening thus much exactly for the subject which is to be treated now, because, as we shall see, the steps of the Greek mathematical argument can be rendered one by one in the language of the modern mathematics; our familiarity with that language together with our being strange to the much more clumsy wording of the Greeks seduces soon to modernize with the form also the thought" (the rendering of Evert M. Bruins in *Janus* (1975), 62, p. 309).

Regarding "geometric algebra" itself, Dijksterhuis has this to say:

Thus it is evident that the use of linesegments [*sic*] and areas enabled the Greeks, notwithstanding their limited concept of number and their lacking an algebra, to treat magnitudes which we would render by positive real numbers and between which we discover relations following an algebraic method. The method applied is generally named, following Zeuthen, by geometric algebra, "da dieselbe," as Zeuthen himself formulates it, "als Algebra teils allgemeiner Grössen, irrationale sowie rationale behandelt, teils andere Mittel als die gewöhnliche Sprache benutzt, um ihr Verfahren anschaulich zu machen und dem Gedächtnis einzuprägen." We shall in this work not follow this usage. As an objection to the use of the word "algebra" in this context can be put forward, that the Greek method does not have in the great majority of the cases the symbolical character, which we nowadays do connect inseparably with the concept of an algebra. (Bruins' translation, *ibid.*, pp. 309-310).

⁸³Cf., n. 12 and n. 34 above.

⁸⁴Freudenthal, "What is Algebra," p. 194.

⁸⁵Unguru, *op. cit.*, pp. 73-76, 88-89, *passim*.

⁸⁶EE, vol. 2, p. 124.

⁸⁷*Ibid.*

⁸⁸*Ibid.*, p. 126. Oskar Becker ("Eudoxos-Studien II. Warum haben die Griechen die Existenz der vierten Proportionale

angenommen?," Q.u.S., B, vol. 2, 1933, pp. 369-387) has amplified this interpretation that the notion of the Dedekind Cut is already implicit in early Greek mathematics, by relating it to the "method of exhaustion." Since Definition V.5 and the "method of exhaustion" are both generally attributed to Eudoxus, Becker suggests that it is he who is behind the development of this (rather too) sophisticated idea.

⁸⁹A sequence (x_n) is Cauchy if, given any $\epsilon > 0$, there exists an integer N , such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$. The assertion that the sequence (x_n) converges means that there exists an x with the property that, given any $\epsilon > 0$, there exists an integer N , such that $|x_n - x| < \epsilon$ for all $n \geq N$.

⁹⁰This criterion is used in a considerable number of the arguments of Book V, e.g., V.4, 7, 9, 13, 16, etc. (cf., EE, vol. 2, pp. 112-186).

⁹¹Ibid., p. 187.

⁹²Ibid., p. 190.

⁹³Ibid., p. 117.

⁹⁴This is Definition V.3 (ibid., p. 114).

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