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ON POLYNOMIAL REACHABILITY OF DISCRETE LINEAR CONTROL SYSTEMS

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1. Abstract

Discrete linear control systems are defined. Definition of reachability is given. Then it is shown that for every reachable discrete control system, there exists at least one polynomial control function which takes the system from any given initial state to any pre-determine final state in a finite time.

2. Introduction

In many system of practical importance the variables are known, or are of interest, only at discrete intervals of time kT, k = 0,1,2,... . Often this will arise from sampling a continuous situation. We shall write x(k), u(k) to denote the values of state and control function at time kT, where k = 0, 1, 2,... Then, the differential model is replaced by the linear difference system

(1) $$x(k+1) = A\,x(k) + B\,u(k),$$

with

(2) $$x(0) = x_0\,, \quad x(N) = x_f,$$

where $x = [x_1, x_2, ..., x_n]^T$ is the n−vector discrete state function. A, B are arbitrary nxn and nxr constant matrices respectively, and u(k) is a r−vector discrete control function.

The main objective of this article is to show the existence of a discrete polynomial vector function for u(k), such that it takes the reachable difference system (1) from any given initial state $x(0) = x_0$ to any pre−assigned final state x_f, in a finite time NT.

First, let us write the general solution of system (1) (See Harnett S., Ref. No. 1, Page 76)

(3) $$x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B\,u(i)\,,$$

where A^k is the nxn transition (fundamental) matrix of $x(k+1) = A\,x(k)$, $x(0) = x_0$. Then, we state the following definition:

3. Definition: The discrete system (1) is <u>completely</u> <u>reachable</u> if for any final state x_f, there exists a finite time $t_f = NT$ and a control sequence $u(0)$, $u(1)$, $u(2)$,..., $u(N-1)$ such that if $x(0) = 0$, then $x(N) = x_f$

Before proceeding to the main theorem of this chapter, we state the following lemma whose result is needed in order to prove the theorem.

4. Lemma 1: The system (1) is completely reachable if and only if

(4) $$\text{rank}\,[B,\ AB,\ A^2B,\ \ldots,\ A^{n-1}] = n.$$

Proof: (See Barnett S., Ref. No. 1, Page91).

We can now proceed to the main result of this chapter.

5. Theorem: If the system (1) is completely reachable, then there exists a discrete polynomial vector fuction for control $u(k)$ of degree less than n, such that it takes the system (1) from $x(0) = x_o$ to any pre-assigned state of x_f in a finite time. Furthermore n intervals of length T are sufficient to fulfill the required time.

Proof: Let us assign an arbitrary discrete polynomial vector function of degree $n-1$ to the control function $u(k)$. Then,

(5) $$u(k) = a_0 + a_1 k + a_2 k^2 + a_3 k^3 + \ldots + a_{n-1} k^{n-1},$$

where a_i, $i = 0, 1, 2, \ldots, n-1$ are arbitrary constant r-vectors. Substituting (5) for $u(k)$ in (3) and let $k = n$, the following is obtained:

(6) $$x(n) = A^n x_o + \sum_{i=0}^{n-1} A^{k-i-1} B\, u(i).$$

Let

(7) $$G = x(n) - A^n x_o\ .$$

Combining (6) and (5) and writing the result in expanded form, we will get:

(8) $$\begin{aligned} G = {} & A^{n-1} B a_0 + A^{n-2} B (a_0 + a_1 + a_2 + \ldots + a_{n-1}) \\ & + A^{n-3} B (a_0 + 2 a_1 + 2^2 a_2 + \ldots + 2^{n-1} a_{n-1}) \\ & + A^{n-4} B (a_0 + 3 a_1 + 3^2 a_2 + \ldots + 3^{n-1} a_{n-1}) \\ & + \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ & + B (a_0 + (n-1) a_1 + (n-1)^2 a_2 + \ldots + (n-1)^{n-1} a_{n-1})\ . \end{aligned}$$

Using matrix notation for equation (8), we obtain the system

$$[\,A^{n-1}B,\ A^{n-2}B, \ldots, B]\,H\,a = G, \tag{9}$$

where $a = [\,a_0, a_1, \ldots, a_{n-1}]^T$ is a nr−vector, and H is the following nr×nr matrix:

$$H = \begin{bmatrix} I & 0 & 0 & . & 0 \\ I & I & I & . & I \\ I & 2\,I & 2^2\,I & . & 2^{n-1}I \\ . & . & . & . & . \\ I & (n-1)I & (n-1)^2I & . & (n-1)^{n-1}I \end{bmatrix} \tag{10}$$

where I is the rxr identity matrix. Since system (1) is completely reachable, from lemma 1

$$\text{rank}\,[B, AB, A^2B, \ldots, A^{n-1}] = n. \tag{11}$$

Equation (11) together with the elementry column operations indicate that

$$\text{rank}\,[\,A^{n-1}B,\ A^{n-2}B, \ldots, B] = n. \tag{12}$$

If matrix H is invertible, system (9) is solvable.

In order to show invertibility of H, we start with nxn matrix F, the special case of H, when r=1, the nxn matrix F is a Vandermond matrix, therefore is invertible.

Now we show the invertibility of H. Since F is non-singular, there exists a nxn matrix W such that

$$WFW^{-1} = Y, \tag{13}$$

where Y is an upper triangular matrix with non-zero diagonal elements (See Corduneanu C., Ref No. 3, Page 100).

Let us define the W and W^{-1} matrices as follows:

$$(14) \qquad W = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdot & \alpha_{1n} \\ \alpha_{21} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n1} & \cdot & \cdot & \alpha_{nn} \end{bmatrix},$$

and

$$W^{-1} = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdot & \beta_{1n} \\ \beta_{21} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \beta_{n1} & \cdot & \cdot & \beta_{nn} \end{bmatrix}.$$

Then the nxn matrix Y will have the following format:

$$(15) \qquad Y = WFW^{-1} = \begin{bmatrix} \lambda_1 & x & x & x \\ 0 & \lambda_2 & x & x \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

where λ_i are non-zero for $i=1,\ldots,n$.
Consider the following two nrxnr matrices;

$$(16) \qquad (WI) = \begin{bmatrix} \alpha_{11}I & \alpha_{12}I & \cdot & \alpha_{1n}I \\ \alpha_{21}I & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n1}I & \cdot & \cdot & \alpha_{nn}I \end{bmatrix},$$

and

$$(WI)^{-1}=\begin{bmatrix} \beta_{11}I & \beta_{12}I & . & \beta_{1n}I \\ \beta_{21}I & . & . & . \\ . & . & . & . \\ \beta_{n1}I & . & . & \beta_{nn}I \end{bmatrix},$$

where I is the rxr identity matrix. Simple multiplication and application of $WW^{-1}=I_n$, demonstrates that $(WI)(WI)^{-1}=I_{nr}$, where I_n and I_{nr} are nxn and nrxnr unity matrices, respectively. Applying equation (15) to $(WI)H(WI)^{-1}$ will result in:

$$(17) \qquad (WI)H(WI)^{-1}=\begin{bmatrix} \lambda_1 I & x & x & x \\ 0 & \lambda_2 I & x & x \\ . & . & . & . \\ 0 & 0 & 0 & \lambda_n I \end{bmatrix},$$

which shows that H is non-singular.

To show the existence of the solution of (9), let us denote

$$(18) \qquad K=[B,AB,A^2B,\dots,A^{n-1}B]\ H.$$

Because H is invertible and rank $[B,AB,\dots,A^{n-1}] = n$, the rank of K is also n. Since n is also the number of rows in A, therefore the augmented matrix [K G] has rank n, and this concludes the existence of solution (See Noble B. and Daniel J. W., Ref. No. 4, Page 92).

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