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THE POTENTIAL BASIS AND THE WEIGHTED SHAPLEY VALUE

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1. Introduction. The Shapley value, one of the main concepts of solution in the theory of cooperative *TU* games, has been one of the main stream topics of research in the last decade, so that a reference volume has been published devoted to this work (A.E. Roth, ed. (1988)). Even though the original definition has been given by L. S. Shapley (1953b) in a very general algebraic framework, the connected research went in an applied direction due to the practical interest of the concept and its extensions. The aim of the present paper is that of returning to the algebraic structure of the Shapley value and trying to use the results for computational purposes. To serve this purpose we shall confine ourselves to the weighted Shapley value, even though the algorithm proposed may be used for other extensions as shown by the author (1990).

In the first section, we introduce a new basis W of the linear space G^N of cooperative *TU* games, a basis associated with a weight vector $\lambda \in R^n_{++}$, and show that the coordinates of any $v \in G^N$ in this basis are the Hart/Mas-Colell potentials of the game and its subgames. Therefore, the new basis is called the potential basis of G^N. As the weighted Shapley value is defined as a linear mapping from G^N to R^n with given values on the unanimity basis U of G^N, this definition gives the matrix representation of the linear mapping relative to bases U in G^N and the standard basis in R^n. In the second section, we obtain first the matrix representation relative to bases W in G^N and the standard basis in R^n. The computational formula by Hart/Mas-Colell (1989) which gives the weighted Shapley value in terms of the potentials is getting an algebraic proof. However, the new matrix representation is allowing us to determine a basis of the null space, the nullity and the range of the linear mapping, as well as an explicit description of the set of games with a given weighted Shapley value. The new basis has the distinguished propety that the weighted Shaply value of any basis vector $W_S \in W$ with $|S| \leq n-2$ is the zero vector. Therefore, we are able to build in the third section a sequence of games with the same weighted Shapley value as a given game, the last game in the sequence being a game for which the Hart/Mas-Colell formulas compute immediately the value. The computational complexity of the algorithm is also given.

2. The potential basis of G^N. A cooperative n-person game with transferable utilities (a TU game) in coalitional form is a pair $G=(N,v)$, where N is the set of players, $|N|=n$, and v is the characteristic function defined on the power set of N with real values and $v(\emptyset)=0$. Any $S \subseteq N$, $S \neq \emptyset$, is a coalition and $v(S)$ is its worth in the game. If the coalitions are ordered, then $(v(S))_{S \subseteq N, S \neq \emptyset}$ is a point in R^{2^n-1}, so that the set of TU games over N, denoted G^N, can be identified with R^{2^n-1}. Therefore, in the following, $v \in G^N$ will mean that the components of the 2^n-1 dimensional vector v are the values of the characteristic function of a game. The notation X_S will be used to represent a vector in R^{2^n-1} associated with coalition S, while $X_S(T)$ denotes the T-coordinate of X_S, as usual in game theory.

Let $D=\{D_S : S \subseteq N, S \neq \emptyset\}$ denote the standard basis of G^N, i.e. $D_S(S)=1$ and $D_S(T)=0$ for $T \neq S$. Let $U=\{U_S : S \subseteq N, S \neq \emptyset\}$ denote the basis of unanimity games, i.e. the basis defined by $u_S(T)=1$ for $T \subseteq S$ and $U_S(T)=0$, otherwise. The representations of $v \in G^N$ in these bases

$$v = \sum_{S \subseteq N, S \neq \emptyset} v(S) D_S, \qquad v = \sum_{S \subseteq N, S \neq \emptyset} \Delta_v(S) U_S, \tag{2.1}$$

show the usual notations of the coordinates; their relationships

$$\Delta_v(S) = \sum_{Q \subseteq S, Q \neq \emptyset} (-1)^{s-q} v(Q), \qquad \forall S \subseteq N,\ S \neq \emptyset, \tag{2.2}$$

where $s = |S|$, $q = |Q|$, can be easily proved (see Owen (1982)).

We define a new basis $W = \{W_S : S \subseteq N, S \neq \emptyset\}$ of G^N, associated with a weight vector $\lambda \in R^n_{++}$ as follows:

$$\text{for } S \subset N,\ W_S(T) = \begin{cases} \lambda(S) & \text{if } T = S, \\ -\lambda_j & \text{if } T = S \cup \{j\}, j \notin S, \\ 0 & \text{otherwise,} \end{cases} \quad \forall T \subseteq N, T \neq \emptyset,$$

(2.3)

$$\text{for } S = N,\ W_N(T) = \begin{cases} \lambda(N) & \text{if } T = N, \\ 0 & \text{otherwise,} \end{cases} \quad \forall T \subseteq N, T \neq \emptyset,$$

where $\lambda(S)$ is the sum of λ_i for $i \in S$. Obviously, W and all W_S depend on λ, even though to keep the notation simple λ was omitted.

Lemma 2.1: For any $\lambda \in R^n_{++}$, $\{W_S\}$ defined by (2.3) is a basis of G^N.

Proof: The $2^n - 1$ vectors W are linearly independent, as it can be seen if we choose an order of coalitions which agrees with their sizes and notice that the matrix (W_S) is lower triangular with positive entries on the main diagonal.

From Lemma 2.1, it follows that for any $v \in G^N$ there is a unique representation

$$v = \sum_{S \subseteq N, S \neq \emptyset} P_\lambda(S) W_S. \tag{2.4}$$

Lemma 2.2: The relationships between the coordinates of v in bases D and W are

$$v(S) = \sum_{i \in S} \lambda_i [P_\lambda(S) - P_\lambda(S - \{i\})], \quad \forall S \subseteq N,\ S \neq \emptyset \tag{2.5}$$

where $P_\lambda(\emptyset) = 0$ by definition.

Proof: From (2.3), the relationships between the bases D and W are

(2.6)

$$W_S = \lambda(S) D_S - \sum_{j \in S} \lambda_j D_{S \cup \{j\}}, \quad \forall S \subset N,\ S \neq \emptyset,$$

$$W_N = \lambda(N) D_N,$$

so that (2.5) is obtained by using (2.6) in (2.4) and comparing with the representation of v in basis D.

Note that (2.5) is exactly the linear system used by Hart/Mas-Colell (1989), to define the weighted potentials of v and its subgames. Therefore, Lemma 2.2 proves that the coordiantes of v in basis W are precisely the Hart/Mas-Colell weighted potentials. Their uniqueness follows here from Lemma 2.1. This shows that W may be called the potential basis of G^N.

Note also that Lemma 2.2 defines recurrently the converse relationships

$$P_\lambda(S) = [v(S) + \sum_{i \in S} \lambda_i P_\lambda(S - \{i\})]/\lambda(S), \qquad \forall S \subseteq N,\ S \neq \emptyset, \tag{2.7}$$

where $P_\lambda(\emptyset) = 0$; the above lemma proves also:

<u>Theorem</u> 2.3: The operator $P_\lambda : G^N \to G^N$ defined by (2.7) is one-to-one and onto.

For the symmetric case $\lambda_i = 1$, $\forall i \in N$, this result and other properties of the operator have been proved by Hart/Mas-Colell (1989). For P_λ they follow easily from the fact that the weighted potentials are the coordinates of v in basis W. The potential basis in the symmetric case has been introduced by Dragan/Potters/Tijs (1989), but the extension to the weighted case was not apparent.

3. <u>The weighted Shapley value</u>.

For a cooperative TU game $v \in G^N$, the weighted Shapley value is a concept of solution that may be derived from L. S. Shapley's work (1953b), in the particular case of a measure defined by means of a weight vector $\lambda \in R^n_{++}$. One can follow Kalai/Samet (1987) to introduce it as follows:

<u>Definition</u> 3.1: The weighted Shapley value is a linear mapping $\phi : G^N \to R^n$, defined by its values on the unanimity basis U

$$\phi_i(U_S, \lambda) = \begin{cases} \lambda_i/\lambda(S) \text{ if } i \in S, \\ 0 \text{ if } i \notin S. \end{cases} \qquad \forall i \in N,\ \forall S \subseteq N,\ S \neq \emptyset. \tag{3.1}$$

It is worthwhile to note the following computational formulas which give $\phi(v,\lambda)$ in terms of the coalitional form $(N, v(S))$, the unanimity (or divident) form $(N, \Delta_v(S))$, or the potential form $(N, P_\lambda(S))$ of the *TU* game v: a) From Shapley's fundamental formula for "imputations" (Shapley, 1953 b, Th. 34, p. 51), we get a computational formula in terms of the coalitional form

$$\phi_i(v,\lambda) = \lambda_i \sum_{S/i \in S} \gamma_S[v(S) - v(S - \{i\})], \qquad \forall i \in N, \tag{3.2}$$

$$\gamma_S = \sum_{Q \subseteq N, Q \cap S = \emptyset} (-1)^{|Q|}/\lambda(S \cup Q), \qquad \forall S \subseteq N,\ S \neq \emptyset.$$

b) another formula, in terms of the unanimity form follows from (3.1) and the linearity property

$$\phi_i(v,\lambda) = \lambda_i \sum_{S/i \in S} \Delta_v(S)/\lambda(S), \qquad \forall i \in N. \tag{3.3}$$

c) A third computational formula in terms of the potential form has been proved by Hart/Mas-Colell (1989), precisely

$$\phi_i(v,\lambda) = \lambda_i[P_\lambda(N) - P_\lambda(N - \{i\})], \qquad \forall i \in N, \tag{3.4}$$

by using an axiomatic approach.

Usually, a game is considered as given in coalitional form. In this case, the computation of ϕ based upon the last two more attractive formulas is not any easier, because it involves computations connected to a change of basis of huge size. In the next section, we shall show how this change can be somehow avoided, in order to give us the opportunity of using (3.4), i.e. we shall derive an algorithm for computing the weighted Shapley value. Here, we compute first the weighted Shapley values of the "potential" games $W_S \in W$ and derive some theoretical results.

<u>Lemma</u> 3.2: For any weight vector $\lambda \in R^n_{++}$, the relationships between the bases U and W are

$$W_S = \sum_{T/T \supseteq S} (-1)^{t-s}\lambda(T)U_T, \qquad \forall S \subseteq N, S \neq \emptyset, \tag{3.5}$$

where $t = |T|$, $s = |S|$.

Proof: The representation of W_S in basis U is

$$W_S = \sum_{T \subseteq N, T \neq \emptyset} \Delta_{W_S}(T) U_T, \qquad \forall S \subseteq N, S \neq 0, \tag{3.6}$$

where $\Delta_{W_S}(T)$ in terms of the coalitional form are given by

$$\Delta_{W_S}(T) = \sum_{Q \subseteq T, Q \neq \emptyset} (-1)^{t-q} W_S(Q), \qquad \forall S \subseteq N, S \neq 0, \tag{3.7}$$

following from (2.1) and it should be computed taking into account (2.3). For $S = N$, if $T \subset N$, then all terms in (3.7) are zero, while if $T = N$, precisely one nonzero term for $Q = N$ gives $\Delta_{W_N}(N) = W_N(N) = \lambda(N)$ and (3.5) for $S = N$ follows from (3.6). For $S \subset N$, if $T \supseteq S$, then all terms in (3.7) are zero, while if $T \supseteq S$, then we get nonzero terms for $Q = S$ and all $Q = S \cup \{j\}$ with $j \in T - S$. In this way, from (2.3) and (3.7) we have

$$\Delta_{W_S}(T) = (-1)^{t-s} \lambda(T), \tag{3.8}$$

so that (3.5) follows from (3.6) and (3.8).

Now we can compute the weighted Shapley values of W_S based upon Lemma 3.2 and Definition 3.1:

Lemma 3.3: For any $\lambda \in R^n_{++}$, the weighted Shapley values of the basis vectors W_S are

$$\begin{aligned} &\phi(W_S, \lambda) = 0, && \text{if } |S| \leq n-2, \\ &\phi(W_S, \lambda) = -\lambda_i e_i, && \text{if } S = N - \{i\},\ i \in N, \\ &\phi(W_S, \lambda) = \lambda, && \text{if } S = N, \end{aligned} \tag{3.9}$$

where vectors e_i, $\forall i \in N$, form the standard basis of R^n.

Proof: From Lemma 3.2, the linearity of ϕ on G^N and Definition 3.1 we get

$$\phi_i(W_S, \lambda) = \lambda_i \sum_{T / T \supseteq S, i \in T} (-1)^{t-s}, \qquad \forall S \subseteq N,\ S \neq 0,\ i \in N. \tag{3.12}$$

If $S = N$, then the last sum equals one and (3.11) follows. If $S = N - \{i\}$, the sum (3.12) for $\phi_h(W_S, \lambda)$ with $h \neq i$ has two terms with opposite signes when $T = N - \{h\}$ and $T = N$, while $\phi_i(W_S, \lambda)$ has only one term equal to -1 when $T = N$, and (3.10) follows. If $|S| \leq n-2$, then two situations may occur: a) for $i \in S$, all $T \supseteq S$ contain i, so that the sum equals $1 - C_{n-s}^{1} + C_{n-s}^{2} - \ldots + (-1)^{n-s} C_{n-s}^{n-s} = 0$; b) for $i \notin S$, only $T \supseteq S \cup \{i\}$ give terms, and the sum equals $-1 + C_{n-s-1}^{1} - \ldots + (-1)^{n-s-1} C_{n-s-1}^{n-s-1} = 0$; hence, (3.9) follows.

From (2.4), Lemma 3.3 and the linearity of ϕ on G^N we get:

<u>Theorem</u> 3.4: (Hart/Mas-Colell, 1989, Th. 5.2). For any weight vector $\lambda \in R_{++}^{n}$ and any game $v \in G^N$, if $P_\lambda(S)$, $S \subseteq N$, $S \neq \emptyset$, are the coordinates of v in basis W and $P_\lambda(\emptyset) = 0$, then the weighted Shapley value of the game is given by

$$\phi_i(v, \lambda) = \lambda_i [P_\lambda(N) - P_\lambda(N - \{i\})], \qquad \forall i \in N.$$

In this way, we got an algebraic proof of Hart/Mas-Colell result. Now we can also get new algebraic results connected to the linear mapping ϕ.

Note that in terms of linear algebra, Lemma 3.2 gives the matrix M for the change of basis from U to W in G^N, while Definition 3.1 gives the matrix representation of ϕ relative to bases U in G^N and e in R^n. Then, Lemma 3.3 is computing matrix $M' = AM$, which by a well-known theorem of linear algebra gives the matrix representation of ϕ relative to bases W in G^N and e in R^n. Moreover, the image $\phi(v, \lambda) = M'P_v$ and this image has been computed in Theorem 3.4. Lemma 3.3 will allow us to derive new results on the properties of ϕ. Kleinberg and Weiss (1985) have introduced an equivalence on G^N: two games are equivalent if they have the same Shapley value. We determine the equivalence classes for the equivalence defined by means of the weighted Shapley value:

<u>Theorem</u> 3.5: Let $\lambda \in R_{++}^{n}$ be any weight vector. For any $\psi \in R^n$, the set of all games $v \in G^N$ with $\phi(v, \lambda) = \psi$ is given by

$$(3.13) \qquad v = \sum_{|S| \leq n-2} p_S W_S + p_N\Big(W_N + \sum_{i \in N} W_{N-\{i\}}\Big) - \sum_{i \in N} (\psi_i / \lambda_i) W_{N-\{i\}},$$

where p_N and p_S, $|S| = n-2$, are any real numbers.

Proof: From Lemma 3.3, we have $\phi(W_N + \sum_{i \in N} W_{N-\{i\}}) = 0$. If $v \in G^N$ is given by (3.13), then by Lemma 3.3 and the linearity of ϕ on G^N we get $\phi(v, \lambda) = \psi$. Conversely, if $\phi(v, \lambda) = \psi$, i.e. we have $P_\lambda(N - \{i\}) = P_\lambda(N) - (\psi_i / \lambda_i)$, $\forall i \in N$, by Theorem 3.4, then the representation (2.4) becomes (3.13) with $p_N = P_\lambda(N)$ and $p_S = P_\lambda(S)$ for $\forall S \subseteq N$, $S \neq \emptyset$, $|S| \leq n-2$.

Note that, beside showing the equivalence classes, Theorem 3.5 is offering an answer to the problem of generating games with a given weighted Shapley value (or a given Shapley value). Moreover, the following result is straightforward:

Theorem 3.6: In G^N, $W_N + \sum_{i \in N} W_{N-\{i\}}$ and W_S with $|S| \leq n-2$ form a basis of the null space for the weighted Shapley value ϕ, the nullity is $n(\phi) = 2^n - n - 1$ and the range of ϕ is R^n.

Note that this last result is an extension of a result obtained by Dragan, Potters and Tijs (1989) for the particular case $\lambda_i = 1$, $\forall i \in N$. For the same case, Kleinberg and Weiss (1985) have obtained a decomposition of the null subspace of the Shapley value into orthogonal subspaces. No new insight can be derived by computing the orthogonal basis in terms of basis W in the symmetric case, because the orthogonal basis has a very complex definition.

4. Algorithm for computing the weighted Shapley value. Consider a weight vector $\lambda \in R^n_{++}$ and a game $v \in G^N$ in coalitional form. The algorithm we are going to derive, based upon Lemma 3.3, is working in two stages. The first stage is an iterative procedure which is building a sequence of games with the same weighted Shapley values as the given game. Moreover, the last game $\overline{v}$ in the sequence, beside $\phi(\overline{v}, \lambda) = \phi(v, \lambda)$, will also have the property shown in the hypothesis of Lemma 4.1 below, so that Theorem 3.4 will be used to compute $\phi(\overline{v}, \lambda)$ in the second stage.

<u>Lemma</u> 4.1: If a game $\bar{v} \in G^N$ has $\bar{v}(S) = 0$ for all $S \subset N$, $S \neq \emptyset$, with $|S| \leq n-2$, then

$$\phi_i(\bar{v}, \lambda) = \lambda_i(x - x_i), \qquad \forall i \in N, \tag{4.1}$$

where

$$x = [\bar{v}(n) + \sum_{j \in N} \lambda_j x_j]/\lambda(N), \quad x_i = \bar{v}(N - \{i\})/\lambda(N - \{i\}), \qquad \forall i \in N. \tag{4.2}$$

<u>Proof</u>: From (2.7) it can be seen that under the hypotheses of the lemma we have $\bar{P}_\lambda(S) = 0$ for all $S \subset N$, $S \neq \emptyset$, with $|S| \leq n-2$; if we denote $x_i = \bar{P}_\lambda(N - \{i\})$, $\forall i \in N$, and $x = \bar{P}_\lambda(N)$, then (4.2) follows also from (2.7) and (4.1) from Theorem 3.4.

The iterative procedure is based upon the remark that for any numbers d_S associated with coalitions S of size at most $n-2$, if $v' = v + \sum_{S/|S| \leq n-2} d_S W_S$, then $\phi(v', \lambda) = \phi(v, \lambda)$. This fact follows from Lemma 3.3 and the linearity of ϕ on G_N. In each step a particular choice of numbers d_S should be done in order to make sure that the game $\bar{v}$ built in the last step satisfies the hypotheses of the above lemma. An appropriate choice is shown by:

<u>Theorem</u> 4.2: Let v be a game such that for some k, $1 \leq k \leq n-2$, we have $v(S) = 0$ for all $S \subset N$ with $|S| < k$. Let T be a coalition with $|T| = k$ for which $v(T) \neq 0$. Then the game v' defined by

$$v' = v - [v(T)/\lambda(T)]W_T \tag{4.3}$$

has the properties: a) $\phi(v', \lambda) = \phi(v, \lambda)$; b) $v'(T) = 0$; c) $v'(S) = 0$, for all $S \subset N$ with $|S| < k$; d) $v'(S) = v(S)$ for all $S \subset N$ with $|S| = k$, $S \neq T$.

<u>Proof</u>: We get (a) by the above remark; from (4.3) and definition (2.3) of W_T we obtain (b), (c) and (d).

This result says that one step performed according to (4.3) is conserving the weighted Shapley value as well as the values of the characteristic function for all coalitions $S \subset N$, $S \neq T$, with $|S| \leq k$, while a new zero value has been obtained for coalition T. In this way, after at most C_n^k steps we shall get a game with the same weighted Shapley value as the game considered before these steps and the current game will have all values of the characteristic function for all coalitions of size at most k equal to zero. If $k<n-2$, then k will be replaced by $k+1$ and a new cycle of steps should be performed; now , if $k=n-2$ the first stage ends followed by the computation shown in Lemma 4.1 in the second stage to obtain the weighted Shapley value.

Note that the maximal number of steps will be 2^n-n-2 and the number of oeprations needed in one step in the worst case is shown by the component-wise form of (4.3), that is

$$(4.4) \qquad v'(S)=\begin{cases} 0 & \text{if } S=T, \\ v(S)+[\Lambda_i/\lambda(T)]v(T) & \text{if } S=T\cup\{i\},\ i\notin T, \\ v(S) & \text{otherwise.}\end{cases}$$

This allows a computation of the complexity of the algorithm. As (4.4) shows, in the worst case each of the C_n^k steps requires 3 $(n-k)$ operations, hence the total number of operations is $3\sum_{k=1}^{n-2}(n-k)C_n^k=3n(2^{n-1}-2)$, i.e. the complexity is $0(m\log_2 m)$, where we have $m=2^n$. This estimate does not change if the comparisons are also taken into account. Summarizing we state:

<u>The Algorithm</u>:

1. Take on $k=1$.

2. Find $T\subset N$ with $|T|=k$ and $v(T)\neq 0$. If none is found go to 4.

3. Set $v(T)=0$, replace $v(S)$ by $v(S)+(\lambda_i/\lambda(T))v(T)$ for all $S=T\cup\{i\}$ with $i\notin T$ and return to 2.

4. If $k=n-2$, go to 5; otherwise, replace k by $k+1$ and go to 2.

5. Compute x_i, $\forall i\in N$, and x by (4.2) and $\phi_i(v,\lambda)$, $\forall i\in N$, by (4.1).

<u>Example</u>: consider the 3-person game given by: $v(1)=1$, $v(2)=2$,

$v(3)=3$, $v(12)=4$, $v(13)=6$, $v(23)=8$, $v(123)=10$, and the weight vector $\lambda=(1, 1, 2)$. The first stage has three steps with $T=\{1\}$, $T=\{2\}$, and $T=\{3\}$, respectively, and $v(T)/\lambda(T)$ equal to 1, 2 and 3/2. The sequence of games built by the algorithm is: $v_1: v_1(1)=0$, $v_1(2)=2$, $v_1(3)=3$, $v_1(12)=5$, $v_1(13)=8$, $v_1(23)=8$, $v_1(123)=10$; $v_2: v_2(1)=0$, $v_2(2)=0$, $v_2(3)=3$, $v_2(12)=7$, $v_2(13)=8$, $v_2(23)=12$, $v_2(123)=10$; $v_3: v_3(1)=0$, $v_3(2)=0$, $v_3(3)=0$, $v_3(12)=7$, $v_3(13)=19/2$, $v_3(23)=27/2$, $v_3(123)=10$. In the second stage we get $x_1=9/2$, $x_2=19/6$, $x_3=37/6$, then $\phi_1=5/3$, $\phi_2=3$, $\phi_3=16/3$.

Note that an algorithm which is also computing a sequence of games proved to end in a game with the same Shapley value as the initial one, has been given for the symetric case by Maschler (1982).

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