SOME OBSERVATIONS ON SURFACES WITH NULL-PROJECTIVE-CURVATURE

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It is proved that only for surfaces with null projective curvature there exist conjugate nets whose curves are normal geodesics. The system determining minimal-projective surfaces with $K = H = 0$ is obtained and so are the invariants of the R_0 surfaces which admit ∞^3 projective deformations and are minimal-projective.

§1. E. Cartan [1] proved the existence of surfaces for which there exist conjugate nets such that

$$\bar{h} = h; \quad \bar{k} = k, \tag{1.1}$$

(h, k, $\bar{h}$, $\bar{k}$, being, respectively, the point- and tangential invariants of the corresponding Laplace equation) and called them $\mathcal{E}$-surfaces and $\mathcal{E}$-nets. Let S be a non-ruled surface referred to its asymptotes, determined near a homography by the complete integrable system

$$x_{uu} = \theta_u x_u + \beta x_v + p_{11}x; \quad x_{vv} = \gamma x_u + \theta_v x_v + p_{22}x. \tag{1.2}$$

According to Cartan an $\mathcal{E}$-surface is characterized by the conditions:

$$\left(\frac{\beta}{\sigma}\right)_u = (\gamma\sigma)_v; \quad (\gamma\sqrt{\sigma})_u = \left(\frac{\beta}{\sqrt{\sigma}}\right)_v, \tag{1.3}$$

$\dfrac{\beta du^3+\gamma dv^3}{2du\, dv}$ being Fubini's projective linear element and

$$dv^2 - \sigma^2 du^2 = 0, \tag{1.4}$$

the corresponding net.

Let

$$F_2 = 2a_{12}dudv; \quad F_3 = a_{12}(\beta du^3 + \gamma dv^3), \tag{1.5}$$

where

$$\omega a_{12}^2 = (x, x_u, x_v, x_{uv}); \quad \omega = \operatorname{sign}(x, x_u, x_v, x_{uv}), \tag{1.6}$$

and

$$|a_{12}| = e^{\theta}, \tag{1.7}$$

the first and second differential forms of Fubini [2,2']. According to him [2], the pangeodesics of a surface are the extremals of the integral

$$I_2 = \int \frac{F_3}{F_2}, \tag{1.8}$$

and their equation is

$$2u'' \frac{\gamma + \beta u'^3}{u'^3} = 2\left(\frac{\gamma_u}{u'} - \beta_v u'\right) + \left(\frac{\gamma_v}{u'^2} - \beta u'^2\right); \quad u' = \frac{du}{dv}. \tag{1.9}$$

We characterize the E-surfaces and the nets by the following results (see [3]):

(a) <u>If the curves of a conjugate net N of a surface S are pangeodesics, S is a E-surface and N a E-net.</u>

(b) <u>Only for E-surfaces[1] there exist conjugate nets whose curves are pangeodesics.</u>

§2. In [4], Fubini adopted his normal differential form:

$$\phi_2 = 2\beta\gamma dudv; \quad \beta\gamma = e^{\theta}, \tag{2.1}$$

[1]*It was shown in [3] that A. Terracini obtained the same conditions (1.3) ten years earlier than Cartan, when studying the applicability of the congruences generated by the tangents of a conjugate net. Accordingly, we will call the E-surfaces "Terracini-Cartan surfaces."*

as a linear element of a metric geometry of S which is invariant for collineations (for correlations and projective deformations). In this geometry, the integral

$$I_1 = \int \sqrt{\phi_2}, \tag{2.2}$$

is invariant and intrinsical and is called Fubini's integral invariant (see [5]). Fubini defined it as a projective notion of the length of an arc of a surface, and considered the geodesics of the metric defined by ϕ_2. The latter are defined by

$$\delta \int \sqrt{\phi_2} = 0. \tag{2.3}$$

They are called projective or normal geodesics, and their equation is:

$$v'' = \theta_u v' - \theta_v v'^2; \quad \theta = \ell n \beta\gamma. \tag{2.4}$$

The curvature of the normal form ϕ_2 is called the projective or normal curvature of the surface, and is given by

$$K = -\frac{1}{\beta\gamma}\frac{\partial^2 \log|\beta\gamma|}{\partial u\, \partial v}. \tag{2.5}$$

Besides K, the expression

$$H = -\frac{1}{\beta\gamma}\frac{\partial^2 \log(\beta:\gamma)}{\partial u\, \partial v}, \tag{2.6}$$

is also important in the theory of surfaces, as $H = 0$ characterizes isothermo-asymptotic surfaces (see [2']). K and H are finite invariants of a non-ruled surface.

§3. Suppose now that the curves of the net

$$dv^2 - \tau^2 du^2 = 0; \quad \tau = \tau(u,v) \tag{3.1}$$

are normal geodesics. Then it follows from (2.4) that

$$\tau_u + \tau\tau_v = \theta_u\tau - \theta_v\tau^2; \quad \tau_u = \tau\tau_v - \theta_u\tau - \theta_v\tau^2. \tag{3.2}$$

Hence

$$(\ln \tau)_u = \theta_u; \quad (\ln \tau)_v = -\theta_v. \tag{3.3}$$

Therefore

$$\theta_{uv} = (\ell_n \beta\gamma)_{uv} = 0, \tag{3.4}$$

and the projective curvature K vanishes. Conversely, suppose that $K = 0$; in that case the transformation

$$\bar{u} = \alpha(u); \quad \bar{v} = \delta(v), \tag{3.5}$$

yields

$$\beta\gamma = 1, \tag{3.6}$$

i.e. $a_{12} = 1$.

It is seen from (2.4) that for S there exists a simple infinite family of conjugate nets, whose curves are normal geodesics, given by

$$dv^2 - c^2du^2 = 0, \quad (c = \text{const.}). \tag{3.7}$$

Accordingly, we arrive at the conclusion that: Only for surfaces with null normal curvature, there exist conjugate nets whose curves are projective geodesics.

According to Cartan [6], there always exist surfaces which admit a given quadratic form. They depend on five arbitrary functions of an argument. Therefore, surfaces with $K = 0$ exist and depend as above.[2]

§4. Surfaces with k = const., and $H = 0$ play an important role in projective differential geometry. In fact, according to Cartan, these surfaces admit ∞^3 projective deformations. (They are the theme of §69, pp. 364-383, in [2]). Fubini and Čech considered first the case $K = H = 0$. It is shown that it is possible to choose the parameters u, v such that

[2] *The same result was obtained by T. Mihăilescu [7], but in the absence of references it is very difficult for a non-specialist reader to differentiate between it and those of Fubini and Cartan, whose names are not mentioned.*

$$\beta = \sqrt{V/U}; \quad \gamma = \sqrt{U/V} \tag{4.1}$$

where

$$U = au^3 + b_1u^2 + c_1u + d_1; \quad V = av^3 + b_2v + c_2v + d_2, \tag{4.2}$$

a, b_i, c_i, d_i being arbitrary constants — provided U and V are not identically zero. Formulas for p_{11} and p_{22} are also available, but are fairly complicated. None is available for L or M. Let

$$L_v + \gamma\beta_u + 2\beta\gamma_u = 0; \quad M_u + \beta\gamma_v + 2\gamma\beta_v, \tag{4.3}$$

$$\beta M_v + 2M\beta_v + \beta_{vvv} = \gamma L_u + 2L\gamma_u + \gamma_{uuu}, \tag{4.3'}$$

be the integrability conditions of system (1.2), where

$$L = \theta_{uu} - \frac{1}{2}\theta_u^2 - \beta\theta_v - \beta_v - 2p_{11}; \; M = \theta_{vv} - \frac{1}{2}\theta_v^2 - \gamma\theta_u - \gamma_u - 2p_{22}. \tag{4.4}$$

Accordint to O. Mayer [8] S is a minimal-projective surface if

$$\beta M_v + 2M\beta_v + \beta_{vvv} = 0, \quad (\gamma L_u + 2L\gamma_u + \gamma_{uuu} = 0). \tag{4.5}$$

As the functions U and V given by (4.2) are sufficiently general, it may be possible to obtain values for them such that the corresponding surfaces would be <u>minimal-projective</u>. From (4.3), noting that $\beta\gamma = 1$, there follows:

$$M = \frac{1}{2}u\frac{V'}{V} + V^*; \quad L = \frac{1}{2}v\frac{U'}{U} + U^*, \tag{4.6}$$

with $V^* = V^*(v)$, $U^* = U^*(u)$. From (4.5), (4.1) and (4.6), we obtain

$$V^{1/2}V^{*\prime} + \frac{V'}{V^{1/2}}V^* + \frac{u}{V^{3/2}}V'' + \frac{1}{8V^{5/2}}(2V^2V''' - 6VV'V'' + 3V'^3) = 0. \tag{4.7}$$

Operating with $\partial/\partial u$, we obtain.

$$V'' = 0, \tag{4.8}$$

whence

$$V = c_2v + d_2. \tag{4.9}$$

In the same way there follows

$$U = c_1 u + d_1. \tag{4.10}$$

Consequently

$$M = -\frac{c_2 u}{2(c_2 v + d_2)} + V^*; \quad L = -\frac{c_1 v}{2(c_1 u + d_1)} + U^*;$$
$$U^* = \frac{C_1^*}{c_1 u + d_1} + \frac{3}{8}\frac{c_1^2}{(c_1 u + d_1)^2}; \quad V^* = \frac{C_2^*}{c_2 v + d_2} + \frac{3}{8}\frac{c_2^2}{(c_2 v + d_2)^2}. \tag{4.11}$$

Without loss of generality, we may suppose $d_1 = d_2 = 0$. Then,

$$\beta = \alpha\sqrt{v/u}; \quad \gamma = \frac{1}{\alpha}\sqrt{u/v}; \quad \theta = 0;$$
$$U^* = \frac{C_1^*}{c_1 u} + \frac{3}{8u^2}; \quad V^* = \frac{C_2^*}{c_2 v} + \frac{3}{8v^2}; \quad \alpha \neq 0, \tag{4.12}$$

$$M = -\frac{u}{2v} + \frac{3}{8v^2} + \frac{C_2^*}{c_2 v};$$
$$L = -\frac{v}{2u} + \frac{3}{8u^2} + \frac{C_1^*}{c_1 u}; \quad (\alpha, C_1^*, C_2^* = \text{const.}), \tag{4.12'}$$

and

$$2p_{11} = -L - \beta_v = \frac{v}{2u} - \frac{3}{8u^2} - \frac{C_1^*}{c_1 u} - \frac{\alpha}{2\sqrt{uv}},$$
$$2p_{22} = -M - \gamma_u = \frac{u}{2v} - \frac{3}{8v^2} - \frac{C_2^*}{c_2 v} - \frac{1}{2\alpha\sqrt{uv}}, \tag{4.12''}$$

which determines minimal-projective surfaces with $K = H = 0$ up to a homography.

For $C_1^* = C_2^* = 0$, the surfaces admit ∞^1 projective deformations. For $C_1^* = C_2^* = 0$ and fixed α_1, the surfaces are projective-undeformable. For $\alpha = 1$, there follows from (4.12):

$$\beta_v = \gamma_u \tag{4.13}$$

and S is an R surface of Tzitzeica-Demoulin.

§5. S is an R_0-surface if (see [2], T.1, p. 359)

$$H \pm K = 0. \tag{5.1}$$

In such a case we can obtain

$$\phi_2 = 2\gamma dudv; \quad \phi_3 = \gamma du^3 + \gamma^2 dv^3, \tag{5.2}$$

or

$$\phi_2 = 2\beta dudv; \quad \phi_3 = \beta^2 dv^3 + \beta dv^3. \tag{5.2'}$$

In §70, R_0-surfaces admitting ∞^3 deformations are considered. Supposing the first case, there results

$$\gamma = UV, \quad U = U(u), \quad V = V(v). \tag{5.3}$$

whence $K = \frac{\partial^2 \log\gamma}{\partial u \partial v} = 0$, i.e. the projective curvature of these surfaces R_0 is null.

These R_0 - surfaces certainly include some minimal-projective ones. To the best of our knowledge, these were not the object of any study. From (4.3) there follows:

$$M = -V'\int Udu + V^*(v); \quad L = -2U'\int Vdu + U^*(u), \tag{5.4}$$

and from (4.5), noting that $UV \neq 0$, we obtain

$$V'' = V^{*\prime} = 0, \tag{5.5}$$

whence

$$V = 2av + b_1; \quad V^* = b_2. \tag{5.6}$$

From (4.5), noting that $\gamma L_u + 2L\gamma_u + \gamma_{uuu} = 0$, there results

$$U = \sqrt[3]{3}(cu+c_1)^{1/3}, \quad U^{*\prime} + 2U^* \frac{U'}{U} + \frac{U'''}{U} = 0. \tag{5.7}$$

Hence

$$U^* = \frac{C^*}{\sqrt[3]{9}(cu+c_1)^{2/3}} + \frac{5c^2}{18(cu+c_1)^2}, \tag{5.8}$$

and

$$M = -\frac{3\sqrt[3]{a}}{2c}(cu+c_1)^{4/3} + b_2; \quad L = -\frac{2c\sqrt[3]{3}(av^2+b_1v+c_2)}{3(cu+c_1)^{2/3}} + U^*. \tag{5.9}$$

Therefore

$$\beta = 1; \ \gamma = \sqrt[3]{c}(cu+c_1)^{1/3}(2av+b_1); \ 2p_{11} = -L; \ 2p_{22} = -M - \frac{\sqrt[3]{3}}{3}\frac{c(2av+b_1)}{(cu+c_1)^{2/3}}. \tag{5.10}$$

Replacement of γ by β evidently yields another class of minimal-projective R_0-surfaces.

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