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ON THE BEHAVIOR OF ITERATIVE METHODS IN THE PRESENCE OF GRID IRREGULARITIES

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1. Introduction. One of the key features of various iterative methods (e.g., S.O.R., multigrid, conjugate gradients) is the serious deterioration in their performance when used to solve algebraic equations arising from the discretization of elliptic boundary value problems on irregular grids. It can happen, for example, that the number of iterations on an irregular grid may be comparable to those required on a uniform grid with mesh spacing equal to the smallest on the irregular grid.

In this paper we shall concentrate on S.O.R., although the ultimate application of the results will be the appropriate modification of the smoothing relaxation in multigrid algorithms. In Section 2 selected numerical results are given which illustrates the deleterious effects of grid irregularities. An important strategy for dealing with these effects is to use spatially varing relaxation parameters ω_i. In Section 3 it is shown that there is an optimal selection for these parameters which minimized the spectral radius of the associated iteration matrix. In this context, the overall method is equivalent to various direct methods (such as Gaussian elimination) depending on the starting values chosen.

At the other extreme are variable relaxation parameters which are determined locally ([1]). While these have been effective for convection-diffusion problems, they do not prevent the significant increase in the number of required iterations as the grid refinement becomes more pronounced.

One interesting aspect that emerged from the numerical study is that boundary refinement with Dirichlet data has very little effect on the iterations. Thus the phenomena described in this paper is relevant only for interior grid refinement.

2. Numerical Results. The effects of grid irregularities are independent of the spatial dimensions, so result from univariate experiments will be reported for simplicity. In particular consider the grid

$$0 = x_1 < x_2 < x_3 < \cdots < x_{N+1} = 1,$$

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and the nodal values $\{u_j\}_{j=0}^{N+1}$ defined by

$$\left\{\frac{2}{x_{j+1}-x_{j-1}}\right\}\left\{\frac{u_{j+1}-u_j}{x_{j+1}-x_j}-\frac{u_j-u_{j-1}}{x_j-x_{j-1}}\right\}=f_j \tag{2.1}$$

for $j=2,\cdots,N$. At $x=x_1=0$ the Neuman condition

$$\frac{u_1-u_0}{x_1-x_0}=0 \tag{2.2}$$

is used. This condition can be viewed as a symmetry condition, hence refinement near $x=0$ is in effect interior refinement. At the right boundary a Dirichlet condition is specified:

$$u_{N+1}=0 \tag{2.3}$$

Note that (2.1)—(2.2)—(2.3) is a linear system with N unknowns $u_1,\cdots,u_N$.

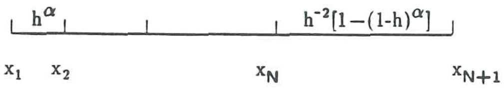
For the numerical experiments a graded grid defined by

$$x_j=[\,(j-1)h]^{\alpha},\quad 1\leq j\leq N+1, \tag{2.4}$$

is used where $h=\frac{1}{N}$ and $\alpha>0$ is a grid control parameter. Note that the ratio of the minimum mesh spaces Δx_{min} to the maximum mesh spacing Δx_{max} is

$$\frac{\Delta x_{max}}{\Delta x_{min}}=N^{\alpha}-(N-1)^{\alpha}=\binom{\alpha}{1}N^{\alpha-1}+0(N^{\alpha-2}) \tag{2.5}$$

where $\binom{\alpha}{\beta}$ is the binomial coefficient. The choice $\alpha=1$ gives a uniform grid and with increasing $\alpha>1$ we get a refinement near $x=0$:



We concentrate on S.O.R. iterations. Given a starting vector $\{u_1^{(0)},\ \cdots,\ u_N^{(0)}\}$ iterates are defined by

$$\frac{2}{(x_{j+1}-x_{j-1})}\left\{\frac{u_{j+1}^{(n)}-\hat{u}_j}{x_{j+1}-x_j}-\frac{\hat{u}-u_{j-1}^{(n+1)}}{x_j-x_{j-1}}\right\}=f_j \tag{2.6}$$

$$u_j^{(n+1)} = \omega_j \hat{u}_j + (1-\omega_j) u_j^{(n)} \tag{2.7}$$

for $n=0, 1, \cdots$. Here $\{\omega_j\}_{j=1}^{N}$ is the set of relaxation parameters, and the choices were used.

The first choice is the best fixed relaxation parameter ($\omega=\omega_j$ all j)

$$\omega = \omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\lambda^2}}\,, \tag{2.8}$$

where λ is the spectral radius of the associated Jacobi matrix [2]. This number was explicitly computed from an eigenvalue routine for the results in Table I. Note the explosive growth in the number of iterations N_α as α is increased for a fixed number N of grid points. For example, increasing α from 1 (uniform grid) to $\alpha=4$ results in increasing the number of iterations by a factor of nearly 20. On the other hand, for fixed α the convergence follows the familiar pattern [2]; i.e., the number of required iterations is proportional to the number of grid points.

Attempts to deal with the effects of irregular grids has centered on relaxation parameters ω_j which varies with the nodal points x_j. Most of the work has been for variable coefficient problems ——mostly notably in the context of convection and diffusion [1], [3]–[5]——however, these are conceptually similar to the variable grid case since a change of variables

$$x = x(t)$$

renders a constant coefficient problem on a variable gbrid into a variable coefficient problem on a uniform grid. Moreover, the grid irregularities introduces convective terms with different spatial scales than the diffusion scales.

The most successful approach to date is that of Botta and Veldman [1]. To derive their scheme the change of variables

$$x(t) = t^{\alpha} \tag{2.9}$$

is used which converts (2.1) into a variable coefficient problem on a uniform grid with spacing h. In particular, (2.1) goes into

$$\tfrac{1}{2}\,(1+\sigma_j h/2)v_{j+1} - v_j + \tfrac{1}{2}\,(1-\sigma_j h/2)v_{j-1} = \frac{h^2}{2}\,F_j, \tag{2.10}$$

where

$$\sigma_j = (1-\alpha)(jh)^{-\frac{1}{\alpha}} \tag{2.11}$$

and F_j, v_j are the images of f_j, u_j. Their idea is to freeze the coefficient σ_j and compute the best relaxation parameter ω_j associated with this constant coefficient problem. Observe that this is a local method in that ω_j depends only on the values of σ near the j^{th} node, or what is same, ω_j depends only on the grid refinement near the j^{th} node.

As can be anticipated by the underlying ellipticity of the problem such local strategies can not prevent the deterioration of the iterations as the grid is refined (i.e., increasing α for fixed N). The numerical results are given in Table I. Moreover, as can be seen from Table I, the locally determined variable ω is also decidedly inferior to the best fixed ω_{opt}. This is to be anticipated since ω_{opt} has a global influence via the eigenvalue λ in (2.8).

Table I. Neumann Boundary data

(a)

	$\omega = \omega_{opt}$		locally determined variable ω		best variable ω	
α	N_α	ρ	N_α	ρ	N_α	ρ
1	36	.72945	39	.79787	11	0
2	68	.86045	47	.86578	11	0
3	153	.94488	179	.96368	11	0
4	386	.98104	591	.99231	11	0

(b)

	$\omega = \omega_{opt}$		locally determined variable ω		best variable ω	
α	N_α	ρ	N_α	ρ	N_α	ρ
1	66	.85450	60	.87464	21	0
2	133	.93362	174	.96286	21	0
3	384	.98039	658	.99333	21	0
4	1254	.99523	2476	.99910	21	0

α versus the number N_α of iterations and the spectral radius ρ of the corresponding iteration matrix with number of intervals N=10 in a) and N=20 in b). The iterations were terminated when the residual reached 10^{-4}.

The best variable $\{\omega_j\}_{j=1}^N$ is totally global, and in fact, equivalent to a direct method. The results using this choice is given in Table I, and the derivation of this relaxation set $\{\omega_j\}_{j=1}^N$ is given in the next section. It is interesting to note that if the Neuman condition (2.2) is replaced with a Dirichlet condition like (2.3), then the results are substantially different. This reflects the fact that boundary refinement has no influence on the iterative scheme. These results are summarized in Table II.

Table II. Dirichlet Boundary data

(a)

	$\omega = \omega_{opt}$		locally determined variable ω		best variable ω	
α	N_α	ρ	N_α	ρ	N_α	ρ
1	17	.52786	20	.66779	10	0
2	14	.45832	13	.44722	10	0
3	13	.39956	11	.40891	10	0
4	11	.34940	9	.36276	9	0

(b)

	$\omega = \omega_{opt}$		locally determined variable ω		best variable ω	
α	N_α	ρ	N_α	ρ	N_α	ρ
1	31	.72945	33	.78272	20	0
2	26	.68052	22	.70688	20	0
3	23	.63657	21	.67808	19	0
4	20	.59655	19	.64417	17	0

α versus the number N_α of iterations and the spectral radius ρ of the corresponding iteration matrix with number of intervals N=10 in a) and N=20 in b). The iterations were terminated when the residual reached 10^{-4}.

3. Optimal Relaxation Set. The goal of this section is to derive the best relaxation parameters $\{\omega_j\}_{j=1}^{N}$; i.e., the ones which minimize the spectral radius of the associated iteration matrix. The result applies to more than the elementary tridiagonal system (2.1) but for a general case of block triadiagonal matrices. We therefore give the derivation for the general case in this section, and discuss specific applications in the next section.

Let

$$A = \begin{bmatrix} A_{11} & A_{12} & & 0 \\ A_{21} & A_{22} & \ddots & \\ & \ddots & \ddots & A_{N-1,N} \\ 0 & & A_{N,N-1} & A_{NN} \end{bmatrix} \tag{3.1}$$

be block tridiagonal with A_{jk} being $r_j \times r_k$ matrices, and each diagonal block A_{ij} being nonsingular. The associated S.O.R. iteration has the form

$$A_{jj-1}u_j^{(n+1)} + A_{jj}\hat{u}_j + A_{jj+1}u_{j+1}^{(n)} = f_j \tag{3.2}$$

$$u_j^{(n+1)} = W_j\hat{u}_j + (I - W_j)u_j^{(n)}, \tag{3.3}$$

where $n \geq 0$ and W_j is the matrix of relaxation parameters.

To describe the iteratin matrix I_W for this scheme let

$$A_\ell = \begin{bmatrix} 0 & & \\ A_{21} & \ddots & \\ & A_{N,N-1} & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} A_{11} & 0 \\ 0 & \ddots \; A_{NN} \end{bmatrix}, \quad A_u = \begin{bmatrix} 0 & A_{12} & 0 \\ & \ddots & A_{N-1,N} \\ 0 & & 0 \end{bmatrix} \tag{3.4}$$

be the block lower, diagonal, and upper parts of A. Also let

$$R = \begin{bmatrix} R_{11} & & 0 \\ & \ddots & \\ 0 & & R_{NN} \end{bmatrix}, \quad R_{jj} = A_{jj} W_j^{-1} \tag{3.5}$$

Then the iteration matrix is

$$I_w = (R + A_\ell)^{-1} (R - A_d - A_u) \tag{3.6}$$

<u>Theorem 1</u>. Let the principal minors of A be nonzero. Then for $1 \leq j \leq N$ there is a nonsingular relaxation matrix W_j for which the spectral radius of I_w is zero. In fact, with this choice I_w is strictly upper triangular:

$$I_w = \begin{bmatrix} 0 & Y_{12} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & Y_{N-1,N} \\ 0 & & & 0 \end{bmatrix} \tag{3.7}$$

for appropriate $r_j \times r_{j+1}$ matrices $Y_{jj+1} (1 \leq j \leq N-1)$.

<u>Proof</u>. By the assumption on the principal minors, it follows that A can be factored into a product of a lower L times a upper U triangular matrix [6]:

$$A = LU. \tag{3.8}$$

The key point, which is special to the (block) triadiagonal case is that L and U can be chosen so that

$$L = \begin{bmatrix} D_1 & & & 0 \\ A_{21} & D_2 & & \\ & \ddots & \ddots & \\ 0 & & A_{N,N-1} & D_N \end{bmatrix}, \quad U = \begin{bmatrix} I & V_{12} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & V_{N-1,N} \\ 0 & & & I \end{bmatrix} \tag{3.9}$$

for suitable $r_j \times r_j$ matrices D_j and $r_j \times r_{j+1}$ matrices V_{jj+1}. That is, the lower block entries of L are exactly those of A so that

$$L = D + A_\ell, \quad D = \begin{bmatrix} D_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & D_N \end{bmatrix} \tag{3.10}$$

and

$$U = I + V, \quad V = \begin{bmatrix} \mathbf{0} & V_{12} & & \\ & \ddots & \ddots & \\ & & \ddots & V_{N-1,N} \\ & & & \mathbf{0} \end{bmatrix} \tag{3.11}$$

Finally, we recall [7] that the diagonal matrices D_j are all nonsingular and that this decomposition is unique given that the diagonal entries of U are identity matrices. Combining (3.8)–(3.11) we arrive at

$$A = (D + A_\ell)(I + V) \tag{3.12}$$

Turning now to (3.6) we note that since

$$A = A_\ell + A_d + A_u \tag{3.13}$$

we have

$$\begin{aligned} I_w &= (R + A_\ell)^{-1}(R + A_\ell - A) \\ &= I - (R + A_\ell)^{-1} A \end{aligned} \tag{3.14}$$

Therefore,

$$A = (R + A_\ell)(I - I_w) \tag{3.15}$$

Comparing (3.12) and (3.15) and using the uniqueness of this form of the LU decomposition we conclude that if the matrix R is chosen so that

$$R = D, \tag{3.16}$$

then

$$I - I_w = I + V \tag{3.17}$$

Thus I_w does indeed have the form (3.7) if (3.16) holds. In fact $Y_{jj+1} = -V_{jj+1}$ for $1 \le j \le N-1$.

Note that (3.16) reduces to

$$A_{jj}W_j^{-1}=D_j \tag{3.18}$$

Thus we have the following

<u>Corollary</u> . Let $D_1,\cdots, D_N$ be the diagonals in the block LU factorization of A. Then the optimal relaxation matrices $W_j(1\leq j\leq N)$ are given by

$$W_j=D_j^{-1}A_{jj} \tag{3.19}$$

It is interesting to explore the relationship between elimination based on the factorization (3.12)(or what is the same (3.15)), and the iteration (3.2)–(3.3) for solving

$$A\underline{u} = \underline{f} \tag{3.20}$$

In the former two back solves are used:

$$(R+A_\ell)\underline{y} = \underline{f} \tag{3.21}$$

and

$$(I+V)\underline{u} = \underline{y} \tag{3.22}$$

If the initial vector is $\underline{u}^{(0)}=\underline{w}$, then (3.2)–(3.3) gives

$$(R+A_\ell)\underline{u}^{(n+1)}=I_w\underline{u}^{(n)}+\underline{f} \tag{3.23}$$

that

$$\underline{u}^{(n+1)}=-(R+A_\ell)^{-1}V\,\underline{u}^{(n)}+\underline{y}\ ,\ n\geq 0, \tag{3.24}$$

It follows that

$$\underline{u}^{(n)}=(I-V+\cdots+(-1)^{k-1}V^{k-1})\underline{y}+(-1)^kV^k\underline{w} \tag{3.25}$$

for $n\geq 0$. Thus if the initial vector $\underline{w}$ is zero the first iterate of S.O.R. gives the vector $\underline{y}$ arising from the first back solve (3.21), and from that point on the Neuman series for

$$(I+V)^{-1}=\sum_{j=0}^{N-1}(-1)^jV^j$$

is generated.

For any choice of the starting vector $\underset{\sim}{w}$ the iterations converge at $n=N$ since

$$V^N=0.$$

4. Examples. For the triadiagonal system (2.1) let

$$a_j=\frac{(x_{j+1}-x_{j-1})}{(x_{j+1}-x_j)(x_j-x_{j-1})},\; b_j=\frac{1}{x_{j+1}-x_j}.$$

Then the diagonals $\{d_j\}_{j=1}^{N}$ in the lower triangular system (3.9) satisfy the well known recursion [6]:

$$d_1=a_1 \tag{4.1}$$

$$d_j=a_j-\frac{b_{j-1}^2}{d_{j-1}},\; j=2,\cdots,N \tag{4.2}$$

The optimal variable relaxation parameters $\{\omega_j\}_{j=1}^{N}$ are given by (3.16), which in this context can be represented as the following continued fraction:

$$\omega_1=1 \tag{4.3}$$

$$\omega_j=\frac{1}{1-\left(\frac{b_{j-1}^2}{a_{j-1}a_j}\right)\omega_{j-1}} \tag{4.4}$$

for $j=2,\cdots,N$.

It is interesting to note that this optimal set starts out at $\omega_1=1$ at the node $x_1=1$ where boundary refinement is present. It then progressively increases to nearly $\omega=2$ at the right hand boundary.

Another interesting feature concerns the closure of the iteration (2.6)–(2.7) with ω_j being given by (4.3)–(4.4). Here $u_j^{(n)}$ is viewed as a time dependent process with iteration time $t=n\Delta t$. Expanding (2.6)–(2.7) and (4.3)–(4.4) in a Taylor series and keeping terms of order Δx and Δt gives the degenerage hyperbolic equation

$$\frac{1}{x}\frac{\partial e}{\partial t} + \frac{\partial^2 e}{\partial x \partial t} = \left(\frac{\Delta x}{\Delta t} \right) \frac{\partial^2 e}{\partial x^2} \tag{4.5}$$

for the error $e(x_j, n\Delta t) = u(x_j) - u_j^n$. This is to be compared with

$$\pi \frac{\partial e}{\partial t} + \frac{\partial^2 e}{\partial x \partial t} = \left(\frac{\Delta x}{\Delta t} \right) \frac{\partial^2 e}{\partial x^2} \tag{4.6}$$

for the optimal fixed relaxation parameter scheme. Totally unlike (4.6), the error in (4.5) behaves like a wave in space-time. See Figure 1

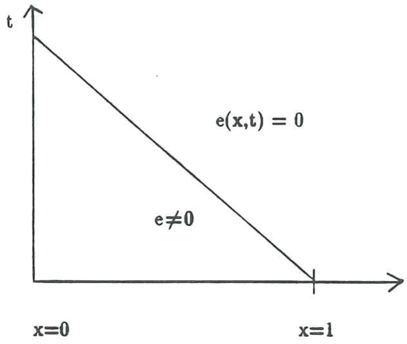


Figure 1. Error propagation in the best variable ω scheme.

The two and three dimensional laplacian can be treated in the same way once they are put in block triadiagonal form. In this case one will obtain a matrix of relaxation parameters W_j. The same is true of 4th order problems and even in one dimension the block form is needed. For example, consider

$$(u_{j-2}+u_{j+2})-4(u_{j+1}+u_{j-1})+6u_j=f_j, \tag{4.7}$$

where $u_1=u_0=0$, $u_{N+1}=u_{N+2}$, and $1 \leq j \leq N$.

This has the block triadiagonal form (3.1), where

$$A_{jj} = A = \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix} \tag{4.8}$$

$$A^{*}_{j+1\,j}=A_{jj+1}=B=\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \tag{4.9}$$

In this setting there are two relation parameters per nodal point; i.e.,

$$W_j = \begin{bmatrix} \omega_{j1} & \omega_{j2} \\ \omega_{j3} & \omega_{j4} \end{bmatrix} \tag{4.10}$$

These 2 × 2 matrix of relaxation parameters admit the following matrix continued fraction expansion

$$W_1 = I \tag{4.11}$$

$$W_j = (I-A^{-1}B^{*}W_{j-1}A^{-1}B)^{-1} \tag{4.12}$$

for $j \geq 2$.

REFERENCES

[1] Botta, Eugen F.F. and Veldman,Arthur E.P., "On local relaxation methods and their application to convection-diffusion equations", Journal of Computational Physics, Vol. 48, (1981), 127-149.

[2] Varga, R. S., Matrix Iterative Analysis, Prentice-Hall, 1962.

[3] Khosla, P.K. and Rubin, S.G., "Filtering of nonlinear instabilities", J. Engrg. Math. 13, (1979), 127-141.

[4] Benjamin, A.S. and Denny, V.E., "On the convergence of numerical solutions for 2-D flows in a cavity at large re", J. Comput. Phys. 33, (1979), 340-358.

[5] Strikwerda, J., "Iterative methods for the numerical solution of second order elliptic equations with large first order terms", SIAM J. Sci. Stat. Comput. 1, (1980), 119-130.

[6] Franklin, J.N., Matrix Theory, Prentice-Hall, 1968.

[7] Field, D.A., "Convergence theorems for matrix continued fractions", Siam J. Math. Anal., Vol 15, No. 6, (1984), 1220-1227.