INVOLUTIVITY OF CONSERVATION LAWS FOR A FLUID OF FINITE DEPTH AND BENJAMIN-ONO EQUATIONS

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We prove that all conservation laws of the Finite Depth (FD) and Benjamin-Ono (B-O) equations commute. Furthermore we prove that the system of integrals of the FD equation is complete and that the system of integrals of KdV equation is also complete (the latter was not known).

I. INTRODUCTION

The equation describing the propagation of long waves in a stratified fluid layer of finite depth is written in a dimensionless form as [1], [2]

$$u_t = 2uu_x + T(u_{xx}), \tag{1.1}$$

where the nonlocal operator T is defined by

$$(Tf)(x) = -\frac{1}{2\lambda^2} P\int_{-\infty}^{\infty} [\coth \frac{\pi(x-\xi)}{2\lambda} - \operatorname{sgn}(x-\xi)]f(\xi)d\xi \tag{1.2}$$

and the physical variables in (1.1) have been rescaled for convenience.

Equation (1.1) has a number of remarkable properties: (1) If $\lambda \to 0$ and $\partial \equiv d/dx$, we consider T as a formal power series in λ, then we have [1], [6]

$$T = \lambda^{-1} \frac{e^{\lambda\partial}+e^{-\lambda\partial}}{e^{\lambda\partial}-e^{-\lambda\partial}} - \lambda^{-2}\partial^{-1} = \frac{\partial}{3} + O(\lambda^2). \tag{1.3}$$

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Thus (1.1) tends to the KdV equation when $\lambda \to 0$, (2) Equation (1.1) has an infinite number of integrals [1]; (3) If we rescale u in (1.1) and make λ go to ∞ then (1.1) tends to

$$v_t = 2vv_x + H(v_{xx}), \quad v(x,t) = \lambda u(x,\lambda t), \tag{1.4}$$

where H is the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\, d\xi. \tag{1.5}$$

Equation (1.4) is the B-O equation. This equation also has an infinite number of integrals which come from the limit of FD-integrals as $\lambda \to \infty$. In what follows we shall look at the FD-equation as a Hamiltonian system and all the statements (except completeness) can automatically be carried over to the B-O-case which we leave here for good, (4) Equation (1.1) is Hamiltonian [7], i.e. can be rewritten as

$$u_t = \frac{1}{6}\partial \frac{\delta}{\delta u} H_3, \quad H_3 = 2u^3 + 3uT(u_x). \tag{1.6}$$

Therefore one can ask whether the Poisson brackets

$$\{H_n, H_m\} = \frac{\delta H_m}{\delta u} \partial \frac{\delta H_n}{\delta u}, \tag{1.7}$$

are trivial (i.e. do they belong to the image of the operator ∂) or not. In the latter case we would have new integrals, $\{H_n, H_m\}$, because the Poisson bracket of integrals is again an integral; — this follows from the Hamiltonian property. It is the purpose of this paper to prove that all the integrals H_n of equation (1.1) commute, and that they form a complete family, i.e. there are no other independent integrals of equation (1.1) which are regular in λ. For various other topics concerning the solutions of equation (1.1) and related equations the reader can consult [1], [3]-[5].

II. INTEGRALS AND THEIR PROPERTIES

If one considers equation (1.1) as regular deformation with parameter λ of the KdV equation

$$u_t = 2uu_x + \frac{1}{3} u_{xxx}, \tag{2.1}$$

then it is an ideologically important problem to find how facts and constructions concerning the KdV equation (2.1) carry over to the "deformed" FD-equation (1.1). An initial step in this direction was made in [6]; from that paper we need the following fact: if

$$q_t = \frac{q_x}{2\lambda^2}[2\lambda\varepsilon q - (1-\frac{\lambda}{\varepsilon})(e^{2\lambda\varepsilon q}-1)] + T(q_{xx}) + \varepsilon\lambda q_x T(q_x) \quad (2.2)$$

then

$$u = \frac{1}{2\lambda^2}[2\lambda\varepsilon q - (1-\frac{\lambda}{\varepsilon})(e^{2\lambda\varepsilon q}-1)] - \varepsilon q_x + \varepsilon\lambda T(q_x) \quad (2.3)$$

is a solution of (1.1). Note that when $\lambda \to 0$, equations (2.2) and (2.3) tend to

$$q_t = 2q_x(q-\varepsilon^2 q^2) + \frac{1}{3}q_{xxx}, \quad (2.2')$$

$$u = q - \varepsilon q_x - \varepsilon^2 q^2 \quad (2.3')$$

respectively, i.e. the Gardner transformation for KdV.

Now q in (2.2) is conservation law, i.e. $q_t = \partial(\cdots)$; therefore inverting (2.3) in the ring $A[[\lambda]][[\varepsilon]]$ of formal power series in ε, where A denotes the ring of polynomials in variables $(u,u_x,u_{xx},\ldots)$, we get

$$q = \sum_{n\geq 0} \varepsilon^n H_n, \quad (2.4)$$

where the H_n are conservation laws of the FD-equation. Note that $H_n \in A[[\lambda]]$ and that we can write

$$H_n = \sum_{i=0}^{\infty} h_{n,i}\lambda^i, \quad h_{n,i} \in A. \quad (2.5)$$

To understand the structure of the H_n's it is helpful to look at the KdV equation (2.1) first. Again q is a conservation law (c.ℓ.) in (2.2'); hence inverting (2.3') one gets

$$q = \sum_{n=0}^{\infty} \varepsilon^n h_n, \quad h_n \in A, \quad (2.4')$$

where h_n in (2.4') is equal to $h_{n,0}$ in (2.5).

We observe that the map (2.3') is homogeneous if the following weights are prescribed: $w(\varepsilon) = -1$, $w(u) = w(q) = 2$, $w(\partial) = 1$ (therefore $w(q^{(n)}) = n + 2$, where $q^{(n)} = \partial^n(q)$). Thus

$$w(h_n) = n + 2. \tag{2.6'}$$

In the same manner one can court the map (2.3): $w(u^{(n)}) = w(q^{(n)}) = n + 2$, $w(\varepsilon) = w(\lambda) = -1$. Then

$$w(H_n) = n + 2, \quad w(h_{n,i}) = n + i + 2. \tag{2.6}$$

At this point some notations will be useful. If $f,g \in A$, say $f = f(u,u^{(1)},\dots,u^{(n)})$, $g = g(u,u^{(1)},\dots,u^{(m)})$, we write: $f \sim 0$ if $f = \partial g$; and $f \approx g$ if $f(u,0,\dots,0) = g(u,0,\dots,0)$.

Proposition 2.7. $h_{2n+1} \sim 0$.

Proof. Write (2.4') as $q = q^+ + q^-$ where $q^+ = \sum \varepsilon^{2n} h_{2n}$, $q^- = \sum \varepsilon^{2n+1} h_{2n+1}$ and substitute this into (2.3'). Then the part that is odd in ε yields $q^- - \varepsilon q^+_x - 2\varepsilon^2 q^+ q^- = 0$, or $q^- = -(2\varepsilon)^{-1}\partial \, \ell n(1-2\varepsilon^2 q^+)$.

Proposition 2.8. If $f \sim 0$ then $f \approx 0$. This is evident.

Proposition 2.9. $h_{2n+1} \approx 0$, $h_{2n} \not\approx 0$.

Proof. The first part follows from (2.7), (2.8). Rewriting (2.3') as

$$u \approx q - \varepsilon^2 q^2 \tag{2.10}$$

we get $q \approx (2\varepsilon^2)^{-1}[1-(1-4\varepsilon^2 u)^{1/2}] = \sum_{n=0}^{\infty} c_n \varepsilon^{2n} u^{n+1}$; all the c_n are different from zero since they are binomial coefficients. Thus

$$h_{2n} \approx c_n u^{n+1}, \quad c_n \neq 0. \tag{2.11}$$

To treat the completeness problem for the KdV equation we need some extra information about the polynomials h_n. Note that substitution of (2.4') into (2.3') yields

$$h_0 = u, \quad h_1 = u^{(1)}, \quad h_{s+2} = \sum_{n+m=s} h_n h_m + \partial h_{s+1}, \quad s \geq 0. \tag{2.12}$$

Let $\ell_i(f)$ denote the part of degree i in a polynomial $f \in A$.

<u>Proposition 2.13.</u> $\ell_0(h_n) = 0;\ \ \ell_1(h_n) = u^{(n)}.$

$$\ell_2(h_{2s+2}) \sim (-1)^s u^{(s)2}. \tag{2.14}$$

<u>Proof.</u> The first two formulae follow directly from (2.12), which also yields $\ell_2(h_{2s+s}) \sim \sum_{n+m=2s} \ell_1(h_n)\ell_1(h_m) = \sum_{n+m=2s} u^{(n)}u^{(m)}$. Therefore

$$\frac{\delta}{\delta u}\,\ell_2(h_{2s+2}) = \sum_k (-\partial)^k \frac{\partial}{\partial u^{(k)}} \sum_{n+m=2s} u^{(n)}u^{(m)}$$

$$= 2\sum_{k=0}^{2s} (-\partial)^k u^{(2s-k)} = 2u^{(2s)} \sum_{k=0}^{2s} (-1)^k = 2u^{(2s)},$$

which gives (2.14).

Now let $f = f(x,u,u^{(1)},\ldots,u^{(n)})$ be a function: polynomial or rational or algebraic ... or smooth. We shall write $f(< n+1)$ to indicate that f does not depend upon $u^{(m)}$ with $m > n$. Then Proposition 2.13 yields

<u>Lemma 2.15.</u> $h_{2s+2} \sim (-1)^2 u^{(s)2} + g_s(< s)$, with some $g_s \in A$.

<u>Theorem 2.16.</u> Let $f(< n+1)$ *be a conservation law for* KdV *equation (2.1). Then* $f \sim \sum_{s=0}^{n+1} d_s h_{2s}$, d_s *being constants. In other words, there are no* c.ℓ-s *of the* KdV *equation besides the* h_n's. *This is a particular case of the following*

<u>Theorem 2.17.</u> Let $k > 0$ *and* $H = (-1)^k u^{(k)2}/2 + (< k)$. *Denote by* X_H *the evolution equation (field)*

$$\dot{u} = \partial\delta H, \quad \delta H = \frac{\delta H}{\delta u}. \tag{2.18}$$

Let $f(\leq m)$ *be an integral of (2.18),* $m > 0$. *Then* $f = du^{(m)2} + (< m)$ *with some constant* d.

Corollary 2.17'. There are no new c.ℓ.-s for any equation in either the KdV or MKdV hierarchies-only those already known.

Proof of Theorem 2.17. If f is a c.ℓ. of (2.18) then $\{H,f\} = X_H(f) \sim 0$, which is equivalent to $\delta\{H,f\} = 0$. We need the following formula ((7.14), Ch. I [7])

$$\delta\{H,f\} = X_H(\delta f) - X_f(\delta H), \tag{2.19}$$

which now turns into

$$X_H(\delta f) = X_f(\delta H) \tag{2.20}$$

We want to compare the $u^{(2k+2m)}$ terms in the left and right sides of (2.20). We write $\partial_s = \partial/\partial u^{(s)}$ and remark that (obviously)

$$\partial_s \partial^n = \sum_\alpha \binom{n}{\alpha} \partial^{n-\alpha} \partial_{s-\alpha}. \tag{2.21}$$

Then $X_H(\delta f) = \sum_{s=0}^{2m} \partial^s [u^{(2k+1)} + (\leq 2k-1)] \cdot \partial_s \delta f$ so

$$\partial_{2k+2m} X_H(\delta f) = \partial_{2m-1} \delta f. \tag{2.22}$$

On the other hand

$$\begin{aligned} X_f(\delta H) &= \sum_{s=0}^{2k} \partial^s (\partial\delta f) \cdot \partial_s [u^{(2k)} + (\leq 2k-2)] \\ &= \partial^{2k+1}\delta f + \sum_{s=0}^{2k-2} \partial^{s+1} \delta f \cdot \partial_s (\leq 2k-2), \end{aligned}$$

so $\partial_{2m+2k} X_f(\delta H) = \partial_{2m+2k} \partial^{2k+1} \delta f = \sum_\alpha \binom{2k+1}{\alpha} \partial^{2k+1-\alpha} \partial_{2m+2k-\alpha} \delta f (\leq 2m)$ $= (\alpha = 2k+1) \partial_{2m-1} \delta f + (\alpha = 2k)(2k+1) \partial \partial_{2m} \delta f$. Comparing this with (2.22) we arrive at

$$\partial \partial_{2m} \delta f = 0, \tag{2.23}$$

But $\partial_{2m} \delta f = \partial_{2m} \sum_{k=0}^{m} \partial^k (-1)^k \partial_k f = (-1)^m \partial_m^2 f$, therefore (2.23) can be read as $\partial_m^2 f = 2d(-1)^m = \text{const.}$ □

Now we are prepared to analyze the FD-integrals H_n.

Proposition 2.24. If $f,g \in A[[\lambda]]$ then $\{f,g\} \approx 0$.

Proof. $\{f,g\} = \delta f \cdot \partial\delta g$ and $\partial\delta g \approx 0$.

Lemma 2.24. Let $f \in A[[\lambda]]$. If $f \approx 0$ and f is c.ℓ. of the FD equation, then $f \sim 0$.

Corollary 2.25. All the integrals H_n commute.

Proof of the Lemma 2.24. Let $f = \sum_{i=0}^{\infty} f_i\lambda^i$. If we prove that $f_0 \sim 0$ then $f \sim \lambda \sum_{i=0}^{\infty} f_{i+1}\lambda^i$ and we can repeat the argument. Since f_0 is a c.ℓ. for the KdV equation and $f_0 \approx 0$, then $f_0 \sim 0$ by (2.11) and (2.16).

Remark 2.26. As we mentioned before, from the corollary 2.25 written in the form

$$\int_{-\infty}^{\infty} \{H_n,H_m\}dx \equiv 0, \tag{2.27}$$

it follows that formula (2.27) is true also for the c.ℓ.-s of the B-O equation.

Theorem 2.28. Let $f \in A[[\lambda]]$. *Suppose that* f *is a* c.ℓ. *of the* FD *equation. Then* f *belongs to the linear space generated by the* H_{2n}*'s.*

Proof. Let $f = \sum_{i=0}^{\infty} f_i\lambda^i$. It's enough to prove that we can find a linear combination $\sum c_k H_{2k}$ such that $(f-\sum c_k H_{2k}) \in O(\lambda)$: then we can repeat the procedure. For this note that f_0 is as c.ℓ. for the KdV equation, so $f_0 \sim \sum c_k h_{2k}$. Since $h_{2k,0} = h_{2k}$, so $\sum c_k H_{2k}$ provides the desired combination. □

Remark 2.29. Of course, the same proof goes through for all higher Fd equations

$$u_t = \partial \frac{\delta H_n}{\delta u}, \tag{2.30}$$

because they are regular deformations of the higher KdV equations

$$u_t = \partial \frac{\delta h_n}{\delta u} . \tag{2.31}$$

Remark 2.30. In the same manner as above, one can easily prove both commutativity and completeness properties for the Benney's long wave equations [7]

$$\dot{A}_n = \partial A_{n+1} + nA_{n-1}\partial A_0 ;$$

where we have infinite number of functions $A_n(x,t)$, $n = 0,1,\ldots$.

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REFERENCES

[1] Satsuma, J., Ablowitz, M. J., and Kodama, Y. (1979). *Phys. Lett. 73A*, 283.
[2] Joseph, R. I. (1977). *J. Phys. A10*, L225; *11* (1978), L97.
[3] Chen, H. H., and Lee, Y. C. (1979). *Phys. Rev. Lett. 43*, 264.
[4] Matsuno, Y. (1979). *Phys. Lett. 74A*, 233.
[5] Matsuno, Y. (1980). *J. Phys. Soc. Japan, 48*, 663; 1024.
[6] Gibbons, J., and Kupershmidt, B. (1980). *Phys. Lett. 79A*, 31.
[7] Manin, Yu. I. (1979). *J. Sov. Math. 11*, 1.