MONOTONIC PROPERTIES OF SOME SPECTRAL RESOLVENTS

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The spectral resolvent concept for a closed linear operator T on a complex Banach space X was introduced in [2] and further developed in [3] and [7].

Since a spectral resolvent E varies between the two extreme values $E(\emptyset) = \{0\}$ and $E(G) = X$ for G (open) $\supset \sigma(T)$, the question arises whether such variation obeys any monotonic rule. It would certainly be useful to know that by expanding an open set G we can increase the corresponding subspace $E(G)$. It was proved in [7] that if a bounded linear operator T has a spectral resolvent E then T has a maximal spectral resolvent E_m, in the sense that for any open set $G \subset \mathcal{C}$ and all spectral resolvents E of T,

$$E(G) \subset E_m(G) = X_T(\overline{G}).$$

This maximal spectral resolvent X_T, originally defined under the name of spectral maximal space [4], is monotonic in the broader sense that for any pair of closed sets F_1, F_2 with $F_1 \subset F_2$, we have $X_T(F_1) \subset X_T(F_2)$. We have no means of deriving a general monotonic property such as $G_1 \subset G_2 \Rightarrow E(G_1) \subset E(G_2)$, which may not be valid for any type of open sets G_1, G_2 and every spectral resolvent E.

In this paper we shall examine two cases in which monotonic properties do hold for spectral resolvents. Two types of open sets will be considered and when necessary, restrictions on the ranges of E will have to be imposed.

Given a Banach space X over the field $\mathcal{C}$ of complex numbers, we denote the Banach algebra of bounded linear operators on X by $B(X)$. For a set $S \subset \mathcal{C}$, we denote by $\overline{S}$ the closure, S^c the complement and $\mathrm{cov}(S)$ the family of all finite open covers of S. $\mathcal{G}$ stands for the family of all open subsets of $\mathcal{C}$. For $T \in B(X)$, we use the notations $\sigma(T)$, $\rho(T)$ and $R(\cdot;T)$ for the spectrum, the resolvent set and the resolvent operator, respectively. We denote by $\rho_\infty(T)$ the unbounded component of $\rho(T)$. For $x \in X$, $\sigma_T(x)$ is the local spectrum, $\rho_T(x)$ is the local resolvent set and $\tilde{x}$ denotes the local resolvent operator, characterized by the property $(\lambda - T)\tilde{x}(\lambda) = x$ for all $\lambda \in \rho_T(x)$.

The existence of $\sigma_T(x)$, $\rho_T(x)$ and $\tilde{x}$ is subjected to the single valued extension property (abbreviated SVEP) of the given T. We denote by $\mathrm{Inv}(T)$ the family of invariant subspaces under T and we write $T|Y$ for the restriction of T to $Y \in \mathrm{Inv}(T)$.

Let $T \in B(X)$ be given throughout this paper.

Definition 1 [2,3]. A spectral decomposition of X by T is a finite system $\{(G_i, Y_i)\} \subset \mathcal{G} \times \mathrm{Inv}(T)$ with the following properties:

(i) $\{G_i\} \in \mathrm{cov}[\sigma(T)]$;

(ii) $X = \sum_i Y_i$;

(iii) $\sigma(T|Y_i) \subset \overline{G}_i$ for all i.

Definition 2 [2,3]. A mapping $E : \mathcal{G} \to \mathrm{Inv}(T)$ is called a spectral resolvent of T if it verifies the following conditions:

(I) $E(\emptyset) = \{0\}$;

(II) for every $\{G_i\} \subset \mathrm{cov}[\sigma(T)]$, $\{(G_i, E(G_i))\}$ is a spectral decomposition of X by T.

The proof of the SVEP for operators which have a spectral resolvent is given in [3]. For T having the SVEP and $S \subset \mathbb{C}$, we denote the linear manifold in X

$$X_T(S) = \{x \in X : \sigma_T(x) \subset S\}.$$

Definition 3 [8]. $Y \in \mathrm{Inv}(T)$ is said to be a T-absorbent space if for every $y \in Y$ and all $\lambda \in \sigma(T|Y)$, the equation

$$(\lambda - T)x = y$$

has all solutions (if any) x in Y.

Some preliminary properties of the T-absorbent spaces will be needed in the subsequent theory. Bartle and Kariotis [1] called an invariant subspace Y a ν-space if

$$\sigma(T|Y) \subset \sigma(T).$$

The following property is well-known.

Proposition 4 [6]. Given T, for every $Y \in \mathrm{Inv}(T)$ the following assertions are equivalent:

$$\sigma(T|Y) \subset \sigma(T);$$

$$R(\lambda;T)Y \subset Y \quad \text{for all} \quad \lambda \in \rho(T).$$

It is easy to see that every T-absorbent space Y is a ν-space for T. For if $\sigma(T|Y) \not\subset \sigma(T)$ then there is a $y \in Y$ such that $R(\lambda;T)y \notin Y$ for some $\lambda \in \rho(T) \cap \sigma(T|Y)$. But then the equation

$$(\lambda - T)x = y$$

has a solution

$$x = R(\lambda;T)y \notin Y$$

which contradicts the definition of Y.

Thus, by a result of Jafarian [5], if $\{(G_i, Y_i)\}$ is a spectral decomposition of X by T such that the Y_i's are T-absorbent spaces (or more generally ν-spaces), then

$$\sigma(T) = \bigcup_i \sigma(T|Y_i) \tag{1}$$

Lemma 5. Let $\{(G_i, Y_i)\}_{i=1,2}$ be a spectral decomposition of X by T in terms of T-absorbent spaces Y_1, Y_2. Then

$$\sigma(T|Y_1 \cap Y_2) = \sigma(T|Y_1) \cap \sigma(T|Y_2).$$

Proof. Let $y \in Y_1 \cap Y_2 = Y$ be arbitrary. Since Y_1 and Y_2 are ν-spaces, Proposition 4 implies

$$R(\lambda;T)y \in Y \quad \text{for all} \quad \lambda \in \rho(T). \tag{2}$$

For $\lambda \in \rho(T|Y_1) \cap \rho(T|Y_2)$, (1) implies that $\lambda \in \rho(T)$ and in view of (2), we have

$$R(\lambda;T|Y_1)y = [R(\lambda;T)|Y_1]y = R(\lambda;T)y \in Y. \tag{3}$$

For $\lambda \in \rho(T|Y_1) \cap \sigma(T|Y_2)$, since Y_2 is T-absorbent,

$$(\lambda-T)R(\lambda;T|Y_1)y = y$$

implies that

$$R(\lambda;T|Y_1)y \in Y_2.$$

On the other hand, $R(\lambda;T|Y_1)y \in Y_1$ and hence

$$R(\lambda;T|Y_1)y \in Y. \tag{4}$$

Thus, by (3) and (4),

$$R(\lambda;T|Y_1)Y \subset Y \quad \text{for all} \quad \lambda \in \rho(T|Y_1)$$

and hence Proposition 4 applied to $Y \in \mathrm{Inv}(T|Y_1)$, gives

$$\sigma(T|Y) \subset \sigma(T|Y_1).$$

By symmetry, $\sigma(T|Y) \subset \sigma(T|Y_2)$ and the proof is concluded. □

The technical difficulty in treating the monotony problem in the general case, lies in the wide variety of shapes and topological structures of open sets. The two cases under consideration here, open disks and complements of compact disks can be extended immediately by some continuous deformations to more general types of open sets.

Theorem 6. Let T *have a spectral resolvent* E *and let* $G = K^c$, *where* K *is any compact disk in* $\mathbb{C}$. *Then* $G_1 \in \mathcal{G}$ *and* $\overline{G} \subset G_1$ *imply* $E(G) \subset E(G_1)$.

Proof. To avoid repetitions, we divide the proof in three parts.

Part A. There is an open set G_2 such that

$$\overline{G} \cap \overline{G}_2 = \emptyset \quad \text{and} \quad \sigma(T) \subset G_1 \cup G_2. \tag{5}$$

Then $\{(G_i,E(G_i))\}_{i=1,2}$ is a spectral decomposition of X by T. Let $x \in E(G)$ be arbitrary and let

$$x = y_1 + y_2 \quad \text{with} \quad y_i \in E(G_i), \quad i = 1,2$$

be a representation of x. We have

$$\sigma_T(x) \subset \sigma[T|E(G)] \subset \overline{G} \quad \text{and} \quad \sigma_T(y_i) \subset \sigma[T|E(G_i)] \subset \overline{G}_i, \quad i = 1,2.$$

Also,

$$\sigma_T(y_1) = \sigma_T(x-y_2) \subset \sigma_T(x) \cup \sigma_T(y_2) \subset \overline{G} \cup \overline{G}_2$$

and hence

$$\sigma_T(y_1) \subset \overline{G}_1 \cap (\overline{G} \cup \overline{G}_2) = \overline{G} \cup (\overline{G}_1 \cap \overline{G}_2).$$

Thus, $\sigma_T(y_1)$ is the disjoint union of the spectral sets

$$\sigma_1 = \sigma_T(y_1) \cap \overline{G}, \quad \sigma_2 = \sigma_T(y_1) \cap (\overline{G}_1 \cap \overline{G}_2).$$

Part B. Since $\sigma_T(x)$ and σ_1 are compact sets contained in $\overline{G}$, by (5) there is a connected open set V such that

$$V \cap \overline{G}_2 = \emptyset \quad \text{and} \quad \sigma_T(x) \cup \sigma_1 \subset V.$$

The set

$$W = V \cap [\sigma_T(x) \cup \sigma_1]^c \subset \rho_T(x) \cap \rho_T(y_1) \cap \rho_T(y_2)$$

and hence the following equation is defined on W:

$$\tilde{x}(\lambda) = \tilde{y}_1(\lambda) + \tilde{y}_2(\lambda).$$

Let Γ be a closed finite system of positively oriented reactifiable Jordan arcs surrounding $\sigma_T(x) \cup \sigma_1$ and contained in W. We have

$$2\pi i x = \int_C R(\lambda;T)x\,d\lambda = \int_\Gamma \tilde{x}(\lambda)\,d\lambda = \int_\Gamma \tilde{y}_1(\lambda)\,d\lambda + \int_\Gamma \tilde{y}_2(\lambda)\,d\lambda = \int_\Gamma \tilde{y}_1(\lambda)\,d\lambda$$

where $C = \{\lambda : |\lambda| = \|T\| + 1\}$.

There are functions $f_i : W \to E(G_i)$, $i = 1,2$ such that

$$\tilde{y}_1(\lambda) = f_1(\lambda) + f_2(\lambda) \quad \text{on} \quad W.$$

Then, a function $g : W \to E(G_1) \cap E(G_2)$ is defined by

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