SOME FIXED POINT THEOREMS FOR CONVEX CONTRACTION MAPPINGS AND CONVEX NONEXPANSIVE MAPPINGS (I)

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0. INTRODUCTION

Let X be a complete metric space with the metric d. In the recent years, a great number of papers present generalizations of the well-known Banach contraction principle. Sometimes, these generalizations refer also to results containing the Schauder fixed point theorem. The purpose of the present paper is to consider a generalization of the Banach contraction principle by introducing a "convexity condition." This condition can be adapted for other classes of mappings, as we show in the last part of the paper.

I. CONVEX CONTRACTION MAPPINGS OF ORDER 2

Let X be a complete metric space with the metric d.

Definition 1.1. A continuous mapping $T : X \to X$ is called a convex contraction mapping of order 2 if there exist a, b in $(0,1)$ such that:

(1) $a + b < 1$,

(2) $d(T^2x,T^2y) \leq ad(Tx,Ty) + bd(x,y)$.

Concerning this class of mappings, which obviously contains the class of contraction mappings, we can prove the following theorem.

Theorem 1.2. Any convex contraction mapping has a unique fixed point.

Proof. Since the uniqueness is trivial, we prove only the existence. Let x_0 be an arbitrary but fixed point in X, and consider further the orbit of x_0, i.e., the sequence $(x_n)_0^\infty$, where $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$. Let

$$K = \max(d(x_0,Tx_0),d(Tx_0,T^2x_0)),$$

and thus for any m,

$$d(T^{2m+1}x_0,T^{2m}x_0) \leq ad(T^{2m}x_0,T^{2m-1}x_0) + bd(T^{2m-1}x_0,T^{2(m-1)}x_0),$$

as well as

$$d(T^{2m}x_0,T^{2m-1}x_0) \leq ad(T^{2m-1}x_0,T^{2(m-2)}x_0) + bd(T^{2(m-1)}x_0,T^{2m-3}x_0).$$

From these we obtain the following inequalities:

(1) $d(T^2x_0,T^3x_0) \leq ad(Tx_0,T^2x_0) + bd(x_0,Tx_0) \leq K(a+b)$,

(2) $d(T^3x_0,T^4x_0) \leq ad(T^2x_0,T^3x_0) + bd(Tx_0,T^2x_0)$
$\leq (aK+bK(a+b)) = K(a+b)$,

(3) $d(T^4x_0,T^5x_0) \leq ad(T^3x_0,T^4x_0) + bd(T^2x_0,T^3x_0) \leq aK(a+b)$
$+ bK(a+b) = K(a+b)^2$,

....

An induction argument shows that

$$d(T^{2m}x_0,T^{2m+1}x_0) \leq K(a+b)^m,$$

and from these estimates we get easy that (x_n) is a Cauchy sequence. Indeed, for $m < n$,

$$d(T^mx_0,T^nx_0) \leq d(T^mx_0,T^{m+1}x_0) + d(T^{m+1}x_0,T^{m+2}x_0)+\cdots+d(T^{n-1}x_0,T^nx_0)$$
$$\leq K(a+b)^{m/2} + K(a+b)^{m/2\ +1}+\cdots \leq K(a+b)^{m/2}\cdot 1/(1-(a+b)).$$

Similar estimates we obtain in the case $m = 2p + 1$, $n = 2\ell$, $m = 2p$, $n = 2\ell + 1$. Clearly, these imply that (x_n) is a Cauchy sequence. Let $x^* = \lim x_n$. Since

$$Tx^* = \lim T^{n+1}x_0 = x^*$$

The existence of the fixed points is proved.

We give now an example to show that there exist mappings which are convex contraction of order 2, but they are not contractions.

Example 1.3[†]. Let $X = [0,1]$ and $T : X \to X$ defined by

$$Tx = \begin{cases} \frac{x}{4} & \text{if } x \in [0,\frac{1}{2}) = A, \\ \frac{x}{5} & \text{if } x \in [\frac{1}{2},1] = B. \end{cases}$$

One easily obtains

$$|T^2x - T^2y| \leq \frac{1}{4}|Tx - Ty|,$$

and thus T is a convex contraction of order 2. From the form of T, it is easy to see that it is not a contraction mapping.

We can consider now a more general family of mappings related to the contraction mappings.

Definition 1.4. A continuous mapping $T : X \to X$ is said to be convex contraction of order n if there exist positive constants $a_0,\ldots,a_{n-1}$, such that

(1) $a_0 + \cdots + a_{n-1} < 1$.

(2) $d(T^nx,T^ny) \leq a_0d(x,y) + a_1d(Tx,Ty) + \cdots + a_{n-1}d(T^{n-1}x,T^{n-1}y)$ hold for all x, y in X.

Theorem 1.2 can be extended to this class of mappings, and since the proof is essentially the same, but with tedious calculations, we mention it only.

Theorem 1.5. Any convex contraction mapping of order n *has a unique fixed point.*

As it is well known, some results about contraction mappings were extended to a larger class of mappings, the so called contractive mappings. We recall that a mapping $S : X \to X$ is called contractive if the following inequality

$$d(Sx,Sy) < d(x,y)$$

holds for all $x \neq y$ in X.

We define now the corresponding class of mappings.

[†]*This example is due to R. Kannan.*

Definition 1.6. A continuous mapping $T : X \to X$ is called convex contractive of order 2 if there exist the constants a_0 and a_1 in $(0,1)$, such that:

(1) $a_0 + a_1 = 1$,

(2) $d(T^2x,T^2y) < a_0d(x,y) + a_1d(Tx,Ty)$ for all $x \neq y$ in X.

A well known result of V. Nemytskii [7] states that if T is a contractive mapping, defined on compact X, then there exist fixed points for T. The following result is an extension of Nemytskii's result for convex contractive mappings.

Theorem 1.7. Let $T : X \to X$ *be a convex contractive mapping of order* 2, *and suppose* X *is compact. Then* T *has a unique fixed point.*

Proof. Let x_0 be an arbitrary but fixed point of X and consider the orbit of x_0 under T, i.e., the set $(T^nx_0)_0^\infty$. From the compactness of X, there follows the existence of a subsequence (n_k), such that

$$x^* = \lim T^{n_k}x_0.$$

From the continuity of T it follows that

$$Tx^* = \lim T^{n_k+1}x_0,$$

$$T^2x^* = \lim T^{n_k+2}x_0,$$

$$T^3x^* = \lim T^{n_k+3}x_0,$$

Let us consider now the function

$$v(x) = \max(d(x,Tx),d(Tx,T^2x)),$$

which is clearly a continuous function on X. Since T is convex contractive of order 2, we obtain that v is nonincreasing with respect to T, i.e.,

$$v(Tx) = v(x).$$

Now the continuity of v and the above formulas for x^*, Tx^*, T^2x^*, T^3x^* give that

$$v(x^*) = v(Tx^*) = v(T^2x^*) = v(T^3x^*).$$

If we suppose that $v(x^*)$ is strictly positive, then the convex contractive property of T implies that

$$v(x^*) = v(T^2x^*) < v(x^*).$$

Thus, we have $v(x^*) = 0$, and this implies that x^* is a fixed point of T. The uniqueness is obvious, and the theorem is proved.

Similarly, we can prove the following theorem which generalizes a result of Edelstein [2].

Theorem 1.8. Let $T : X \to X$ *be a convex contractive mapping of order* 2, *and suppose that any orbit* $(T^n x)_0^\infty$, $x \in X$, *has a limit point* ξ. *Then* ξ *is the unique fixed point of* T.

Another possibility to obtain an extension of the Banach existence theorem, for larger classes of mappings, is to localize different conditions. The following definition introduces the localization for the convex contraction mappings of order 2.

Definition 1.9. A continuous mapping $T : X \to X$ is called locally convex contractive of infinite order if there exists a sequence of positive numbers (a_i), $\sum_0^\infty a_i < 1$, such that for each $x \in X$, there exists an integer $n = n(x)$ with the property

$$d(T^n x, T^n y) \leq a_0 d(x,y) + a_1 d(Tx,Ty) + \cdots + a_{n-1} d(T^{n-1}x, T^{n-1}y).$$

Remark 1.10. If $a_1 = a_{1+i}$, $i = 0,1,2,\ldots$, then the above class of mappings reduces to the class defined by Sehgal [9].

An easy modification in the method of proof of Sehgal's result gives us the following theorem.

Theorem 1.11. Any locally convex contraction mapping of infinite order has a unique fixed point.

II. TWO-SIDED CONVEX CONTRACTION MAPPINGS

We consider now another class of mappings, suggested by the class of mappings satisfying the following condition: S is defined on a complete metric space, and for some a and b in $(0,1)$, the following inequalities hold:

(1) $a + b < 1$,

(2) $d(Tx,Ty) = ad(x,Tx) + bd(y,Ty)$.

We note that there exists a great number of papers in which results about this class, or related to it, are presented. Our class is considered in the following definition.

Definition 2.1. A continuous mapping $T : X \to X$ is said to be two-sided convex contraction, provided that there exist a_1, a_2 and b_1, b_2 in $(0,1)$, such that the following inequalities hold:

(1) $a_1 + a_2 + b_1 + b_2 < 1$,

(2) $d(T^2x,T^2y) \leq a_1d(x,Tx) + a_2d(Tx,T^2x) + b_1d(y,Ty) + b_2d(Ty,T^2y)$, for all $x \neq y$ in X.

A related class of mappings containing the convex contraction mappings is considered in the following definition.

Definition 2.2. A continuous mapping $T : X \to X$ is said to be of convex type 2, if there exist positive numbers c_0, c_1, a_1, a_2, b_1, b_2, such that the following inequalities hold:

(1) $c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1$,

(2) $d(T^2x,T^2y) = c_0d(x,y) + c_1d(Tx,Ty) + a_1d(x,Tx) + a_2d(Tx,T^2y) + b_1d(y,Ty) + b_2d(Ty,T^2y)$.

It is clear that this class of mappings contains the mappings defined in Defiontions 1.1 and 2.1.

Concerning the fixed points for mappings considered in Definition 1.1 we can prove the following theorem.

Theorem 2.3. Any two-sided convex contraction mapping has a unique fixed point.

Proof. Let x_0 be an arbitrary but fixed point in X, and consider the orbit of x_0 under T, i.e., the set $(T^nx_0)_0^\infty$. We set

$$K = \max(d(x_0,Tx_0),d(Tx_0,T^2x_0)).$$

Then we have,

$$d(T^2x_0,T^3x_0) \le a_1d(x_0,Tx_0) + a_2d(Tx_0,T^2x_0)$$
$$+ b_1d(T^2x_0,Tx_0) + b_2d(T^2x_0,T^3x_0),$$

and thus

$$d(T^2x_0,T^3x_0) \le ((a_1+a_2+b_1)/(1-b_2)) \cdot K.$$

Similarly,

$$d(T^3x_0,T^4x_0) \le ((a_1+a_2+b_1)/(1-b_2)) \cdot K$$

as well as

$$d(T^4x_0,T^5x_0) \le ((a_1+a_2+b_1)/(1-b_2))^2 \cdot K.$$

An induction argument gives that

$$d(T^mx_0,T^{m+1}x_0) \le ((a_1+a_2+b_1)/(1-b_2))^{m-2} \cdot K$$

holds for all $m \ge 4$. From these estimates we obtain easy that the sequence $(T^nx_0)_0^\infty$ is a Cauchy sequence. Then clearly $x^* = \lim T^nx_0$ is a fixed point for T. Since the uniqueness is obvious, the theorem is proved.

Using a similar method we can prove the following result about the class of mappings considered in the Definition 2.2.

Theorem 2.4. Any mapping which is of convex type 2 has a unique fixed point.

We note that we can consider another class of mappings containing the mappings considered in the Definition 2.2 and related to the so called "generalized contractions." For the reader's convenience we recall that a mapping $T : X \to X$ is called generalized contraction if there exists a function $\alpha(\cdot)$ defined on X, and with values in $[0,1)$, such that for any $x \in X$ and all $y \in X$

$$d(Tx,Ty) \le \alpha(x)d(x,y).$$

Our class is defined as follows.

Definition 2.5. A continuous mapping $T : X \to X$ is called a generalized convex type 2 if there exist positive functions $c_0(\cdot)$, $c_1(\cdot)$, $a_1(\cdot)$, $a_2(\cdot)$, $b_1(\cdot)$, $b_2(\cdot)$, such that the following inequalities hold:

(1) $c_0(x) + c_1(x) + a_1(x) + a_2(x) + b_1(x) + b_2(x) < 1$,

(2) $d(T^2x,T^2y) \leq c_0(x)d(x,y) + c_1(x)d(Tx,Ty) + a_1(x)d(x,Tx) + a_2d(Tx,T^2x) + b_1d(y,Ty) + b_2d(Ty,T^2y)$.

Clearly, this class of mappings reduces, for an appropriate selection of the functions $c_i(\cdot)$, $a_i(\cdot)$ and $b_i(\cdot)$, to the classes of mappings considered above.

We close with some remarks about an extension of a result in [8], concerning the fixed points of certain maps on an interval into itself.

First, we note that the arguments in the proof of Theorem 1 in [8] give also the following result.

Theorem 2.6. Let $T : [a,b] \to [a,b]$ *with the property that* $a,b \in T([a,b])$. *Suppose that for some positive* r,s, $r+s = 1$ *the following inequality holds:*

$$|Tx - Ty| \leq r|x - Tx| + s|y - Ty|.$$

In this case, the midpoint of $[a,b]$ *is a fixed point of* T.

The same argument (i.e. the use of the fact that $[a,b]$ is a subset which is convex, closed and bounded in an uniformly convex space) permits to prove the following result.

Theorem 2.7. Let C *be a compact, convex and closed subset of an uniformly convex Banach space* X. *Suppose that* $T : C \to C$ *satisfies the following property:*

$$\|Tx - Ty\| \leq r\|x - Tx\| + s\|y - Ty\|, \quad r,s \in [0,1], \quad r + s = 1,$$

for all x, y *in* C, *and* T *has the property that* $\partial C \subseteq T(C)$ *(*∂C *is the boundary of* C*). Then* T *has a fixed point in* C.

It is not difficult to see that any point of C, which is the midpoint of any diametral segment in C (the segment $[x,y]$ is called diametral if $\|x - y\| = d(C) = \mathrm{diam}(C)$), is a fixed point for T.

We conjecture that Theorem 2.7 is true without assumption on compactness of C.

III. A FIXED POINT THEOREM FOR MAPPING DIMINISHING DIAMETERS

Let $T : X \to X$ be a contraction mapping. In this case it is easy to see that for any bounded set M in X, we have the following relation $d(TM) \leq kd(M)$, where $TM = \{Tx, x \in M\}$.

We consider now a class of mappings related to this inequality.

Definition 3.1. A mapping $T : X \to X$ is said to be with locally diminishing diameter property, if there exists $k \in (0,1)$ such that for any bounded set M in X, there exists an integer $n = n(M)$, such that $d(T^nM) \leq kd(M)$. From just the definition, it is clear that any contraction is a mapping with the property stated in Definition 3.1. We show now that this class contains the mappings considered in [9].

Indeed, let M be any bounded set in X, pick an x in M, and let $T : X \to X$ with the property that there exists $k \in (0,1)$, and for each $x \in X$ there exists $n = n(x)$ with the property that for all $y \in X$,

$$d(T^nx, T^ny) \leq kd(x,y).$$

We choose an integer m such that $2k^m < 1$.

Consider the following sequence of points

$$n_1 = n(x) \Rightarrow x_1 = T^{n_1}(x);\quad n_2 = n(x_1) \Rightarrow x_2 = T^{n_2}(x_1);\ \ldots;$$

$$n_m = n(x_{m-1}) \Rightarrow x_{m+1} = T^{n_{m+1}}(x_m), \ldots .$$

Now, if y, z are arbitrary points in M, we have,

$$d(T^{n_1+\cdots+n_m}(y), T^{n_1+\cdots+n_m}(z)) \leq d(T^{n_1+\cdots+n_m}(y), x_m)$$
$$+ d(x_m, T^{n_1+\cdots+n_m}(z)) \leq kd(T^{n_1+\cdots+m_{m-1}}(y), x_{m-1})$$
$$+ kd(x_{m-1}, T^{n_1+\cdots+n_{m-1}}(z)) \leq \cdots \leq k^md(x,y) + k^md(x,z) \leq 2k^md(M),$$

and thus the assertion is proved.

For the fixed point theorem which follows, we suppose that the space X is bounded.

Theorem 3.2. Let $T : X \to X$ *with locally diminishing diameter property. Then* T *has a unique fixed point in* X.

Proof. Since it is obvious that the set of fixed points of T contains at most one point, we prove only the existence of fixed points. To this end, we consider the following sequence of sets:

$$\overline{X} \supseteq \overline{T(X)} \supseteq \overline{T^2(X)} \supseteq \overline{T^3(X)} \supseteq \cdots \supseteq \overline{T^n(X)} \supseteq \overline{T^{n+1}(X)} \supseteq \cdots$$

We show that $\lim d(T^n(X)) = 0$. We remark that our sequence contains the following subsequence

$$X_1 = \overline{T^{n_1}(X)}, \quad n_1 = n(X), \quad X_2 = \overline{T^{n_2}(X_1)}, \quad n_2 = n(X_1), \ldots .$$

Since we have the estimate

$$d(X_m) \leq k^m d(X),$$

we obtain that $\lim d(X_m) = 0$. Since the diameter of any set M satisfies the relation $d(M) = d(\overline{M})$ we get that $\lim d(T^n(X)) = 0$. Thus by Cantor's theorem

$$\bigcap_n \overline{T^n(X)} = (x^*).$$

We note also that we have the important relation: for each x, $\lim T^n x = x^*$. If T is supposed continuous, then from this relation we conclude that x^* is a fixed point of T.

We consider now the set

$$G = (x^*, Tx^*, T^2x^*, \ldots)$$

and from the above relation we get that it is closed and invariant for T, and the image of G is exactly G. If G contains more than one point, this is a contradiction. The theorem is proved.

IV. CONVEX CONTRACTION MAPPINGS AND GENERALIZED METRIC SPACES

We consider the extended real line, which consists of all points of the real line, and two points denoted by $-\infty$ and ∞, with the usual order relation and $-\infty \leq x \leq \infty$.

Definition 4.1. A function $d : X^2 \to R_e$, (R_e the extended real line) on an abstract set X is called a generalized metric, if the following assertions hold:

(1) $d(x,y) = d(y,x)$,
(2) $d(x,y) = 0$ iff $x = y$,
(3) $d(x,z) \leq d(x,y) + d(y,z)$ (if $d(x,y) = \infty$ or $d(y,z) = \infty$ then we consider that $d(x,y) + d(y,z) = \infty$).

Any set X, with a generalized metric, is called, after Luxemburg, a generalized metric space. Of course we can define in this setting all notions known in the theory of metric spaces.

The following result presents a fixed point theorem for a class of mappings on generalized complete metric spaces.

Theorem 4.1. Let (X,d) *be a generalized complete metric space and* $T : X \to X$ *be a map with the following properties:*

(1) $d(T^2x,T^2y) \leq ad(x,y) + bd(Tx,Ty)$, $a + b < 1$, x,y *in* X;
(2) for any point $x \in X$, *there exists an integer* n, *such that* $d(T^nx,T^{n+1}x)$ *and* $d(T^{n+1}x,T^{n+2}x)$ *are less than* ∞;
(3) if $Tx = x$, $Ty = y$, *then* $d(x,y) < \infty$.

Then T *has a unique fixed point in* X.

Proof. Let x_0 be an arbitrary, but fixed point in X, and consider the orbit of x_0 under T. According to 2 there exists an integer n such that

$$d(T^nx_0,T^{n+1}x_0) < \infty, \quad d(T^{n+1}x_0,T^{n+2}x_0) < \infty,$$

and thus, according to (1) we get that

$$d(T^{n+2}x_0,T^{n+3}x_0) \leq ad(T^nx_0,T^{n+1}x_0) + bd(T^{n+1}x_0,T^{n+2}x_0) \leq (a+b)K$$

where

$$K = \max(d(T^n x_0, T^{n+1} x_0), d(T^{n+1} x_0, T^{n+2} x_0)).$$

We get further

$$d(T^{n+3}, x_0, T^{n+4} x_0) \leq ad(T^{n+1} x_0, T^{n+2} x_0) + bd(T^{n+2} x_0, T^{n+3} x_0)$$

$$\leq aK + b(a+b)K = (a+b)K$$

and

$$d(T^{n+4} x_0, T^{n+5} x_0) \leq ad(T^{n+2} x_0, T^{n+3} x_0) + bd(T^{n+3} x_0, T^{n+4} x_0)$$

$$\leq a(a+b)K + b(a+b)K = (a+b)^2 K.$$

An induction argument shows that we have the following inequality:

$$d(T^{n+m} x_0, T^{n+m+1} x_0) \leq (a+n)^{m-2} K, \quad m \geq 4.$$

From this estimate, it is clear that the sequence $(T^n x_0)_0^\infty$ is a Cauchy sequence. Clearly $x^* = \lim T^n x_0$ is a fixed point of T. Now, the uniqueness follows from the properties 1 and 3 of the mapping T.

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