ON NIKODYM'S BOUNDEDNESS THEOREM

Corneliu Constantinescu

Let $\mathfrak{R}$ be a σ-ring and let M be a set of real valued measures on $\mathfrak{R}$. The theorem referred in the title ([5], page 118) asserts that $\{\mu(A)|\mu \in M, \ A \in \mathfrak{R}\}$ is a bounded set provided $\{\mu(A)|\mu \in M\}$ is bounded for any $A \in \mathfrak{R}$. Assume X is a Hausdorff topological space, $\mathfrak{K}$ is the set of compact sets of X, $\mathfrak{R}$ is the σ-ring of Borel sets of X, $\mathfrak{T}$ is a subset of $\mathfrak{R}$, and M is a set of $\mathfrak{K}$-regular real valued measures on $\mathfrak{R}$. In this case it is sometimes possible to prove that the above hypothesis "$\{\mu(A)|\mu \in M\}$ is bounded for any $A \in \mathfrak{R}$" follows from the weaker one "$\{\mu(A)|\mu \in M\}$ is bounded for any $A \in \mathfrak{T}$." Such a result was obtained by J. Dieudonné ([1], Proposition 9) for X compact and $\mathfrak{T}$ the set of open sets of X. It was extended by P. Gänssler ([3], Theorem 3.1) for X regular, and $\mathfrak{T}$ the set of regular open sets of X. We show in this paper (Corollary 2.9) that Gänssler's theorem holds for group valued measures too, and prove a similar result for normal spaces and closed regular measures (Corollary 2.10). In arbitrary topological groups the bounded and the precompact sets do not coincide (as in the real case) and so we prove the above propositions separately for the bounded and for the precompact case. In order to obtain the boundedness or the precompactness of the set $\{\mu(A)|\mu \in M, \ A \in \mathfrak{A}\}$ one may use the following result.

<u>*Theorem 1.7.*</u> *Let $\mathfrak{R}$ be a ring of sets, let G be a Hausdorff topological additive group, and let M be a set of G-valued measures on $\mathfrak{R}$ such that:*

(a) $\{\mu(A)|\mu \in M\}$ is a bounded set of G ($\{\mu(B)|\mu \in M, B \in \mathfrak{R}, B \subset A\}$ is a precompact set of G) for any $A \in \mathfrak{R}$;

(b) for any disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{R}$, there exists an infinite subset M of $\mathbb{N}$ such that $\bigcup_{n\in M} A_n \in \mathfrak{R}$.

Then $\{\mu(A)|\mu \in M, \ A \in \mathfrak{R}\}$ is a bounded (precompact) set of G.

The following special case deserves a separate mention.

<u>Corollary 1.4.</u> Let G be a Hausdorff topological additive group, let I be a set, and let $\mathfrak{J}$ be a set of subsets of I such that:

(a) any finite subset of I belongs to $\mathfrak{J}$;
(b) for any disjoint sequence $(J_n)_{n\in\mathbb{N}}$ of finite subsets of I, there exists an infinite subset M of $\mathbb{N}$, such that $\bigcup_{n\in M} J_n \in \mathfrak{J}$.

Let further F be a set of G-valued functions on I such that:
(c) $(f(\iota))_{\iota\in J}$ is a summable for any $J \subset I$ and for any $f \in F$;
(d) $\{\sum_{\iota\in J} f(\iota) \mid f \in F\}$ is a bounded (precompact) set of G for any $J \in \mathfrak{J}$.

Then $\{\sum_{\iota\in J} f(\iota) \mid f \in F,\ J \subset I\}$ is a bounded (precompact) set of G.

Notations. We denote by $\mathbb{N}$ the set of natural numbers (0 will not be considered a natural number), by $\mathbb{R}$ the set of real numbers, and for any set X, by $\mathfrak{P}(X)$ the power set of X (i.e. the set $\{A \mid A \subset X\}$). A ring of sets is a set $\mathfrak{R}$ of sets such that $A \cup B$, $A \setminus B \in \mathfrak{R}$ for any $A,B \in \mathfrak{R}$. A σ-ring is a ring of sets $\mathfrak{R}$ such that the union of any sequence in $\mathfrak{R}$ belongs to $\mathfrak{R}$. A disjoint family of sets is a family $(A_\iota)_{\iota\in I}$ of sets such that $A_\iota \cap A_\kappa = \emptyset$ for any different $\iota,\kappa \in I$. A quasi-σ-ring is a ring of sets $\mathfrak{R}$, such that for any disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{R}$, there exists an infinite subset M of N with $\bigcup_{n\in M} A_n \in \mathfrak{R}$. A quasi-$\delta$-ring is a ring of sets $\mathfrak{R}$, such that for any disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{R}$ whose union is contained in a set of $\mathfrak{R}$, there exists an infinite subset M of $\mathbb{N}$ with $\bigcup_{n\in M} A_n \in \mathfrak{R}$.

Let X be a topological space and let A be a subset of X. We denote by $\bar{A}$ and by $\mathring{A}$ the closure and the interior of A respectively. An open set U of X is called regular, if $U = \mathring{\bar{U}}$. A relatively σ-compact set of X is a subset of X which is contained in the union of a countable family of compact sets of X.

Let G be a topological additive (i.e. commutative) group. We set for any subsets A, B of G

$$A + B := \{x+y \mid x \in A,\ y \in B\}, \quad A - B := \{x-y \mid x \in A,\ y \in B\},$$

$$-A := \{-x \mid x \in A\}$$

and define inductively nA for any $n \in \mathbb{N}$ by

$$1\,A := A, \quad (n+1)\,A := A + nA.$$

A bounded set of G is a subset A of G such that for any 0-neighbourhood U in G there exist $n \in \mathbb{N}$ and a finite subset P of G, such that $A \subset P + nU$. A precompact set is a subset A of G such that for any 0-neighbourhood U in G there exists a finite subset P of G such that $A \subset P + U$. As these definitions show, there is a certain parallelism among the two notions. In order to unify their handling we introduce a map of $\mathbb{N}$ into itself defined by

$$\rho(n) := \begin{cases} n & \text{in the bounded case} \\ 1 & \text{in the precompact case.} \end{cases}$$

Throughout this paper we shall denote by G a Hausdorff topological additive group, by $\mathfrak{R}$ a ring of sets, and by $\mathfrak{K}$ a subset of $\mathfrak{R}$ closed with respect to finite unions (then $\emptyset \in \mathfrak{K}$).

Let μ be a map of $\mathfrak{R}$ into G. μ is called additive, if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for any disjoint sets A, B of $\mathfrak{R}$. μ is called a measure, if it is countable additive, i.e. if

$$\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for any disjoint sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathfrak{R}$, whose union belongs to $\mathfrak{R}$. μ is called $\mathfrak{K}$-regular, if for any $A \in \mathfrak{R}$, and for any 0-neighbourhood U in G there exists $K \in \mathfrak{K}$, such that $K \subset A$ and

$$\mu(B) - \mu(A) \in U$$

for any $B \in \mathfrak{R}$, with $K \subset B \subset A$. Any map of $\mathfrak{R}$ into G is $\mathfrak{R}$-regular.

We say, that a subset $\mathfrak{T}$ of $\mathfrak{R}$ separates $\mathfrak{K}$, if for any disjoint sets $K', K'' \in \mathfrak{K}$ there exist disjoint sets $T', T'' \in \mathfrak{T}$ with $K' \subset T'$, $K'' \subset T''$.

I. GENERAL MEASURES

Lemma 1.1. Let U be a 0-neighbourhood in G, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in G, and let $(p_n)_{n \in \mathbb{N}}$ be an increasing unbounded sequence in $\mathbb{N}$ such that

$$x_n \notin \{x_m | m \in \mathbb{N},\ m < n\} + \rho(p_n)U$$

for any $n \in \mathbb{N}$. Then $\{x_n | n \in M\}$ is not a bounded (precompact) set of G for any infinite subset M of $\mathbb{N}$.

Let M be an infinite subset of $\mathbb{N}$ and assume $\{x_n | n \in M\}$ is bounded (precompact). Let V be a 0-neighbourhood in G such that $V - V \subset U$. There exist a $p \in \mathbb{N}$ and a finite subset P of G, such that

$$\{x_n | n \in M\} \subset P + \rho(p)V.$$

Then there exist $x \in P$, and an infinite subset M' of M, such that

$$\{x_n | n \in M'\} \subset x + \rho(p)V.$$

Let $m,n \in M'$ such that $m < n$, and $p \leq p_n$. Then

$$x_n - x_m \in \rho(p)V - \rho(p)V \subset \rho(p)U \subset \rho(p_n)U,$$

and this is a contradiction. □

<u>Proposition 1.2.</u> Let $\mathcal{M}$ be a set of additive maps of $\mathfrak{R}$ into G such that

(a) $\{\mu(A) | \mu \in \mathcal{M}\}$ is a bounded set of G ($\{\mu(B) | \mu \in \mathcal{M},\ B \in \mathfrak{R},\ B \subset A\}$ is a precompact set of G) for any $A \in \mathfrak{R}$;

(b) for any sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, and for any disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{R}$, there exists an infinite subset M of $\mathbb{N}$ such that $\{\mu_n(A_n) | n \in M\}$ is a bounded (precompact) set of G.

Then $\{\mu(A) | \mu \in \mathcal{M},\ A \in \mathfrak{R}\}$ is a bounded (precompact) set of G.

We want to show first that

$$\{\mu(B) | \mu \in \mathcal{M},\ B \in \mathfrak{R},\ B \subset A\}$$

is a bounded set of G for any $A \in \mathfrak{R}$. Assume the contrary. Then there exist $A \in \mathfrak{R}$ and a symmetric 0-neighbourhood U in G such that for any finite subset P of G, and for any $p \in \mathbb{N}$, there exists $(\mu,B) \in \mathcal{M} \times \mathfrak{R}$, such that $B \subset A$ and

$$\mu(B) \notin P + pU.$$

We construct inductively a sequence $(\mu_n)_{n\in\mathbb{N}}$ in M and a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{R}$ such that we have for any $n \in \mathbb{N}$:

(1) $A_n \subset A \setminus \bigcup_{m=1}^{n-1} A_m$;

(2) $\mu_n(A_n) \notin \{\mu_m(A_m) \mid m \in N, \quad m < n\} + nU$;

(3) for any finite subset P of G, and for any $p \in \mathbb{N}$, there exists $(\mu, B) \in M \times \mathfrak{R}$ such that $B \subset A \setminus \bigcup_{m-1}^{n} A_m$, and

$$\mu(B) \notin P + pU.$$

Let $n \in \mathbb{N}$ and assume the sequences were constructed up to $n-1$. By (a)

$$\{\mu(A \setminus \bigcup_{m=1}^{n-1} A_m) \mid \mu \in M\}$$

is a bounded set of G. Hence there exist a finite subset P of G, and a $p \in \mathbb{N}$, such that

$$\{\mu(A \setminus \bigcup_{m=1}^{n-1} A_m) \mid \mu \in M\} \subset P + pU.$$

By (3) (or by the hypothesis in the proof if $n = 1$) there exists $(\mu_n, B) \in M \times \mathfrak{R}$, such that $B \subset A \setminus \bigcup_{m=1}^{n-1} A_m$ and

$$\mu_n(B) \notin P + \{\mu_m(A_m) \mid m \in \mathbb{N}, \quad m < n\}$$
$$- \{\mu_m(A_m) \mid m \in \mathbb{N}, \quad m < n\} + (n+p)U.$$

If there exist a finite subset Q of G, and a $q \in \mathbb{N}$ such that

$$\{\mu(C) \mid \mu \in M, \quad C \in \mathfrak{R}, \quad C \subset B\} \subset Q + qU,$$

then we set $A_n := B$; otherwise we set

$$A_n := (A \setminus \bigcup_{m=1}^{n-1} A_m) \setminus B.$$

If $A_n = B$ then the conditions (1), (2), (3), are trivially fulfilled. If $A_n \neq B$ then (1) and (3) are fulfilled. Assume (2) is not fultilled. Then

$$\mu_n(B) = \mu_n(A \setminus \bigcup_{m=1}^{n-1} A_m) - \mu_n(A_n) \in P + pU - \{\mu_m(A_m) | m \in \mathbb{N},\ m < n\} - nU,$$

which is a contradiction. Hence (2) is fulfilled in this case too and the inductive construction is finished.

By (2) and Lemma 1.1 $\{\mu_n(A_n) | n \in M\}$ is not bounded for any infinite subset M of $\mathbb{N}$, and this contradicts (b).

We prove now the assertion of the proposition. Assume it does not hold. Then there exists a symmetric 0-neighbourhood U in G such that for any finite subset P of G and for any $p \in \mathbb{N}$ there exists $(\mu, A) \in \mathcal{M} \times \mathfrak{R}$ with

$$\mu(A) \notin P + \rho(p)U.$$

We construct inductively a disjoint sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathfrak{R}$, and a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$, such that

$$\mu_n(A_n) \notin \{\mu_m(A_m) | m \in \mathbb{N},\ m < n\} + \rho(n)U$$

for any $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and assume the sequences were constructed up to $n - 1$. By the above considerations (by (a))

$$\{\mu(A) | \mu \in \mathcal{M},\quad A \in \mathfrak{R},\quad A \subset \bigcup_{m=1}^{n-1} A_m\}$$

is bounded (precompact). Hence

$$\{\mu(A) | \mu \in \mathcal{M},\quad A \in \mathfrak{R},\quad A \cap (\bigcup_{m=1}^{n-1} A_m) = \emptyset\}$$

is not bounded (precompact), and there exists therefore $(\mu_n, A_n) \in \mathcal{M} \times \mathfrak{R}$ such that $A_n \cap (\bigcup_{m=1}^{n-1} A_m) = \emptyset$ and

$$\mu_n(A_n) \notin \{\mu_m(A_m) | m \in \mathbb{N},\quad m < n\} + \rho(n)U .$$

This finishes the inductive construction.

By Lemma 1.1 $\{\mu_n(A_n) \mid n \in M\}$ is not bounded (precompact) for any infinite subset M of $\mathbb{N}$ and this contradicts (b). $\square$

Remark. Let μ be an additive map of $\mathfrak{R}$ into G, such that

$$\lim_{n\to\infty} \mu(A_n) = 0$$

for any disjoint sequence in $\mathfrak{R}$. Then $\{\mu(A_n) \mid n \in \mathbb{N}\}$ is a bounded set of G, and by the above proposition $\{\mu(B) \mid B \in \mathfrak{R}\}$ is a bounded set of G. This result was proved by M. P. Katz ([4], page 1160) for μ a measure and by L. Drewnowski ([2], Corollary 1) for G locally convex.

Theorem 1.3. Let X *be a set, let* $\mathfrak{K}$ *be the set of finite subsets of* X*, let* $\mathfrak{R}$ *be a ring of subsets of* X *containing* $\mathfrak{K}$*, and let* $\mathcal{M}$ *be a set of* $\mathfrak{K}$*-regular* G*-valued measures on* $\mathfrak{R}$*, such that for any disjoint sequence* $(K_n)_{n\in\mathbb{N}}$ *in* $\mathfrak{K}$ *there exists an infinite subset* M *of* $\mathbb{N}$*, such that:*

(a) $\{\mu(K_1) \mid \mu \in \mathcal{M}\}$ *is a bounded (precompact) set of* G*;*

(b) $\bigcup_{n\in M} K_n \in \mathfrak{R}$*;*

(c) $\{\mu(\bigcup_{n\in M} K_n) \mid \mu \in \mathcal{M}\}$ *is a bounded (precompact) set of* G*.*

Then $\{\mu(A) \mid \mu \in \mathcal{M},\ A \in \mathfrak{R}\}$ *is a bounded (precompact) set of* G*.*

Let $(K_n)_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{K}$ and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. By (b) there exists an infinite subset M of $\mathbb{N}$ such that $\bigcup_{n\in M} K_n \in \mathfrak{R}$. Assume $\{\mu_n(K_n) \mid n \in M\}$ is not a bounded (precompact) set of G. Then there exists a 0-neighbourhood U in G such that

$$\{\mu_n(K_n) \mid n \in M\} \not\subset P + \rho(p)U$$

for any $p \in \mathbb{N}$, and for any finite subset P of G. Let V be a symmetric 0-neighbourhood in G such that $5V \subset U$. By (a) there exist an increasing sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ and an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of finite subsets of G such that

$$\{\mu(K) \mid \mu \in \mathcal{M},\ K \subset \bigcup_{m=1}^{n} K_m\} \subset P_n + \rho(p_n)V$$

for any $n \in \mathbb{N}$. Since the measures of $\mathcal{M}$ are $\mathfrak{K}$-regular, there exists an increasing sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$, such that

$$\{\mu_n(A) \mid A \in \mathfrak{R},\ A \subset \bigcup_{\substack{m \in M \\ m \geq k_n}} K_m\} \subset V$$

for any $n \in \mathbb{N}$.

We may construct inductively an increasing sequence $(l_n)_{n \in \mathbb{N}}$ in M, such that we have for any $n \in \mathbb{N} \setminus \{1\}$:

(1) $l_n \geq k_{l_{n-1}}$;

(2) $\mu_{l_n}(K_{l_n}) \notin \{\mu_{l_m}(K_{l_m}) \mid m \in \mathbb{N},\ m < n\} + P_{l_{n-1}} - P_{l_{n-1}} + \rho(n+2p_{l_{n-1}}+2)U$.

Then there exists an infinite subset M_0 of $\{l_n \mid n \in \mathbb{N}\}$, such that

$$A := \bigcup_{n \in M_0} K_n \in \mathfrak{R},$$

and such that $\{\mu(A) \mid \mu \in \mathcal{M}\}$ is a bounded (precompact) set of G. Let $m,n \in \mathbb{N}$ with $m < n$ and $l_m, l_n \in M_0$. Assume

$$\mu_{l_n}(A) - \mu_{l_m}(A) \in \rho(n)V.$$

Then by (1)

$$\begin{aligned}\mu_{l_n}(K_{l_n}) - \mu_{l_m}(K_{l_m}) &= (\mu_{l_n}(A) - \mu_{l_n}(\bigcup_{\substack{p \in M_0 \\ p < l_n}} K_p) - \mu_{l_n}(\bigcup_{\substack{p \in M_0 \\ p > l_n}} K_p)) \\ &\quad - (\mu_{l_m}(A) - \mu_{l_m}(\bigcup_{\substack{p \in M_0 \\ p < l_m}} K_p) - \mu_{l_m}(\bigcup_{\substack{p \in M_0 \\ p > l_m}} K_p)) \\ &\in \rho(n)V - P_{l_{n-1}} - \rho(p_{l_{n-1}})V - V + P_{l_{m-1}} + \rho(p_{l_{m-1}})V + V \\ &\subset P_{l_{m-1}} - P_{l_{n-1}} + \rho(n+2p_{l_{n-1}}+2)U,\end{aligned}$$

and this contradicts (2). Hence

$$\mu_{\iota_n}(A) \notin \{\mu_{\iota_m}(A) \mid m \in \mathbb{N},\ \ m < n,\ \ \iota_m \in M_0\} + \rho(n)V$$

for any $n \in \mathbb{N}$ such that $\iota_n \in M_0$. By Lemma 1.1 this relation is a contradiction. Hence

$$\{\mu_n(K_n) \mid n \in M\} \text{ is a bounded (precompact) set of } G.$$

Let U be a closed 0-neighbourhood in G. By the above proof and by Proposition 1.2 there exist $n \in \mathbb{N}$ and a finite subset P of G such that

$$\{\mu(K) \mid \mu \in \mathcal{M},\ \ K \in \mathfrak{K}\} \subset P + \rho(n)U.$$

Since U is closed and since every measure of $\mathcal{M}$ is $\mathfrak{K}$-regular we get

$$\{\mu(A) \mid \mu \in \mathcal{M},\ \ A \in \mathfrak{R}\} \subset P + \rho(n+1)U.$$

U being arbitrary, $\{\mu(A) \mid \mu \in \mathcal{M},\ \ A \in \mathfrak{R}\}$ is a bounded (precompact) set of G. □

Corollary 1.4. Let I be a set, let $\mathfrak{J}$ be a set of subsets of I, and let $\mathcal{F}$ be a set of G-valued functions on I such that:

(a) any finite subset of I belongs to $\mathfrak{J}$;

(b) for any disjoint sequence $(J_n)_{n \in \mathbb{N}}$ of finite subsets of I there exists an infinite subset M of $\mathbb{N}$, such that $\bigcup_{n \in M} J_n \in \mathfrak{J}$;

(c) $(f(\iota))_{\iota \in J}$ is summable for any $J \subset I$, and for any $f \in \mathcal{F}$;

(d) $\{\sum_{\iota \in J} f(\iota) \mid f \in \mathcal{F}\}$ is a bounded (precompact) set of G for any $J \in \mathfrak{J}$.

Then $\{\sum_{\iota \in J} f(\iota) \mid f \in \mathcal{F},\ \ J \subset I\}$ is a bounded (precompact) set of G. □

Lemma 1.5. There exists an uncountable set $\mathfrak{M}$ of infinite subsets of $\mathbb{N}$, such that $M' \cap M''$ is finite for any different M', M'' of $\mathfrak{M}$.

Let $\mathfrak{N}$ be the set of infinite subsets of $\mathbb{N}$, and let Ω be the set of infinite subsets $\mathfrak{M}$ of $\mathfrak{N}$ such that $M' \cap M''$ is finite for any different sets M', M'' of $\mathfrak{M}$. Ω being inductively ordered by the inclusion relation, it possesses a maximal element $\mathfrak{M}_0$ (Zorn). Assume $\mathfrak{M}_0$ countable. Then there exist a bijection $\phi : \mathbb{N} \to \mathfrak{M}_0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathfrak{N}$ such that

$$x_n \in \phi(n) \setminus \bigcup_{m=1}^{n-1} \phi(m)$$

for any $n \in \mathbb{N}$. The set $\{x_n | n \in \mathbb{N}\}$ belongs to $\mathfrak{N} \setminus \mathfrak{M}_0$ and has a finite intersection with any set of $\mathfrak{M}_0$, which contradicts the maximality of $\mathfrak{M}_0$. □

The aim of the next proposition is to give an example of a set $\mathfrak{J}$, fulfilling the conditions (a), (b) of Corollary 1.4.

Proposition 1.6. Let I be a set, let Φ be a set of free ultrafilters on I, such that $\{\mathfrak{F} \in \Phi | K \in \mathfrak{F}\}$ is countable for any countable subset K of I, and let $\mathfrak{J}$ be the set

$$\{K | K \text{ countable subset of } I\} \setminus \bigcup_{\mathfrak{F}\in\Phi} \mathfrak{F}.$$

Then:

(a) $\mathfrak{J}$ contains any finite subset of I;
(b) $K \subset J \in \mathfrak{J} \Rightarrow K \in \mathfrak{J}$;
(c) for any sequence $(J_n)_{n\in\mathbb{N}}$ in $\mathfrak{J}$ there exists an infinite subset M of $\mathbb{N}$ such that $\bigcup_{n\in M} J_n \in \mathfrak{J}$.

(a) and (b) are trivial. Let $(J_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{J}$. By (b) we may assume $(J_n)_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{J}$. By Lemma 1.5 there exists an uncountable set $\mathfrak{M}$ of infinite subsets of $\mathbb{N}$, such that $M' \cap M''$ is finite for any different $M', M'' \in \mathfrak{M}$. Then

$$(\{\mathfrak{F} \in \Phi | \bigcup_{n\in M} J_n \in \mathfrak{F}\})_{M\in\mathfrak{M}}$$

is an uncountable disjoint family of subsets of Φ whose union is countable. Hence there exists $M \in \mathfrak{M}$ for which $\{\mathfrak{F} \in \Phi | \bigcup_{n\in M} J_n \in \mathfrak{F}\}$ is empty and therefore $\bigcup_{n\in M} J_n \in \mathfrak{J}$. □

Theorem 1.7. Let $\mathfrak{R}$ *be a quasi-σ-ring and let* M *be a set of* G*-valued measures on* $\mathfrak{R}$*, such that* $\{\mu(A) | \mu \in M\}$ *is a bounded set of* G *(*$\{\mu(B) | \mu \in M, B \in \mathfrak{R}, B \subset A\}$ *is a precompact set of* G*) for any* $A \in \mathfrak{R}$*. Then* $\{\mu(A) | \mu \in M, A \in \mathfrak{R}\}$ *is a bounded (precompact) set of* G.

Let $(A_n)_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{R}$. We set

$$\mathfrak{S} := \{M \subset \mathbb{N} \mid \bigcup_{n\in M} A_n \in \mathfrak{R}\},$$

$$\phi : \mathfrak{S} \to \mathfrak{R}, \quad M \mapsto \bigcup_{n\in M} A_n.$$

$\mathfrak{S}$ is a quasi-σ-ring containing any finite subset of $\mathbb{N}$ and $\{\mu\circ\phi \mid \mu \in \mathcal{M}\}$ is a set of G-valued measures on $\mathfrak{S}$ such that $\{\mu\circ\phi(M) \mid \mu\in\mathcal{M}\}$ is a bounded (precompact) set of G for any $M \in \mathfrak{S}$. By Theorem 1.3

$$\{\mu\circ\phi(M) \mid \mu\in\mathcal{M},\quad M\in\mathfrak{S}\}$$

is a bounded (precompact) set of G. In particular $\{\mu_n(A_n) \mid n\in\mathbb{N}\}$ is a bounded (precompact) set of G for any sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$. By Proposition 1.2 $\{\mu(A) \mid \mu\in\mathcal{M},\quad A\in\mathfrak{R}\}$ is a bounded (precompact) set of G. □

Remark. The above result is a generalization of Nikodym's boundedness theorem ([5], page 418).

II. MEASURES ON TOPOLOGICAL SPACES

Proposition 2.1. Let μ be a $\mathfrak{K}$-regular, additive map of $\mathfrak{R}$ into G, let $A \in \mathfrak{R}$, and let $\mathfrak{T}$ be a subset of $\mathfrak{R}$ such that:

(a) $\{T\in\mathfrak{T} \mid A\subset T\}$ is lower directed;

(b) for any $K\in\mathfrak{K}$ with $A\cap K = \emptyset$, there exist disjoint sets $S,T\in\mathfrak{T}$ such that $K\subset S$, $A\subset T$.

Then for any 0-neighbourhood U in G there exist $T\in\mathfrak{T}$ such that $A\subset T$, and

$$\{\mu(B) \mid B\in\mathfrak{R},\quad B\subset T\setminus A\} \subset U.$$

Since $\emptyset\in\mathfrak{K}$, there exists by (b) a $T_0\in\mathfrak{T}$ such that $A\subset T_0$. μ being $\mathfrak{K}$-regular and additive, there exists $K\in\mathfrak{K}$ such that $K\subset T_0\setminus A$ and

$$\{\mu(B) \mid B\in\mathfrak{R},\quad B\subset (T_0\setminus A)\setminus K\} \subset U.$$

By (b) there exist disjoint sets $T',T''\in\mathfrak{T}$ such that $A\subset T'$, $K\subset T''$. By (a) there exists $T\in\mathfrak{T}$ with

$$A \subset T \subset T_0 \cap T'.$$

Let $B \in \mathfrak{R}$ such that $B \subset T \setminus A$. Then $B \subset (T_0 \setminus A) \setminus K$, and therefore $\mu(B) \in U$. $\square$

Proposition 2.2. Let M be a set of $\mathfrak{K}$-regular, additive maps of $\mathfrak{R}$ into G, let $\mathfrak{T}$ be a subset of $\mathfrak{R}$, and let ϕ be a map defined on $\mathfrak{T}$ such that:

(a) for any $K \in \mathfrak{K}$ and for any $T', T'' \in \mathfrak{T}$ with $K \subset T' \cap T''$ there exist $T \in \mathfrak{T}$ and $S \in \phi(T)$ such that

$$K \subset T \subset S \subset T' \cap T'';$$

(b) $\mathfrak{T}$ separates $\mathfrak{K}$;

(c) for any sequence $(\mu_n)_{n \in \mathbb{N}}$ in M and for any sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathfrak{T}$ whose union is contained in a set of $\mathfrak{T}$, and for which there exists a disjoint sequence $(S_n)_{n \in \mathbb{N}}$ of sets such that $T_n \subset S_n \in \phi(T_n)$ for any $n \in \mathbb{N}$, there exists an infinite subset M of $\mathbb{N}$ such that $\{\mu_n(T_n) \mid n \in M\}$ is a bounded (precompact) set of G.

Then for any $K \in \mathfrak{K}$ and for any 0-neighbourhood U in G there exist $T \in \mathfrak{T}$, $n \in \mathbb{N}$, and a finite subset P of G such that $K \subset T$ and

$$\{\mu(A) \mid (\mu, A) \in M \times \mathfrak{R},\ A \subset T \setminus K\} \subset P + \rho(n)U.$$

Assume the contrary. Then there exist $K \in \mathfrak{K}$ and a 0-neighbourhood U in G such that for any $T \in \mathfrak{T}$ with $K \subset T$, for any $n \in \mathbb{N}$, and for any finite subset P of G there exists $(\mu, A) \in M \times \mathfrak{R}$ such that $A \subset T \setminus K$ and

$$\mu(A) \notin P + \rho(n)U.$$

Let V be a 0-neighbourhood in G such that $3V \subset U$. Since $\emptyset \in \mathfrak{K}$ there exists by (b) a $T_0' \in \mathfrak{T}$ with $K \subset T_0'$. We construct inductively a sequence $(\mu_n)_{n \in \mathbb{N}}$ in M, two sequences $(T_n)_{n \in \mathbb{N}}$, $(T_n')_{n \in \mathbb{N}}$ in $\mathfrak{T}$, and a sequence $(S_n)_{n \in \mathbb{N}}$ such that we have for any $n \in \mathbb{N}$:

(1) $K \subset T_n' \subset T_{n-1}'$;

(2) $T_n \subset S_n \subset T_{n-1}' \setminus T_n'$;

(3) $S_n \in \phi(T_n)$;

(4) $\mu_n(T_n) \notin \{\mu_m(T_m) \mid m \in \mathbb{N},\ m < n\} + \rho(n)V.$

Let $n \in \mathbb{N}$ and assume the sequences were constructed up to $n-1$. By the hypothesis of the proof there exists $(\mu_n, A) \in M \times \mathfrak{R}$ such that $A \subset T'_{n-1} \setminus K$ and

$$\mu_n(A) \notin \{\mu_m(T_m) \mid m \in \mathbb{N}, \quad m < n\} + \rho(n+1)U.$$

Since μ_n is $\mathfrak{K}$-regular there exists $L \in \mathfrak{K}$ such that $L \subset A$ and

$$\mu_n(A) - \mu_n(L) \in V.$$

By (b) there exist disjoint sets $T', T'' \in \mathfrak{T}$ with $K \subset T'$, $L \subset T''$. By (a), (b), and Proposition 2.1 there exists $T \subset \mathfrak{T}$ such that $L \subset T$ and

$$\mu_n(L) - \mu_n(B) \in V.$$

for any $B \in \mathfrak{R}$ with $L \subset B \subset T$. By (a) and (b) there exist $T_n, T'_n \in \mathfrak{T}$ and $S_n \in \phi(T_n)$ such that

$$K \subset T'_n \subset T' \cap T'_{n-1}, \quad L \subset T_n \subset S_n \subset T \cap T'' \cap T'_{n-1}.$$

The above conditions (1), (2), (3) are obviously fulfilled. We have

$$\mu_n(A) = (\mu_n(A)-\mu_n(L)) + (\mu_n(L)-\mu_n(T_n))+\mu_n(T_n) \in \mu_n(T_n) + 2V$$

and this implies (4). This finishes the inductive construction. By (c) there exists an infinite subset M of $\mathbb{N}$ such that $\{\mu_n(T_n) \mid n \in M\}$ is a bounded (precompact)set of G and by Lemma 1.1 this contradicts (4). □

<u>Proposition 2.3.</u> Let M be a set of $\mathfrak{K}$-regular, additive maps of $\mathfrak{R}$ into G, let $\mathfrak{T}$ be a subset of $\mathfrak{R}$, and let ϕ be a map defined on $\mathfrak{T}$ such that:

(1) for any $K \in \mathfrak{K}$ and for any $T', T'' \in \mathfrak{T}$ with $K \subset T' \cap T''$ there exist $T \in \mathfrak{T}$ and $S \in \phi(T)$ such that

$$K \subset T \subset S \subset T' \cap T'';$$

(2) $\mathfrak{T}$ separates $\mathfrak{K}$;

(3) for any sequence $(\mu_n)_{n \in \mathbb{N}}$ in M for any continuous group homomorphism u of G into a metrizable topolo- ical additive group H, and for any sequence $(T_n)_{n \in \mathbb{N}}$

in $\mathfrak{T}$ whose union is contained in a set of $\mathfrak{T}$, and for which there exists a disjoint sequence $(S_n)_{n \in \mathbb{N}}$ of sets such that $T_n \subset S_n \in \phi(T_n)$ for any $n \in \mathbb{N}$, there exist an infinite subset M of $\mathbb{N}$ and a map $\psi : \mathfrak{P}(M) \to \mathfrak{T}$ such that $\psi(\{n\}) = T_n$ and such that $u \circ \mu_n \circ \psi$ is an H-valued measure on $\mathfrak{P}(M)$ for any $n \in \mathbb{N}$;

(4) $\{\mu(T) | \mu \in \mathcal{M}\}$ is a bounded (precompact)set of G for any $T \in \mathfrak{T}$.

Then:

(a) for any $K \in \mathfrak{K}$ and for any 0-neighbourhood U in G there exist $T \in \mathfrak{T}$, $n \in \mathbb{N}$, and a finite subset P of G such that $K \subset T$ and

$$\{\mu(A) | (\mu,A) \in \mathcal{M} \times \mathfrak{R},\ A \subset T \backslash K\} \subset P + \rho(n)U;$$

(b) $\{\mu(K) | \mu \in \mathcal{M}\}$ is a bounded (precompact) set of G for any $K \in \mathfrak{K}$.

Proof(a). Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}$, and let $(T_n)_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{T}$ whose union is contained in a set of $\mathfrak{T}$ and for which there exists a disjoint sequence $(S_n)_{n \in \mathbb{N}}$ of sets such that $T_n \subset S_n \in \phi(T_n)$ for any $n \in \mathbb{N}$. There exist a surjective continuous group homomorphism u of G into a metrizable topological additive group H, and a 0-neighbourhood V in H such that $2u^{-1}(V) \subset U$. By (3) there exist an infinite subset M of $\mathbb{N}$ and a map $\psi : \mathfrak{P}(M) \to \mathfrak{T}$ such that $\psi(\{n\}) = T_n$, and such that $u \circ \mu_n \circ \psi$ is an H-valued measure on $\mathfrak{P}(M)$ for any $n \in \mathbb{N}$. By (4) and Theorem 1.3

$$\{u \circ \mu_n(T_n) | n \in M\}$$

is a bounded (precompact) set of H. By (1), (2), and Proposition 2.2 there exist $T \in \mathfrak{T}$, $n \in \mathbb{N}$, and a finite subset Q of H such that $K \subset T$, $n \neq 1$, and

$$\{u \circ \mu(A) | (\mu,A) \in \mathcal{M} \times \mathfrak{R},\ A \subset T \backslash K\} \subset Q + \rho(n-1)V.$$

Since u is surjective there exists a finite subset P of G such that $u(P) = Q$. Let $(\mu,A) \in \mathcal{M} \times \mathfrak{R}$ with $A \subset T \setminus K$. Then there exist $x \in Q$ and a family $(x_m)_{1 \le m \le \rho(n-1)}$ in V, such that

$$u \circ \mu(A) = x + \sum_{m=1}^{\rho(n-1)} x_m .$$

Let $y \in P$ with $u(y) = x$. Since u is surjective there exists a family $(a_m)_{1 \leq m \leq \rho(n-1)}$ in $u^{-1}(V)$ such that $u(y_m) = x_m$ for any m. We get

$$u(\mu(A) - y - \sum_{m=1}^{\rho(n-1)} y_m) = 0,$$

and therefore

$$\mu(A) - y - \sum_{m=1}^{\rho(n-1)} y_m \in u^{-1}(V),$$

$$\mu(A) \in P + \rho(n)U.$$

Hence

$$\{\mu(A) \mid (\mu,A) \in \mathcal{M} \times \mathfrak{R},\ \ A \subset T \setminus K\} \subset P + \rho(n)U.$$

Proof(b). Let U be a 0-neighbourhood in G and let $K \in \mathfrak{K}$. Let further V be a 0-neighbourhood in G, such that $V - V \subset U$. By (a) there exist $T \in \mathfrak{T}$, $n \in \mathbb{N}$, and a finite subset P of G such that $K \subset T$ and

$$\{\mu(T \setminus K) \mid \mu \in \mathcal{M}\} \subset P + \rho(n)V.$$

By (4) there exist $n' \in \mathbb{N}$ and a finite subset P' of G such that

$$\{\mu(T) \mid \in \mathcal{M}\} \subset P' + \rho(n')V.$$

We get

$$\{\mu(K) \mid \mu \in \mathcal{M}\} \subset P' - P + \rho(n')V - \rho(n)V \subset P' - P + \rho(n+n')U.$$

Hence $\{\mu(K) \mid \mu \in \mathcal{M}\}$ is a bounded (precompact) set of G. □

Proposition 2.4. Let $\mathcal{M}$ be a set of $\mathfrak{K}$-regular, additive maps of $\mathfrak{R}$ into G, let $\mathfrak{T}$ be a subset of $\mathfrak{R}$, and let ϕ be a map defined on $\mathfrak{T}$ such that

(a) for any $K \in \mathfrak{K}$ and for any T', T'' with $K \subset T' \cap T''$ there exist $T \in \mathfrak{T}$ and $S \in \phi(T)$ such that

$$K \subset T \subset S \subset T' \cap T'';$$

(b) $\mathfrak{T}$ separates $\mathfrak{K}$;
(c) for any $(K,T) \in \mathfrak{K} \times \mathfrak{T}$ for which there exists $S \in \phi(T)$ with $T \subset S$ and $K \cap S = \emptyset$, there exist $T' \in \mathfrak{T}$ with $K \subset T'$ and $T \cap T' = \emptyset$;
(d) any set of $\mathfrak{R}$ is contained in a set of $\mathfrak{T}$;
(e) for any sequence $(\mu_n)_{n \in \mathbb{N}}$ in M, for any continuous group homomorphism u of G into a metrizable topological additive group H, and for any sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathfrak{T}$ whose union is contained in a set of $\mathfrak{T}$, and for which there exists a disjoint sequence $(S_n)_{n \in \mathbb{N}}$ of sets such that $T_n \subset S_n \in \phi(T_n)$ for any $n \in \mathbb{N}$, there exist an infinite subset M of $\mathbb{N}$ and a map $\psi : \mathfrak{P}(M) \to \mathfrak{T}$ such that $\psi(\{n\}) = T_n$ and such that $u \circ \mu_n \circ \psi$ is an H-valued measure on $\mathfrak{P}(M)$ for any $n \in \mathbb{N}$;
(f) $\{\mu(T) | \mu \in M\}$ is a bounded (precompact) set of G for any $T \in \mathfrak{T}$.

Then $\{\mu(A) | \mu \in M\}$ is a bounded (precompact) set of G for any $A \in \mathfrak{R}$.

Assume the contrary. Then there exist A and a 0-neighbourhood U in G such that for any $n \in \mathbb{N}$ and for any finite subset P of G there exists $\mu \in M$ with

$$\mu(A) \notin P + \rho(n)U.$$

By (d) there exists $T \in \mathfrak{T}$ containing A. Let V be a 0-neighbourhood in G such that $3V \subset U$. We construct inductively a sequence $(\mu_n)_{n \in \mathbb{N}}$ in M, three sequences $(T_n)_{n \in \mathbb{N}}$, $(T'_n)_{n \in \mathbb{N}}$, $(T''_n)_{n \in \mathbb{N}}$ in $\mathfrak{T}$, and two sequences $(S_n)_{n \in \mathbb{N}}$, $(S'_n)_{n \in \mathbb{N}}$ such that we have for any $n \in \mathbb{N}$:

(1) $T_n \subset S_n \in \phi(T_n)$;
(2) $S_n \subset T'_n \subset S'_n \in \phi(T'_n)$;
(3) $S'_n \subset T''_n \subset T \setminus \bigcup_{m=1}^{n-1} S_m$;
(4) $\mu_n(T_n) \notin \{\mu_m(T_m) | m \in \mathbb{N},\ m < n\} + \rho(n)V$;
(5) for any finite subset P of G and for any $p \in \mathbb{N}$ there exists $\mu \in M$ such that

$$\mu(A \setminus \bigcup_{m=1}^{n} T''_m) \notin P + \rho(p)U.$$

Let $n \in \mathbb{N}$ and assume the sequences were constructed up to $n - 1$. By (5) there exists $\mu_n \in \mathcal{M}$ such that

$$\mu_n(A \setminus \bigcup_{m=1}^{n-1} T''_m) \notin \{\mu_m(T_m) \mid m \in \mathbb{N},\ m < n\} + \rho(n+1)U.$$

Since μ_n is $\mathfrak{K}$-regular, there exists $K \in \mathfrak{K}$ such that $K \subset A \setminus \bigcup_{m=1}^{n-1} T''_m$ and

$$\mu_n(A \setminus \bigcup_{m=1}^{n-1} T''_n) - \mu_n(K) \in V.$$

By (a), (b), (e), (f) and Proposition 2.3 there exist $T' \in \mathfrak{T}$, $p \in \mathbb{N}$, and a finite subset P of G such that $K \subset T'$ and

$$\{\mu(B) \mid (\mu,B) \in \mathcal{M} \times \mathfrak{R},\ B \subset T' \setminus K\} \subset P + \rho(p)V,$$

$$\{\mu(K) \mid \mu \in \mathcal{M}\} \subset P + \rho(p)V.$$

By (a), (b) and Proposition 2.1 there exists $T'' \in \mathfrak{T}$ such that $K \subset T''$ and

$$\mu_n(K) - \mu_n(B) \in V$$

for any $B \in \mathfrak{R}$ with $K \subset B \subset T''$. By (c), (2), and (3) there exists a family $(T'''_m)_{1 \le m \le n-1}$ in $\mathfrak{T}$ such that

$$K \subset T'''_m, \quad T'_m \cap T'''_m = \emptyset$$

for any $m \in \mathbb{N}$, $m < n$. By (a) there exist $T_n, T'_n, T''_n \in \mathfrak{T}$, $S_n \in \phi(T_n)$, and $S'_n \in \phi(T'_n)$ such that

$$K \subset T_n \subset S_n \subset T'_n \subset S'_n \subset T''_n \subset T \cap T' \cap T'' \cap (\bigcap_{m=1}^{n-1} T'''_m).$$

Hence (1), (2), and (3) are fulfilled. Assume

$$\mu_n(T_n) \in \{\mu_m(T_m) \mid m \in \mathbb{N},\ m < n\} + \rho(n)V.$$

Then

$$\mu_n(A\setminus \bigcup_{m=1}^{n-1} T''_m) = (\mu_n(A\setminus \bigcup_{m=1}^{n-1} T''_m)-\mu_n(K)) + (\mu_n(K)-\mu_n(T_n)) + \mu_n(T_n)$$

$$\in \{\mu_m(T_m) \mid m \in \mathbb{N},\ \ m < n\} + \rho(n)V + V + V$$

$$\subset \{\mu_m(T_m) \mid m \in \mathbb{N},\ \ m < n\} + \rho(n+1)U$$

which is a contradiction. Hence (4) holds too. We have

$$\mu(A\setminus \bigcup_{m=1}^{n-1} T''_m) = \mu(A\setminus \bigcup_{m=1}^{n} T''_m) + \mu(K) + \mu((A\setminus \bigcup_{m=1}^{n-1} T''_m)\cap(T''_n\setminus K))$$

$$\in \mu(A\setminus \bigcup_{m=1}^{n} T''_m) + 2P + 2\rho(p)V$$

for any $\mu \in \mathcal{M}$. By (5) if $n \neq 1$ and by the hypothesis of the proof if $n = 1$ for any finite subset Q of G, and for any $m \in \mathbb{N}$, there exists $\mu \in \mathcal{M}$ such that

$$\mu(A\setminus \bigcup_{m=1}^{n} T''_m) \notin Q + \rho(m)U.$$

Hence (5) holds too. This finishes the inductive construction.

There exist a continuous surjective group homomorphism u of G into a metrizable topological additive group H, and a 0-neighbourhood W in H such that $2u^{-1}(W) \subset V$. By (e), (1), (2), and (3) there exist an infinite subset M of $\mathbb{N}$ and a map $\psi : \mathfrak{P}(M) \to \mathfrak{T}$ such that $\psi(\{n\}) = T_n$, and such that $u \circ \mu_n \circ \psi$ is an H-valued measure on $\mathfrak{P}(M)$ for any $n \in \mathbb{N}$. Since $\{u \circ \mu(T) \mid \mu \in \mathcal{M}\}$ is a bounded (precompact) set of H, for any $T \in \mathfrak{T}$, we deduce by Theorem 1.3 that

$$\{u \circ \mu_n(T_n) \mid n \in M\}$$

is a bounded (precompact) set of H too. By Lemma 1.1 there exists $n \in \mathbb{N}$ such that

$$u \circ \mu_n(T_n) \in \{u \circ \mu_m(T_m) \mid m \in \mathbb{N},\ \ m < n\} + \rho(n)W.$$

Hence there exist $m \in \mathbb{N}$, with $m < n$, and a family $(x_i)_{1\le i\le \rho(n)}$ in W such that

$$u \circ \mu_n(T_n) = u \circ \mu_m(T_m) + \sum_{i=1}^{\rho(n)} x_i .$$

Since u is surjective there exists a family $(y_i)_{1 \le i \le \rho(n)}$ in $u^{-1}(W)$ such that $u(y_i) = x_i$ for any $i \in \mathbb{N}$, with $i \le \rho(n)$. We get

$$u(\mu_n(T_n) - \mu_m(T_m) - \sum_{i=1}^{\rho(n)} y_i) = 0,$$

and therefore

$$\mu_n(T_n) - \mu_m(T_m) - \sum_{i=1}^{\rho(n)} y_i \in u^{-1}(W).$$

Hence

$$\mu_n(T_n) \in \{\mu_m(T_m) \mid m \in \mathbb{N},\ m < n\} + (\rho(n)+1)u^{-1}(W)$$

$$\subset \{\mu_m(T_m) \mid m \in \mathbb{N},\ m < n\} + \rho(n)V,$$

which contradicts (4). □

Lemma 2.5. Let $(x_\iota)_{\iota \in I}$ be a family in G such that any infinite subset J of I contains an infinite subset K such that $(x_\iota)_{\iota \in K}$ is summable. If $\{0\}$ is a G_δ-set of G, then $\{\iota \in I \mid x_\iota \neq 0\}$ is countable.

Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of open sets of G with

$$\bigcap_{n \in \mathbb{N}} U_n = \{0\}.$$

Then for any $n \in \mathbb{N}$ the set $\{\iota \in I \mid x_\iota \notin U_n\}$ is finite and therefore

$$\{\iota \in I \mid x_\iota \neq 0\} = \bigcup_{n \in \mathbb{N}} \{\iota \in I \mid x_\iota \notin U_n\}$$

is countable. □

Proposition 2.6. Let X be a topological space, let $\mathfrak{R}$ be a ring of subsets of X, let $(\mu_\iota)_{\iota\in I}$ be a countable family of G-valued measures on $\mathfrak{R}$, and let $(T_n)_{n\in\mathbb{N}}$ be a disjoint sequence of open sets of X. We set

$$A(M) := \overset{\circ}{\overline{\bigcup_{n\in M} T_n}} \setminus \bigcup_{n\in M} T_n$$

for any $M \subset \mathbb{N}$, and assume:

(a) $\bigcup_{n\in M} T_n$ is regular for any finite subset M of N;

(b) $A(M) \in \mathfrak{R}$ for any $M \subset \mathbb{N}$;

(c) for any sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathfrak{P}(\mathbb{N})$, such that $(A(M_n))_{n\in\mathbb{N}}$ is a disjoint sequence, there exists an infinite subset J of $\mathbb{N}$ with $\bigcup_{n\in J} A(M_n) \in \mathfrak{R}$;

(d) $\{0\}$ is a G_δ-set of G.

Then there exists an infinite subset M_0 of $\mathbb{N}$, such that $\mu_\iota(A(M)) = 0$ for any $M \subset M_0$ and for any $\iota \in I$.

G^I endowed with the product topology is a Hausdorff topological additive group for which $\{0\}$ is a G_δ-set and

$$\mu : \mathfrak{R} \to G^I, \quad B \mapsto (\mu_\iota(B))_{\iota\in I}$$

is an G^I-valued measure on $\mathfrak{R}$ such that we have

$$\mu(B) = 0 \Leftrightarrow (\text{for all } \iota \in I \Rightarrow \mu_\iota(B) = 0)$$

for any $B \in \mathfrak{R}$. Hence it is sufficient to prove the above proposition for a G-valued measure μ only.

By Lemma 1.5 there exists an uncountable set $\mathfrak{M}$ of infinite subsets of $\mathbb{N}$, such that $M' \cap M''$ is finite for any different M', M'' of $\mathfrak{M}$. By (a)

$$A(M') \cap A(M'') = \emptyset$$

for any different $M', M'' \in \mathfrak{M}$. Assume for any $M \in \mathfrak{M}$ there exists $M' \subset M$ with

$$\mu(A(M')) \neq 0.$$

Then $(A(M'))_{M\in\mathfrak{M}}$ is an uncountable disjoint family in $\mathfrak{R}$ and by (c) for any sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathfrak{M}$ there exists an infinite subset J of $\mathbb{N}$ such that

$$\bigcup_{n\in J} A(M'_n) \in \mathfrak{R}.$$

By (d) and Lemma 2.5 the set

$$\{M \in \mathfrak{M} \mid \mu(A(M')) \neq 0\}$$

is countable and this is a contradiction. Hence there exists $M_0 \in \mathfrak{M}$ with $\mu(A(M)) = 0$ for any $M \subset M_0$. $\square$

Theorem 2.7. Let X be a topological space, let $\mathfrak{R}$ be a quasi-δ-ring of subsets of X, let $\mathfrak{T}$ be a set of regular open sets of X belonging to $\mathfrak{R}$, and let M be a set of $\mathfrak{K}$-regular G-valued measures on $\mathfrak{R}$ such that:

(1) for any $K \in \mathfrak{K}$ and for any $T', T'' \in \mathfrak{T}$ with $K \subset T' \cap T''$ there exists $T \in \mathfrak{T}$ with

$$K \subset T \subset \overline{T} \subset T' \cap T'';$$

(2) $\mathfrak{T}$ separates $\mathfrak{K}$;
(3) for any $(K,T) \in \mathfrak{K} \times \mathfrak{T}$ with $\overline{K} \cap \overline{T} = \emptyset$, there exists $T' \in \mathfrak{T}$ with $K \subset T'$ and $T \cap T' = \emptyset$;
(4) any set of $\mathfrak{R}$ is contained in a set of $\mathfrak{T}$;
(5) any open set of X which is contained in a set of $\mathfrak{T}$ belongs to $\mathfrak{R}$;
(6) $\{\mu(T) \mid \mu \in M\}$ is a bounded (precompact) set of G for any $T \in \mathfrak{T}$.

Then $\{\mu(A) \mid \mu \in M\}$ is a bounded (precompact) set of G for any $A \in \mathfrak{R}$. If in addition $\mathfrak{R}$ is a quasi-σ-ring then $\{\mu(A) \mid \mu \in M, A \in \mathfrak{R}\}$ is a bounded set of G.

We denote by ϕ the map

$$\mathfrak{T} \to \mathfrak{P}(\mathfrak{P}(X)), \quad T \mapsto \{S \mid \overline{T} \subset S, \ S \text{ open set of } X\}.$$

The conditions (a), (b), (c), (d) and (f) of Proposition 2.4 are obviously fulfilled. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in M, let u be a continuous group homomorphism of G into a metrizable topological additive group H, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{T}$ whose union is contained in a set of $\mathfrak{T}$, and for which there exists a disjoint sequence $(S_n)_{n\in\mathbb{N}}$ of sets such that $T_n \subset S_n \in \phi(T_n)$, for any $n \in \mathbb{N}$. By (5)

$$\overline{\bigcup_{m\in M} T_m}^{\circ} \setminus \bigcup_{m\in M} T_m \in \mathfrak{R}$$

for any $M \subset \mathbb{N}$ and therefore by Proposition 2.6 there exists an infinite subset M_0 of $\mathbb{N}$ such that

$$u \circ \mu_n(\overline{\bigcup_{m\in M} T_m}^{\circ} \setminus \bigcup_{m\in M} T_m) = 0$$

for any $n \in \mathbb{N}$, and for any $M \subset M_0$. We denote by ψ and ψ' the maps

$$\mathfrak{P}(M_0) \to \mathfrak{T}, \quad M \mapsto \overline{\bigcup_{m\in M} T_m}^{\circ}$$

and

$$\mathfrak{P}(M_0) \to \mathfrak{R}, \quad M \mapsto \bigcup_{m\in M} T_m$$

respectively. By the above relation $u \circ \mu_n \circ \psi = u \circ \mu_n \circ \psi'$, and therefore $u \circ \mu_n \circ \psi$ is an H-valued measure for any $N \in \mathbb{N}$. Hence condition (e) of Proposition 2.4 is fulfilled too and the assertion follows from this proposition. □

Corollary 2.8. Let X be a regular space, let $\mathfrak{K}$ be the set of compact sets of X, let $\mathfrak{T}$ be the set of regular open sets of X, let $\mathfrak{R}$ be a quasi-σ-ring of subsets of X containing any open set of X, and let $\mathcal{M}$ be a set of $\mathfrak{K}$-regular G-valued measures on $\mathfrak{R}$ such that $\{\mu(T) | \mu \in \mathcal{M}\}$ is a bounded (precompact) set of G for any $T \in \mathfrak{T}$. Then $\{\mu(A) | (\mu,A) \in \mathcal{M} \times \mathfrak{R}\}$ is a bounded set of G ($\{\mu(A) | \mu \in \mathcal{M}\}$ is a precompact set of G for any $A \in \mathfrak{R}$). □

Remark. This result was proved by P. Gänssler ([3], Theorem 3.1) for $G = \mathbb{R}$ and $\mathfrak{R}$ a σ-ring.

Corollary 2.9. Let X be a normal space, let $\mathfrak{K}$ be the set of closed sets of X, let $\mathfrak{T}$ be the set of regular open sets of X, let $\mathfrak{R}$ be a quasi-σ-ring of subsets of X containing any open set of X, and let $\mathcal{M}$ be a set of $\mathfrak{K}$-regular G-valued measures on $\mathfrak{R}$ such that $\{\mu(T) | \mu \in \mathcal{M}\}$ is a bounded (precompact) set of G for any $T \in \mathfrak{T}$. Then

$$\{\mu(A) | (\mu,A) \in \mathcal{M} \times \mathfrak{R}\}$$

is a bounded set of G ($\{\mu(A) | \mu \in \mathcal{M}\}$ is a precompact set of G for any $A \in \mathfrak{R}$). □

Corollary 2.10. Let X be a locally compact paracompact space, let $\mathfrak{R}$ be a σ-ring of relatively σ-compact sets of X containing any compact set of X, let $\mathfrak{K}$ be the set of closed sets of X belonging to $\mathfrak{R}$, let $\mathfrak{T}$ be the set of regular open sets of X belonging to $\mathfrak{R}$, and let $\mathcal{M}$ be a set of $\mathfrak{K}$-regular G-valued measures on $\mathfrak{R}$ such that $\{\mu(T) | \mu \in \mathcal{M}\}$ is a bounded (precompact) set of G for any $T \in \mathfrak{T}$. Then $\{\mu(A) | (\mu, A) \in \mathcal{M} \times \mathfrak{R}\}$ is a bounded set of G ($\{\mu(A) | \mu \in \mathcal{M}\}$ is a precompact set of G for any $A \in \mathfrak{R}$). □

REFERENCES

[1] Dieudonné, J. (1951). "Sur la convergence des suites de mesures de Radon." *Ann. Acad. Brasil. Ciencias 23,* 21-38.

[2] Drewnowski, L. (1973). "Uniform boundedness principle for finitely additive vector measures." *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astr. Phys. 21,* 115-118.

[3] Gänssler, P. (1971). "A convergence theorem for measures in regular Hausdorff spaces." *Math. Scand. 29,* 237-244.

[4] Katz, M. P. (1972). "On extention of measures" (Russian). *Sibirskii Math. J. 13,* 1158-1168.

[5] Nikodym, O. (1933). "Sur les familles bornées de fonctions parfaitement additives d'ensemble abstrait." *Monatshefte für Math. 40,* 418-426.