# FLUID FLOW AND PRESSURE DISTRIBUTION IN FERROMAGNETIC FILMS

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NOMENCLATURE

$A$ = nondimensional relationship, defined in (2)

$A_n$ = integration constant, dimensionless

$A_{cn}, A_{sn}$ = values of $A_n$ defined in (23), dimensionless

$A_0(X_1)$ = function defined in A(1), dimensionless

$A_{c,2n-1}(X_1)$ = function defined in A(1), dimensionless

$A_{s,2n}(X_1)$ = function defined in A(1), dimensionless

$a$ = constant defined in (31), dimensionless

$B$ = nondimensional relationship, defined in (2)

$B_n$ = integration constant, dimensionless

$B_{cn}, B_{sn}$ = values of $B_n$ defined in (23), dimensionless

$b$ = bearing width (extent of the active zone); also reference length (for journal bearings $b = r_1$), m

$C_j$ = integration constant ($j = 1,2$), dimensionless

$C_M$ = integration constant defined in (9), $\mathrm{PamH}^{-1}$

$\overline{C}_j$ = integration constant ($j = 1,2$), dimensionless

$c$ = bearing radial clearance, m

$c_m^{(i)}$ = coefficients, defined in (35), dimensionless

$e$ = eccentricity (distance between journal and bearing axes), m

$F_{in}$ = function defined in (35), $i = 1,2$, dimensionless

$H$ = $h/h_2$ = dimensionless film thickness

$H_j$ = value of $H$ at $X_1 = X_{1j}$, dimensionless

$\mathcal{H}$ = magnetic field strength, $Am^{-1}$

$\mathcal{H}_j$ = value of $\mathcal{H}$ along $X_{1j}(X_3)$ boundary, $Am^{-1}$

$h$ = film thickness, m

$h_2$ = minimum film thickness, m

$I(X_1)$ = indefinite integral, shown in (A.5)

$I_1(n\xi_1)$ = first kind, first order, modified Bessel function, dimensionless

$\mathcal{I}_{0i}$ = integral defined in (3), $HA^2m^{-3}$(Pa)

$i$ = subscript

$j$ = subscript

$K_1(n\xi_1)$ = second kind, first order, modified Bessel function, dimensionless

$M$ = $\mu/\mu_{1\mathcal{H}}$ = nondimensional viscosity

$M^*$ = magnetization strength, $ATm^{-1}$

$\mathcal{M}^*$ = integral defined in (9), $PamH^{-1}$

$\mathcal{M}^*_j$ = value of $\mathcal{M}^*$ at $X^*_{1j}$ ($j = 1,2$), $PamH^{-1}$

$\mathcal{M}^*_\infty(X_1)$ = value of $\mathcal{M}^*$ for $\lambda = \infty$, $PamH^{-1}$

$N$ = number, maximum $n$, dimensionless

$n$ = integer; also exponent or subscript, dimensionless

$P$ = $ph_2^2/6\mu_1Vb$ = dimensionless pressure

$P_0$ = ambient pressure, dimensionless

$P^{(0)}$ = dimnesionless hydrodynamic pressure

$P^*$ = dimensionless magnetic pressure

$P^*_j$ = value of $P^*$ at $X_{1j}$ $(j = 1,2)$, dimensionless

$P^*_\infty$ = dimensionless magnetic pressure in infinitely long bearings

$\overline{P}^*(X_1)$ = value of $P^*(X_1,\lambda/2)$ at the borders of the bearing, dimensionless

$p$ = pressure, Pa

$q_i$ = constants, $i = 1,2$, defined in (33), dimensionless

$r_1$ = journal radius, m

$T$ = temperature, K

$U$ = dimensionless function, defined in (15)

$U_0(X_1)$ = part of $U$, depending only on $X_1$ coordinate, dimensionless

$U_{cn}(X_1)$ = function defined in (17), dimensionless

$U_{sn}(X_1)$ = function defined in (17), dimensionless

$V$ = reference velocity, $ms^{-1}$

$V_{ji}$ = $v_{ji}/V$, nondimensional velocity at $x_2 = 0$ $(j = 1)$ and $x_2 = h$ $(j = 2)$, parallel to $x_i$ $(i = 1,2,3)$

$V^*_i$ = $v^*_i/V$ = nondimensional value of $v^*_i$

$V^*_{ji}$ = values of $V^*_i$ for $x_2 = 0$ $(j = 1)$ and $x_2 = h$ $(j = 2)$

$v_{ji}$ = velocity parallel to $x_i$ axis, at $x_2 = 0$ $(j = 1)$, and $x_2 = h$ $(j = 2)$, $ms^{-1}$

$v^*_i$ = integral defined in (3), $VAN^{-1}(ms^{-1})$

$v^*_{ji}$ = values of $v^*_i$ at $x_2 = o, j = 1$ and $x_2 = h, j = 2$, $VAN^{-1}(ms^{-1})$

$X_i$ = $x_i/b$ = nondimensional coordinate, $i = 1,3$

$X_2$ = $x_2/h_2$ = nondimensional coordinate

$X_{ij}$ = abscissas of boundaries of bearing's load carrying zone $(j = 1,2)$, dimensionless

$X^*_{1j}$ = abscissa of the upstream ($j = 1$) and downstream ($j = 2$) boundary of $P^*$, dimensionless

$x_1$ = coordinate parallel to the relative velocity of solid surfaces, m

$x_2$ = coordinate perpendicular to a solid surface, m

$x_3$ = coordinate perpendicular to $x_1$ and $x_2$, m

$Y_n(X_1)$ = function standing either for $U_{cn}(X_1)$ or $U_{sn}(X_1)$, dimensionless

$\alpha$ = constant defined in (33), dimensionless

$\beta$ = constant defined in (33), dimensionless

$\gamma_i$ = constants defined in (33), $i = 1,2$, dimensionless

$\varepsilon = \frac{e}{c}$ = bearing eccentricity ratio, dimensionless

$\theta$ = angle measured from the centerline, dimensionless

$\lambda$ = bearing length-width (active zone extent) ratio; bearing length-journal radius ratio, for journal bearings, dimensionless

$\mu$ = viscosity, Pas

$\mu_0 = 4\pi \cdot 10^{-7}$ = free space permeability, $Hm^{-1}$

$\mu_{1H}$ = reference viscosity under an applied magnetic field $H$, Pas

$\xi$ = variable defined in (31), dimensionless

$\xi_1$ = variable defined in (20), dimensionless

$\xi_{1j}$ = value of $\xi_1$ for $X_1 = X_{1j}$ ($j = 1,2$), dimensionless

$\sigma(T)$ = temperature function shown in (7), $A^{2-n}Tm^{n-2}$

$\sigma_0$ = constant value of $\sigma(T)$, $A^{2-n}Tm^{n-2}$

## I. INTRODUCTION

Although the importance of ferromagnetic fluids in the areas of sealing problems, reduction of friction, energy conversion, levitation and other applications has often been emphasized, the lubrication with ferrofluids has not been treated to any appreciable extent. Some work recently carried out at the GM Research Laboratories has shown that bearings lubricated with ferrofluids may offer very attractive advantages. Therefore, a complete theory of lubrication with fluids carrying magnetic particles has been formulated.

In a former paper [1]*, the general momentum equation for ferrofluids was developed by introducing the magnetic stresses. Under the assumption that the pressure is constant across the film, some compatiblity conditions for the magnetic effects were discussed and the velocity profiles were obtained. Furthermore, a substitute for the usual Reynolds pressure differential equation was developed. Since the former analysis was particularly applied with the short bearing approximation, we will now extend the available theory first to infinitely long bearings, then to general tridimensional lubrication case.

The integration of the pressure equation yields the solution of the problem. Let us first consider the pressure distribution $P^{(0)}$ in a nonmagnetic lubricant with the same viscosity as that of the ferrofluid, but without the effect of magnetic stresses. It was earlier shown that this viscosity, $\mu_{1H}$, is that of the base fluid, increased by the effects of ferrite particles in suspension and by their orientation due to the magnetic field. The total pressure $P = P^{(0)} + P^*$, where $P^*$ represents the contribution of magnetic stresses, is then given by the equation [1]

$$\frac{b}{h_2}(V_{22}-V_{12}) + \frac{\partial}{\partial X_i}\left[(V_{2i}-V_{1i}+V^*_{2i}-V^*_{1i})\int_0^H A\,dX_2\right] + H\frac{\partial V_{1i}}{\partial X_i} - (V_{2i}-V_{1i})\frac{\partial H}{\partial X_i}$$
$$- \frac{\partial}{\partial X_i}\int_0^H V^*_i\,dX_2 + \frac{\partial(HV^*_{1i})}{\partial X_i} + \frac{\partial}{\partial X_i}\left(\frac{\partial P}{\partial X_i}\int_0^H B\,dX_2\right) = 0; \quad i = 1,3, \qquad (1)$$

where

*Numbers in brackets designate references at end of paper.*

$$A = \left\{ \int_0^{x_2} \frac{dX_2}{M} \right\} \Big/ \left\{ \int_0^{H} \frac{dX_2}{M} \right\}; \quad A(H) = 1$$

$$B = 6\left\{ \int_0^{x_2} \frac{X_2 dX_2}{M} - \left[ \left( \int_0^{H} \frac{X_2 dX_2}{M} \right) \Big/ \left( \int_0^{H} \frac{dX_2}{M} \right) \right] \int_0^{x_2} \frac{dX_2}{M} \right\}; \quad B(H) = 0. \tag{2}$$

Because this equation preserves the linear character with respect to P, one can consider separately the known solution $P^{(0)}$, for nonmagnetic fluids, and $P^*$ which fulfills the part of Eq. (1) where the pressure and the magnetic effects are involved. Therefore, with

$$I_{0i} = \mu_0 \int M^* \frac{\partial H}{\partial x_i} dx_2; \quad v_i^* = \int \frac{I_{0i}}{\mu} dx_2;$$
$$v_{1i}^* = v_i^* \quad (x_2 = 0); \quad v_{2i}^* = v_i^* \quad (x_2 = h) \tag{3}$$

where $v_i^*$ represents the velocity components induced by the magnetic field, the differential equation in $P^*$, derived from (1), is

$$\frac{\partial}{\partial X_i}\left[ (V_{2i}^* - V_{1i}^*) \int_0^H A dX_2 - \int_0^H V_i^* dX_2 + HV_{1i}^* + \frac{\partial P^*}{\partial X_i} \int_0^H B dX_2 \right] = 0; \quad i = 1,3. \tag{4}$$

The subscript $i = 1,3$ means summation in Eqs. (1) and (4).

## II. THE BIODIMENSIONAL PROBLEM

For unidirectional flow, $i = 1$, $\partial P^*/\partial X_1 = dP^*/dX_1$, Eq. (4) yields

$$P_\infty^* = \int \frac{\int_0^H [V_i^* - (V_{2i}^* - V_{1i}^*) A] dX_2 - HV_{1i}^* + C_1}{\int_0^H B dX_2} dX_1 + C_2. \tag{5}$$

Let us now assume both magnetization $M^*$ and field $H$ as independent of $x_2$. This is a reasonable hypothesis due to the smallness of the gap between the journal and the bearing: however, a certain approximation is introduced, due to the temperature profile across the film, while the magnetization generally varies with the temperature. As shown earlier [1], the field gradient across the film may be considered as negligible when compared to the derivatives $\partial H/\partial x_i$ (i = 1,3) and, for certain magnetic fluids, the pyromagnetic coefficient vanishes. Therefore, the components of the magnetic induced velocity are

$$V^*_i = \frac{\mu_0 M^* h_2^2}{2\mu_1 Vb}\frac{\partial H}{\partial X_i}X_2^2;\quad V^*_{1i} = 0;\quad V^*_{2i} = \frac{\mu_0 M^* h_2^2}{2\mu_1 Vb}\frac{\partial H}{\partial X_i}H^2;\ i = 1,3. \tag{6}$$

In a limited range of field magnitudes, the magnetization may be expressed as [2]

$$M^* = \sigma(T)H^{n-1} = \sigma_0 H^{n-1}, \tag{7}$$

with $\sigma(T) = \sigma_0$ = constant under the previous assumptions. At the saturation point $M^* = \sigma_0$ = const and $n = 1$.

In further developments Eq. (7) may be used, or the more general expression,

$$M^* = \sigma_0 f(H) \tag{8}$$

and with $i = 1,3$ standing for summation

$$\int M^* \frac{\partial H}{\partial X_i} dX_i = \int M^*(H)dH = M^*(H) + C_M, \tag{9}$$

where $C_M$ = const.

For constant viscosity, $M = 1$, and formula (5) becomes

$$P^* = \int \frac{\frac{\mu_0\sigma_0 h_2^2}{2\mu_1 Vbn}\int_0^H (X_2-H)X_2\frac{\partial H^n}{\partial X_1}dX_2 + C_1}{-\frac{1}{2}H^3}dX_1 + C_2. \tag{10}$$

The boundary conditions for the solution $P^*_\infty$ are

$$\begin{aligned} X_1 &= X_{11};\quad H = H_1;\quad P^*_\infty = 0,\\ X_1 &= X_{12};\quad H = H_2;\quad P^*_\infty = 0. \end{aligned} \tag{11}$$

Introducing now the value of $M^*$ (7) in Eq. (10), we obtain the values of $C_1$ and $C_2$ and finally

$$P^*_\infty = \frac{\mu_0\sigma_0 h_2^2}{6\mu_1 Vbn}\left[H^n - H_1^n - (H_2^n - H_1^n)\frac{\int_{x_{11}}^{x_1}\frac{dX_1}{H^3}}{\int_{x_{11}}^{x_{12}}\frac{dX_1}{H^3}}\right]. \qquad (12)$$

If $H_2 = H_1$, formula (12) simply becomes

$$P^*_\infty = \frac{\mu_0\sigma_0 h_2^2}{6\mu_1 Vbn}(H^n - H_1^n), \qquad (13)$$

which, as already found, shows that $P^*_\infty > 0$ if $H > H_1$, i.e., a supplementary magnetic pressure is superposed to the pressure built-up by viscous effects, if the field inside the active zone is higher than that at the boundaries.

A similar solution may be obtained, under the same assumptions, for the tridimensional problem. Equations (1) and (6) yield

$$\frac{\partial}{\partial X_i}\left[H^3\left(\frac{\mu_0 M^* h_2^2}{6\mu_1 Vb}\frac{\partial H}{\partial X_i} - \frac{\partial P^*}{\partial X_i}\right)\right] = 0; \quad i = 1,3, \qquad (14)$$

where $i$ also shows summation. Let us introduce a new function U

$$\frac{\mu_0 h_2^2}{6\mu_1 Vb}M^* - P^* = U. \qquad (15)$$

This substituion is always possible if the value of $M^*$ as given by formulae (7) or (8) is valid, and the viscosity is constant. Hence, from Eq. (14), we deduce

$$\frac{\partial}{\partial X_i}(H^3\frac{\partial U}{\partial X_i}) = 0; \quad i = 1,3 \qquad (16)$$

Using a procedure developed in [3] and [4], U is assumed to be of the form

$$U = U_0(X_1) + \sum_{n=1}^{\infty}[U_{cn}(X_1)\cos\frac{n\pi}{\lambda}X_3 + U_{sn}(X_1)\sin\frac{n\pi}{\lambda}X_3] \qquad (17)$$

Introducing the expression (17) of $U$ in Eq. (16) and equating to $0$ the coefficients of $\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\pi X_3/\lambda$, the following equations are obtained, if $H$ depends only on $X_1$

$$\frac{d}{dX_1}\left(H^3\frac{dU_0}{dX_1}\right) = 0; \quad \frac{d}{dX_1}\left(H^3\frac{dY_n}{dX_1}\right) - \frac{n^2\pi^2}{\lambda^2}H^3Y_n = 0. \tag{18}$$

The first relationship corresponds to the equation already analyzed of one dimensional flow, while the second expression is valid for every function $Y_n = U_{cn}$ and $Y_n = U_{sn}$.

## III. SLIDER BEARINGS OF FINITE LENGTH

We consider a linear variation of film thickness, i.e.,

$$H = H_1 - \frac{H_1-H_2}{X_{12}-X_{11}}(X_1-X_{11}); \quad X_{11} \le X_1 \le 1 + X_{11}, \tag{19}$$

where $\quad - X_{11} = 1, \; H_2 = 1.$ Let us introduce a new variable

$$\xi_1 = \frac{\pi}{\lambda}\frac{X_{12}-X_{11}}{H_1-H_2}H = \frac{\pi}{\lambda}\frac{H}{H_1-1}; \tag{20}$$

he last uation of (18) may be written as

$$\frac{d^2Y_n}{d\xi_1^2} + \frac{3}{\xi_1}\frac{dY_n}{d\xi_1} - n^2Y_n = 0 \tag{21}$$

This is a Bessel equation whose solution is, with $A_n$, $B_n$ arbitrary cons ts and $I_1(n\xi_1)$, $K_1(n\xi_1)$ the modified Bessel functions of irst order,

$$Y_n = \frac{1}{\xi_1}[A_nI_1(n\xi_1) + B_nK_1(n\xi_1)] \tag{22}$$

Introduc the solutions (22) in Eq. (17), and also using the value of $U_0(\;)$ as given by the first Eq. (A.3), we obtain from Eq. (15)

$$P^* = \frac{\mu_0 h_2^2}{6\mu_1 Vb} M^* - \frac{1}{2(H_1-1)} \frac{\overline{C}_1}{[H_1-(H_1-1)(X_1-X_{11})]^2} - C_2$$
$$+ \sum_{n=1}^{\infty} \frac{1}{\xi_1} \Big\{ [A_{cn} I_1(n\xi_1) + B_{cn} K_1(n\xi_1)] \cos \frac{n\pi}{\lambda} X_3$$
$$+ [A_{sn} I_1(n\xi_1) + B_{sn} K_1(n\xi_1)] \sin \frac{n\pi}{\lambda} X_3 \Big\} \tag{23}$$

with

$$\xi_1(X_1) = \frac{\pi}{\lambda}\left(\frac{H_1}{H_1-1} + X_{11} - X_1\right); \quad U_0(\xi_1) = \frac{\pi^2}{2\lambda^2 (H_1-1)^3} \frac{1}{\xi_1^2} \overline{C}_1 + \overline{C}_2. \tag{24}$$

The constants $\overline{C}_1$, $\overline{C}_2$, $A_{cn}$, $B_{cn}$, $A_{sn}$, $B_{sn}$ are determined such that $P^*$ matches the required values at the boundaries. Generally speaking, so far as an infinite number of constants are available, any kind of pressure variation upon a given curve can be represented, if $P^*$ is continuous and finite. However, the physical conditions are already fulfilled by $P^{(0)}$, the solution for nonmagnetic films, such that $P^* = 0$ at the boundaries, i.e.,

$$\xi_1 = \xi_{11} = \frac{\pi}{\lambda} \frac{H_1}{H_1-1}; \quad \xi_1 = \xi_{12} = \frac{\pi}{\lambda}\left(\frac{H_1}{H_1-1}+1\right); \quad P^* = 0$$
$$X_3 = \pm \frac{\lambda}{2}; \quad P^* = 0. \tag{25}$$

Furthermore $M^*$ is a known function given by (A.1) with the usual procedure to obtain the Fourier coefficients (see Appendix A, formulas (A.6)). Now, keeping in mind that the pressure distribution is symmetric with respect to the plane $OX_1X_2$, also considering the last condition (25), then Eq. (23) must contain only odd cosine terms such that $n$ shall be replaced by $2n - 1$ and all coefficients $A_{sn} = B_{sn} = 0$. Hence, the first boundary condition (25) yields the values of $A_{c\,2n-1}$ and $B_{c\,2n-1}$ as follows (Appendix B)

$$A_{c2n-1} = \frac{\dfrac{\mu_0 h_2^2}{6\mu_1 Vb}\left[\xi_{12} A_{c2n-1}(\xi_{12}) - \xi_{11} \dfrac{K_1\{(2n-1)\xi_{12}\}}{K_1\{(2n-1)\xi_{11}\}} A_{c2n-1}(\xi_{11})\right]}{\dfrac{K_1\{(2n-1)\xi_{12}\}}{K_1\{(2n-1)\xi_{11}\}} I_1\{(2n-1)\xi_{11}\} - I_1\{(2n-1)\xi_{12}\}};$$

$$B_{c2n-1} = \frac{\dfrac{\mu_0 h_2^2}{6\mu_1 Vb}\left[\xi_{12}A_{c2n-1}(\xi_{12}) - \xi_{11}\dfrac{I_1\{(2n-1)\xi_{12}\}}{I_1\{(2n-1)\xi_{11}\}}A_{c2n-1}(\xi_{11})\right]}{\dfrac{I_1\{(2n-1)\xi_{12}\}}{I_1\{(2n-1)\xi_{11}\}}K_1\{(2n-1)\xi_{11}\} - K_1\{(2n-1)\xi_{12}\}} \qquad (26)$$

The constants $\overline{C}_1$, $\overline{C}_2$ are obtained as shown in Appendix A, therefore both $A_{c\ 2n-1}$ and $B_{c\ 2n-1}$ are fully determined.

## IV. JOURNAL BEARINGS OF FINITE LENGTH

The variation of $H$ as a function of the position angle with respect to the centerline is

$$H = (1+\varepsilon\cos\theta)/(1-\varepsilon);\quad H_1 = (1+\varepsilon)/(1-\varepsilon);\quad H_2 = 1;\quad \varepsilon = e/c. \qquad (27)$$

With the journal radius as a reference length, we have $\theta = X_1$. The value of $U_0$ as given by (A.3) and formula (27) is

$$U_0(X_1) = \overline{C}_1\left[\frac{2+\varepsilon^2}{(1+\varepsilon)^2}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan^{-1}\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{X_1}{2}\right) - \frac{\varepsilon(1-\varepsilon)}{2(1+\varepsilon)}\frac{\sin X_1}{1+\varepsilon\cos X_1}\left(\frac{1-\varepsilon}{1+\varepsilon\cos X_1}+\frac{3}{1+\varepsilon}\right)\right] + \overline{C}_2. \qquad (28)$$

The constants $\overline{C}_1$, $\overline{C}_2$ are obtained from (A.5), where $I(X_1)$ is the term in brackets in (28).

## V. BOUNDARY CONDITIONS

Unlike slider bearings with surrounding ambient pressure, the boundaries $X_{11}$ and $X_{12}$ are to be calculated as being the loci where the combined effect of magnetic and hydrodynamic pressure, inside the film, reaches the value $P_0$. Because magnetic and hydrodynamic stresses are uncoupled, the hydrodynamic pressure is built-up independently of the applied magnetic field. Therefore, the contribution of viscous forces remains unchanged when $H$ is constant or zero. We can then assume the integration constants of the hydrodynamic solution as known and the hydrodynamic pressure distribution as well. Thus, only the constants of the solution for the magnetic pressure have to be determined.

For hydrodynamic solutions, the Sommerfeld boundary conditions, $P^{(0)}(X_1) = P^{(0)}(2\pi+X_1)$ allowing for negative hydrodynamic pressure are most likely to occur. The Swift-Stieber conditions, $P^{(0)} = \partial P^{(0)}/\partial X_1 = 0$ at the exit, do not seem appropriate for, when combined with positive magnetic pressure, they actually eliminate the exit point, allowing overall positive pressure. Also the Swift-Stieber conditions, applied to the resultant pressure, have the effect of coupling hydrodynamic and magnetic effects by boundary conditions, a situation which physically seems incorrect. Moreover, the position of the inlet should not be dependent upon the magnetic stresses, which in journal bearings may exist upon all the bearing circumference.

Keeping in mind that the magnetic pressure $P^*$ is generated by the variation of the magnetic field (14) and that $M^* = \text{const.}$ if $\partial H/\partial X_i = 0$, $(i = 1,3)$ then $P^*$ is constant too, as shown in Appendix A. Henceforth, if $M^* = M_1^* = \text{const.}$ for $X_1 \leq X_{11}^*$, then $P^* = P_1^*$ and if $M^* = M_2^* = \text{const.}$ for $X_1 \geq X_{12}^*$, we have $P^* = P_2^*$ as shown in Fig. 1. The initial magnetic pressure may be assumed $P_1^* = 0$, because no magnetic effects occur ahead of $X_{11}^*$,

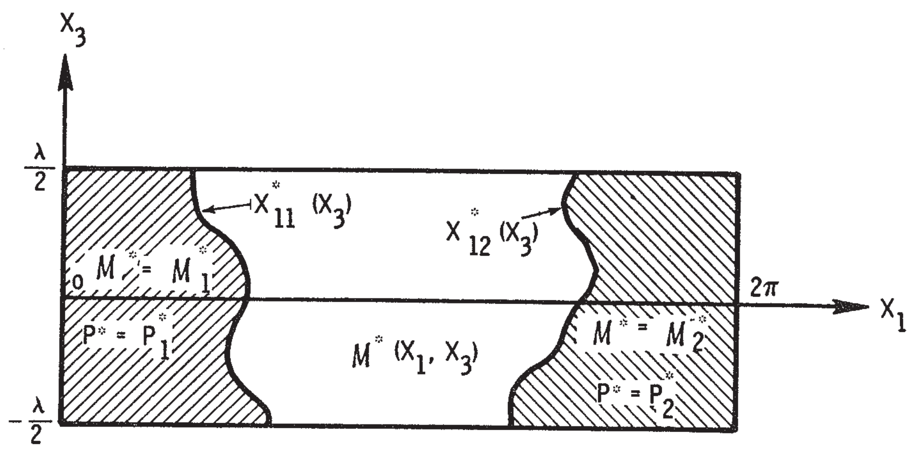


*Fig. 1. Developed bearing surface, showing the regions with $M^*$ variable and $M^* = M_j^*$ = constant, j = 1,2. The curves $X_{1j}(X_3)$ are the boundaries of those regions.*

while $P_2^*$ might have a certain value when $M_1^* \neq M_2^*$. In journal bearings, if $M^*$ is a continuous periodic function, we have $X_{12}^* = X_{11}^* + 2\pi$, $M_1^* = M_2^*$ ($M_1^*$ means then the minimum value of $M^*$ with respect to $X_1$) and $P_1^* = P_2^*(X_3)$. Obviously, when $P_2^*$ does not depend on $X_3$, then, along a line $X_{11}^* = X_{12}^*(X_3)$, $P_1^* = P_2^* = P^*(X_1, \pm\lambda/2) = 0$.

From the previous considerations, the boundary conditions for journal bearings may be written as follows

$$P^*(X_1, \frac{\lambda}{2}) = P^*(X_1, -\frac{\lambda}{2}) = 0;$$

$$P^*[X_{11}^*(X_3)] = 0;\ P^*[X_{12}^*(X_3)] = P_2^*,\ \text{or}\ P^*[X_{11}^*(X_3)+2\pi] = P^*[X_{11}^*(X_3)]; \tag{29}$$

$$P^{(0)}(X_{11}, X_3) + P^*(X_{11}, X_3) = P^{(0)}(X_{12}, X_3) + P^*(X_{12}, X_3) = P_0.$$

VII. THE PRESSURE DISTRIBUTION IN FINITE JOURNAL BEARINGS

The Solution $Y_n$ of the second Eq. (18) is similar to that obtained by Tao [5]. Introducing the value of $H$ from Eq. (27), we have

$$\frac{d^2Y_n}{dX_1^2} - \frac{3\varepsilon \sin X_1}{1+\varepsilon \cos X_1} \frac{dY_n}{dX_1} - \frac{n^2\pi^2}{\lambda^2} Y_n = 0. \tag{30}$$

We set

$$X_1 = 2 \sin^{-1}\sqrt{\xi}\ ; \quad a = \frac{1+\varepsilon}{2\varepsilon}, \tag{31}$$

to obtain from (30) the Heun equation

$$\xi(\xi-1)(\xi-a)\frac{d^2Y_n}{d\xi^2} + [\frac{a}{2} - (\frac{7}{2}+a)\xi + 4\xi^2]\frac{dY_n}{d\xi} + \frac{n^2\pi^2}{\lambda^2}(\xi-a)Y_n = 0. \tag{32}$$

Using the notations [6]

$$\alpha = \frac{3}{2}(1+\sqrt{1-(\frac{2n\pi}{3\lambda})^2}); \quad \beta = \frac{3}{2}(1-\sqrt{1-(\frac{2n\pi}{3\lambda})^2});$$

$$\gamma_1 = \frac{1}{2}; \quad \gamma_2 = \frac{3}{2}; \quad q_1 = \frac{n^2\pi^2}{\lambda^2}a; \quad q_2 = \frac{n^2\pi^2}{\lambda^2}a + \frac{3}{2}, \tag{33}$$

the solution of Eq. (32) may be written as

$$Y_n = A_n F_{1n}(a,q_1,\alpha,\beta,\gamma_1,\frac{1}{2},\xi) + B_n \xi^{1-\gamma_2} F_{2n}(a,q_2,\alpha,\beta,\gamma_2,\frac{1}{2},\xi). \tag{34}$$

The functions $F_{in}$ are expressed in terms of power series, which for $a > 1$ and $\xi < 1$ (31) are always convergent. Therefore [6]

$$F_{in} = 1 + \sum_{m=1}^{\infty} c_m^{(i)} \xi^m; \quad i = 1,2, \tag{35}$$

with

$$c_1^{(i)} = \frac{q_i}{a\gamma_i};$$

$$c_2 = \frac{[a(\gamma_i + \frac{1}{2}) + \alpha + \beta + \frac{1}{2} + q_i]\frac{q_i}{a\gamma_i} - \alpha\beta}{2a(\gamma_i + 1)}$$

$$a(m+1)(\gamma_i+m)c_{m+1}^{(i)} = [a(\gamma_i + m - \frac{1}{2}) + \alpha + \beta - \frac{1}{2} + m + \frac{q_i}{m}]mc_m^{(i)}$$

$$- [(m-1)(m-2) + (m-1)(\alpha+\beta+1) + \alpha\beta]c_{m-1}^{(i)}. \tag{36}$$

It may be observed from (33) that

$$\alpha + \beta = 3; \quad \alpha\beta = \frac{n^2\pi^2}{\lambda^2}, \tag{37}$$

and the coefficients $c_m^{(i)}$ are real, although $\alpha$ and $\beta$ may be conjugate complex numbers. The recurrence formulas (36) yield directly the $c_m^{(i)}$ - s, such that a simple computer program can be used to generate the Heun functions $F_{in}$ (35) with any desired accuracy.

The solution is similar to (23) and, by using the relationships (A.1) and (28), it may be written as

$$P^* = \frac{\mu_0 h_2^2}{6\mu_1 Vb} A_0(X_1) - U_0(X_1) + \sum_{n=1}^{\infty} [A_{2n-1}F_{1,2n-1}(x_1)$$

$$+ B_{2n-1}\sin\frac{X_1}{2}F_{2,2n-1}(X_1) + \frac{\mu_0 h_2^2}{6\mu_1 Vb}A_{c2n-1}(X_1)]\cos(2n-1)\pi\frac{X_3}{\lambda}. \quad (38)$$

The conditions (29) yield the constants $A_{2n-1}$, $B_{2n-1}$, by a procedure already shown in Appendix B for slider bearings:

$$A_{2n-1} = \frac{\mu_0 h_2^2}{6\mu_1 Vb} \times$$

$$\frac{\sin\frac{X^*_{12}}{2}F_{2,2n-1}(X^*_{12})A_{c2n-1}(X^*_{11}) - \sin\frac{X^*_{11}}{2}F_{2,2n-1}(X^*_{11})A_{c2n-1}(X^*_{12})}{\sin\frac{X^*_{11}}{2}F_{2,2n-1}(X^*_{11})F_{1,2n-1}(X^*_{12}) - \sin\frac{X^*_{12}}{2}F_{2,2n-1}(X^*_{12})F_{1,2n-1}(X^*_{11})};$$

$$(39)$$

$$B_{2n-1} = \frac{\mu_0 h_2^2}{6\mu_1 Vb} \times$$

$$\frac{F_{1,2n-1}(X^*_{11})A_{c2n-1}(X^*_{12}) - F_{1,2n-1}(X^*_{12})A_{c2n-1}(X^*_{11})}{\sin\frac{X^*_{11}}{2}F_{2,2n-1}(X^*_{11})F_{1,2n-1}(X^*_{12}) - \sin\frac{X^*_{12}}{2}F_{2,2n-1}(X^*_{12})F_{1,2n-1}(X^*_{11})}.$$

In the above formulas $X^*_{11}$, $X^*_{12}$, are the values of $X_1$ limiting the domain of variation of $M^*$, since constant $M^*$ values do not generate magnetic pressure $P^*$, as shown by Eq. (14) and Appendix A. The particular case $P^* = 0$ for both $X^*_{11}$, $X^*_{12}$ at the ends of the interval of variation of $M^*$ was considered in (A.5) to obtain $\bar{C}_1$, $\bar{C}_2$. More generally, if $M^*_1$, $M^*_2$ (the initial and final values of $M^*$) are not equal, then the constants $\bar{C}_1$, $\bar{C}_2$ in (28) should include this effect, by adding the terms $P^*_2/(\int_{x_{11}}^{x_{12}} dX_1/H^3)$ to $\bar{C}_1$ and $P^*_2 I(X_{11})/(\int_{x_{11}}^{x_{12}} dX_1/H^3)$ to $\bar{C}_2$ (here $P^*_2 = (\mu_0 h_2^2/6\mu_1 Vb)(M^*_2 - M^*_1)$).

The curves in Fig. 2 show, for a given value of $X_3$, the variation of the hydrodynamic pressure $P^{(0)}$, of the magnetic pressure $P^*$, with the abscissas $X^*_{11}$, $X^*_{12}$ limited by the constant values of $M^*$ ($M^*_1$ and $M^*_2$) and the resultant curve $P = P^{(0)} + P^*$. It may be observed that the boundary conditions $P = P_0$, (29), occur at $X_{11}$ and $X_{12}$, which are not known, a priori. These abscissas must be found after addition of $P^{(0)}$ and $P^*$, and they show, as in a previous paper [1], the increase of the active area of the bearing, due to the variation of the magnetic field, while the pressure is higher than that produced only by viscous stresses. Hence, the actual boundary conditions are a result of hydrodynamic and magnetic effects, and they cannot be anticipated, nor used in solving the problem, as in other more usual situations met in lubrication theory.

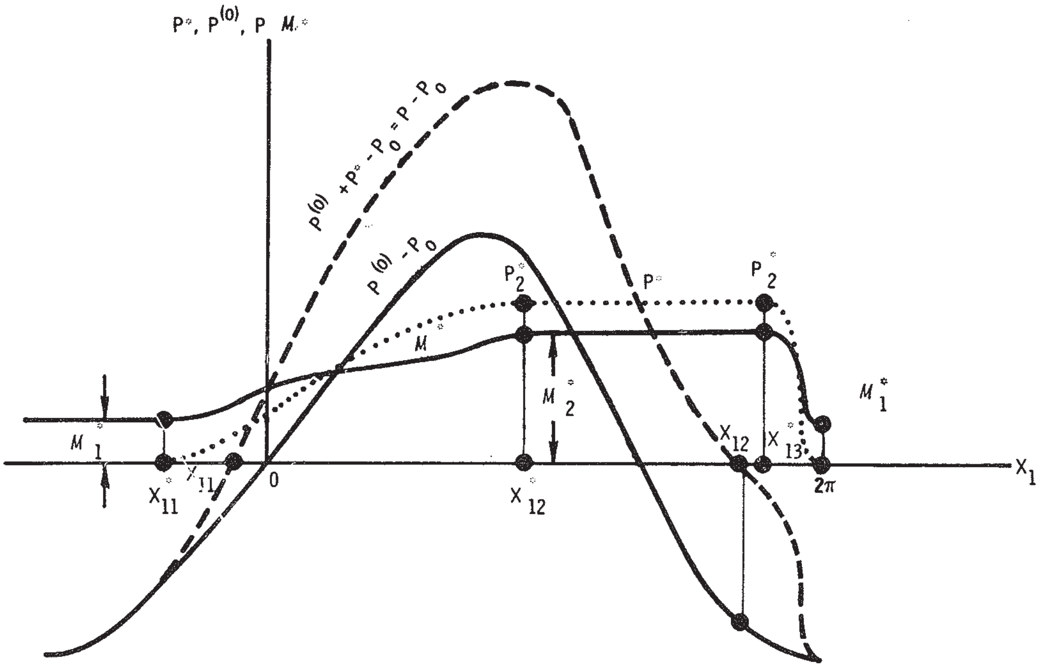


*Fig. 2. Curves $P^{(0)}(X_i)$, $P^*(X_i)$ and $P(X_i)$ showing hydrodynamic, magnetic and resultant pressure variation at $X_3$ = const. The abscissas $X^*_{1j}$ (j = 1,2,3) separate zones of variable and constant $M^*$; $X_{11}$, $X_{12}$ are the boundary values of $X_1$ for the total pressure.*

A more general case is when $P_1^*$ and $P_2^*$ are not located at constant abscissas, but along curves $X_{11}^* = f_1(X_3)$ and $X_{12}^* = f_2(X_3)$. A procedure similar to that already discussed may be used, by introducing the above relationships in (38) while $P^* = 0$ or $P^* = P_2^*$. By taking N discrete values $X_{11}^*$ and $X_{12}^*$, Eq. (38) yields a system of 2N equations with 2N variables $A_{2n-1}$, $B_{2n-1}$ $(1 \leq n \leq N)$, to be solved.

## VIII. CONCLUSIONS

1. Magnetic and viscous effects can be studied independently, due to the linearity of the governing equations.

2. For the unidimensional flow (infinitely long bearings), the two-dimensional magnetic pressure field depends on both film thickness and magnetic field. However, when the initial and final magnitudes of the magnetic field are equal, then the magnetic pressure is independent with respect to film thickness variation. In all cases, simple formulae are obtained for the pressure distribution.

3. The tridimensional problem is solved, both for slider and journal bearings. Analytic formulae and expansions allow easy computer programs for numerical applications. The solutions are expressed in Bessel or Heun functions.

4. The boundary conditions for the resultant pressure (combined magnetic and hydrodynamic effects) are not known, a priori, and depend on the intensity and distribution of the magnetic field and magnetization.

5. The Sommerfeld boundary conditions for the hydrodynamic pressure shall be used, in connection with the specific conditions for the magnetic field, to obtain the real extent of the bearing active area.

6. Magnetic pressure is generated only over the surfaces where a variation of the magnetic field occurs. The geometry of those surfaces is a controlling factor for the solution of the tridimensional lubrication problem and for the mathematical formulation of its boundary conditions.

## REFERENCES


[1] Tipei, N. (1975). "Theory of lubrication with ferrofluids: Application to short bearings." General Motors Research Report EM-269.

[2] Neuringer, J. L., and Rosensweig, R. E. (1969). "Ferrohydrodynamics." *The Physics of Fluids, Vol. 7, No. 12,* 1927.

[3] Michell, A. G. M. (1905). "The lubrication of plane surfaces." *Z. Math. u. Physik 132.*

[4] Fernlund, Ingemar (1962). "On solving Reynolds equation for bearings of finite width with Fourier sine transforms." Report No. 20, Institute of Machine Elements, Chalmers University of Technology, Scandinavian University Books, Akademiförlaget-Gumperts, Göteborg.

[5] Tao, L. N. (1959). "On journal bearings of finite length with variable viscosity." ASME Trans., *Journal of Applied Mechanics.*

[6] Kamke, E. (1948). *Differential Gleichungen Lösungsmethoden und Lösungen.* Bd. 1, Chelsea Publishing Company, NY.


## APPENDIX A

Considering $A_0(X_1) = M^*(X_1, \pm\lambda/2)$, i.e., the part of $M^*$ (see Formula (9)) independent of the coordinate $X_3$, then we can set

$$M^*(X_1,X_3) = A_0(X_1) + \sum_{n=1}^{\infty} [A_{c2n-1}(X_1)\cos(2n-1)\pi\frac{X_3}{\lambda} + A_{s2n}(X_1)\sin 2n\pi\frac{X_3}{\lambda}], \tag{A.1}$$

which shows that for an infinitely long bearing, $\lambda = \infty$, $M^* = M^*_\infty$.

$$M^*_\infty(X_1) = A_0(X_1) + \sum_{n=1}^{\infty} A_{c2n-1}(X_1). \tag{A.2}$$

Assuming now $M^*_\infty(X_1,X_2)$ is symmetric with respect to the plane $X_3 = 0$ (middle of the bearing) we have $A_{s2n} \equiv 0$. Equation (15) and the first Eq. (18) yield

$$U_0(X_1) = \overline{C}_1 \int \frac{dX_1}{H^3} + \overline{C}_2 = \frac{\mu_0 h_2^2}{6\mu_1 Vb} A_0(X_1) - \overline{P}^*(X_1). \tag{A.3}$$

Since the last member of (A.3) does not depend on $X_3$, and $H$ as well, then both $\overline{C}_1$, $\overline{C}_2$ are constants. For $X_3 = \pm\lambda/2$, Eq. (A.1) shows that $A_0(X_1)$ and $\overline{P}^*(X_1)$ actually represent the effect of the magnetic field along the borders of the bearing parallel to $OX_1$ axis, $\overline{P}^*(X_1) = P^*(X_1, \pm\lambda/2)$.

If the pressure at $X_3 = \pm\lambda/2$ is constant, $P = P_0$, and assuming it is given by the usual solution for nonmagnetic fluids, then $\overline{P}^* = 0$ and

$$\frac{6\mu_1 Vb}{\mu_0 h_2^2} U_0(X_1) = A_0(X_1) = \frac{6\mu_1 Vb}{\mu_0 h_2^2}\left(C_1 \int \frac{dX_1}{H^3} + C_2\right) \tag{A.4}$$

The relationship (A.4) yields the condition that the distribution of $M^*$ (hence $M^*$ and $H$) must fulfill, in order that the particular boundary condition $P^*(\pm\lambda/2) = 0$ for pressure to be satisfied. In the general case, when $A_0(X_1)$ is not of the form shown in (A.4), then accordingly to (A.3), a certain pressure distribution over the boundaries $X_3 = \pm\lambda/2$ results, as a consequence of magnetic effects. Since $A_0(X_1)$ is known, the constants $\overline{C}_1$ and $\overline{C}_2$ are determined to match the pressure values at the inlet, $X_1 = X_{11}$, and outlet, $X_1 = X_{12} = X_{11} + 1$. When $\overline{P}^*(X_{11}) = \overline{P}^*(X_{12}) = 0$, we obtain

$$C_1 = \frac{\mu_0 h_2^2}{6\mu_1 Vb}\,\frac{A_0(X_{12}) - A_0(X_{11})}{I(X_{12}) - I(X_{11})};$$

$$C_2 = \frac{\mu_0 h_2^2}{6\mu_1 Vb}\,\frac{A_0(X_{11})I(X_{12}) - A_0(X_{12})I(X_{11})}{I(X_{12}) - I(X_{11})};$$

$$I(X_1) = \int \frac{dX_1}{H^3}. \tag{A.5}$$

In practice, $A_0$ is often either zero or a constant, therefore $\overline{C}_1 = 0$ and $\overline{C}_2 = \mu_0 h_2^2 A_0 / 6\mu_1 Vb$.

The functions $A_{c2n-1}$, $A_{s2n}$ (n integer) are obtained by the usual procedure, with $M^*(X_1, X_3)$ a known function:

$$A_{c2n-1}(X_1) = \frac{1}{\lambda}\int_0^{2\lambda} M^*(X_1, X_3)\cos(2n-1)\pi\frac{X_3}{\lambda}dX_3;$$

$$A_{s2n}(X_1) = \frac{1}{\lambda}\int_0^{2\lambda} M^*(X_1, X_3)\sin 2n\pi\frac{X_3}{\lambda}dX_3. \qquad (A.6)$$

APPENDIX B

We use the relationships (A.4), (A.5), and introduce the development (A.1) in Eq. (23). Afterwards, we let the coefficients of $\cos(2n-1)\pi X_3/\lambda$ vanish, when $X_1 = X_{11}$ and $X_1 = X_{12} = X_{11} + 1$, for $n = 1, 2, \ldots, \infty$, in order to satisfy the first condition (25). At each $n$ value two independent equations are obtained,

$$\frac{\mu_0 h_2^2}{6\mu_1 Vb} A_{c2n-1}(\xi_{1i}) + \frac{1}{\xi_{1i}}[A_{c2n-1} I_1\{(2n-1)\xi_{1i}\}$$
$$+ B_{c2n-1} K_1\{(2n-1)\xi_{1i}\}] = 0; \ i = 1, 2. \qquad (B.1)$$

Solving the system (B.1), gives the values of $A_{c2n-1}$ and $B_{c2n-1}$, as shown by formulas (26).