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ON THE IRREDUCIBILITY OF BESSEL POLYNOMIALS

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Dedicated to Professor
I.J.Schoenberg on the
occasion of his 85-th
birthday.

1. INTRODUCTION. The BP(Bessel Polynomials) were defined by Krall and Frink [11] as the polynomial solutions of the differential equation

$$x^2 y_n'' + 2(x+1) y_n' - n(n+1) y_n = 0,$$

normalized by $y_n(0) = 1$. Also the name BESSEL POLYNOMIALS is due to Krall and Frink.

Explicitly one has

$$y_n(x) = \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!m!} \left(\frac{x}{2}\right)^m .$$

Sporadically the BP had appeared already earlier in the mathematical literature,at least since 1929 (see[1] and [12]).

In [5] (see also [6]) several of the algebraic properties of the BP were studied.Among others,it was shown that,for $n \leq 400$,all BP are irreducible over the rationals.Also the following conjecture was made:

CONJECTURE 1. All BP are irreducible.

However,only much weaker results were proven in [5] .The present paper contains some further progress made on this topic.An abstract of this work [7] has been published in the Notices of the American Mathematical Society.This abstract induced M.Filaseta to take up once more the problem of the irreducibility of BP. In [3] he obtained the strogest result known at present on this problem.He proves the following

THEOREM A(M.Filaseta) .Almost all BP are irreducible.

In his work, M. Filaseta uses rather deep algebraic number theoretic methods. For that reason it seams worthwhile to still publish the present short and simple paper, although the result is weaker than Theorem A, namely

THEOREM B. The set of integers n, for which $y_n(x)$ is irreducible has positive density.

2. SOME KNOWN RESULTS. We recall the following known results that can be found in [5] and [6] .

Theorem 1. If $n = p^m$, or $n = p-1$, or $n = p+1$, where here and in what follows p stands for a rational prime, then $y_n(x)$ is irreducible.

This is part of Theorem 1 in [6], Chapter 11; see also

Theorem 2. Let $p_j < p_{j+1}$ be consecutive primes, let $p_j < n < p_{j+1}$ and set $P = \max\{p \mid p|n(n+1)\}$, i.e., denote by P the largest prime factor of $n(n+1)$; then $y_n(x)$ cannot have a factor of degree less than $P - 1$.

This is Theorem 1(f) in [6], Chapter 11.

In what follows, p with or without subscript stands for a rational prime. To given pair of consecutive primes $p_j < p_{j+1}$ and $p_j < n < p_{j+1}$ set $k_1 = n - p_j$, $k_2 = p_{j+1} - n - 1$, $k(n) = \min(k_1(n), k_2(n))$ and $k = \max_{p_j<n<p_{j+1}} k(n)$. It is easy to verify that $k = (p_{j+1} - p_j)/2 - 1$.

Next, denote by $\partial^o f$ the degree of the polynomial $f(x)$. With these notations, we can state the following theorem.

Theorem 3. For every n, the BP $y_n(x)$ is either irreducible, or else contains an irreducible factor $g(x)$, with $\partial^o g \geq n - k(n)$; $y_n(x)$ contains no factor of degree d with $k(n) < d < n - k(n)$.

This is Theorem 4 in [6], Chapter 11.

3. MAIN THEOREM AND ITS PROOF.

THEOREM C. Let p_j, p_{j+1}, and P be defined as before. If $P > (p_{j+1} - p_j)/2$, then $y_n(x)$ is irreducible.

PROOF. Assume that $y_n(x) = g(x)h(x)$ is a non-trivial factorization of $y_n(x)$, with $g(x)$ irreducible and, hence, by Theorem 3, of degree $\partial^0 g \geqq n - k(n)$. Then $\partial^0 h \leqq k(n)$. Also, by Theorem 2, $\partial^0 h \geqq P - 1$. On the basis of our assumption we now have the following sequence of inequalities:

$\partial^0 h \leqq k(n) \leqq k_{\;} = (p_{j+1} - p_j)/2 - 1 < P - 1 \leqq \partial^0 h.$

This obvious contradiction shows that $y_n(x)$ has no non-trivial factorization.

REMARK. Up to $n \leqq 6.4 \cdot 10^5$ all n satisfy the condition of Theorem C, with 6 exceptions. It is clear that if some n does not satisfy said condition, it does not follow that $y_n(x)$ is reducible. Indeed, as observed by Filaseta, in all 6 cases mentioned, the corresponding polynomial $y_n(x)$ is irreducible by one of the theorems of Section 2.

4. HEURISTIC CONSIDERATIONS. Previous Remark suggests the following conjecture :

CONJECTURE 2. For almost all n, the condition $P > (p_{j+1} - p_j)/2$ holds.

Clearly, this is very likely the best we can hope for, because we already know that "almost all" cannot be replaced by "all".

Conjecture 2 is made plausible by the following two facts.

a) If $\Omega(n)$ is the total number of prime factors of n, then the normal order of $\Omega(n)$ is loglog n (see [9]). This means that, except for a set of integers of density zero, the average prime divisor of n is about $n^{1/\log\log n}$ and that of $n(n+1)$ is about $n^{2/\log\log n}$. In particular, "in general", $P \geqq n^{2/\log\log n}$. This reasoning, however, does not offer an easy way to estimate the number of exceptions to the last inequality, with $n \leqq x$, say. If we are satisfied with a slightly weaker inequality, then it is easy to show that, for almost all n, $P > \mathrm{Log}^2 n$. We shall not actually use this result in any proof. Therefore, in order not to interrupt the flow of the main idea, we give, for completeness, the proof (due to Filaseta [4]) in the Appendix.

b) We also know (see 2) that, for every function $g(u)$ that is monotonically increasing (no matter how slowly) and for almost all primes p_j, the following inequalities hold:

$$\frac{\log p_j}{g(p_j)} < p_{j+1} - p_j < g(p_j) \log p_j .$$

It is now tempting to conclude that, for almost all n,

$$(p_{j+1} - p_j) < \log^2 p_j \leqq \log^2 n < p$$

which implies Conjecture 2. Unfortunately, we are unable to justify this conclusion. Indeed, while $\log^2 n \geqq \log^2 p_j > p_{j+1} - p_j$ holds for almost all primes, it is conceivable that in those exceptional cases, in which $p_{j+1} - p_j$ is exceptionally large, so that $\log^2 p_j < p_{j+1} - p_j$, the number of n in those intervals is so large, that the total density of the set of n, for which the needed inequality fails is no longer of density zero. It is clear that, if we could eliminate this last possibility, we would have proven Theorem A. This, however, would require the proof that the number of $n \leqq x$, in all intervals (p_j, p_{j+1}), in which $\log^2 p_j < p_{j+1} - p_j$, is "small". No such result, sufficiently strong for the present purpose seems known. The best that seems obtainable by the present approach is Theorem B.

5. <u>PROOF OF THEOREM B</u>. If we deny Theorem B, we assert that the set of integers n, such that $p_j < n \leqq p_{j+1}$ and for which $p_{j+1} - p_j \leqq \log^2 p_j$ is of density zero. Let us denote this set by E and set $E(x) = \{n \in E,\ n \leqq x\}$; finally, let $|E(x)|$ be the number of integers in $E(x)$.

Next, we observe that

$$x \geqq \sum_{p_{j+1} \leqq x} (p_{j+1} - p_j) = \sum_{\substack{p_{j+1} \leqq x \\ p_{j+1} - p_j \geqq \log^2 p_j}} (p_{j+1} - p_j) + \sum_{\substack{p_{j+1} \leqq x \\ p_{j+1} - p_j < \log^2 p}} (p_{j+1} - p_j)$$

By assumption the last sum is $o(x)$. Also,

$$\sum_{\substack{p_{j+1}\geq x \\ p_{j+1}-p_j\geq \log^2 p_j}} (p_{j+1}-p_j) \geq \sum_{\substack{p_{j+1}\leq x \\ p_{j+1}-p_j\geq \log^2 p_j}} \log^2 p_j = \sum_{p_{j+1}\leq x} \log^2 p_j - \sum_{\substack{p_{j+1}\leq x \\ p_{j+1}-p_j<\log^2 p_j}} \log^2 p_j .$$

Let us denote the last sum by S. Heuristically, the density of primes in $E(x)$ is $(\log x)^{-1}$ and this implies that the number of primes in $E(x)$ is at most $c_1 \, |E(x)| \, (\log x)^{-1}$. By our present assumption, $|E(x)| = o(x)$ and it follows that $S = o(x \log x)$.

For a proof of that heuristic statement assume that the density of the p_j's in the set of all primes is positive. In that case there exists a constant $c > 0$, such that the number of intervals in $E(x)$ exceeds $c\, x/\log x$. It also is known (see 8) that there exists a constant C (explicitly known) such that the number of pairs of consecutive primes p_j, p_{j+1} with $p_{j+1} \leq x$ and $p_{j+1} - p_j = 2k$ is at most equal to $Cx/\log^2 x$, with the possible exception of a (most likely empty) set of integers k, of density zero. Hence, the number of "small" intervals of $E(x)$, i.e., of intervals of length at most $(\log x)/K$

$$2C \frac{x}{\log^2 x} \cdot \frac{\log x}{K} = \frac{2C}{K} \frac{x}{\log x}$$

The total number of intervals in $E(x)$ is, under the present assumptions, larger than $c\, x/\log x$. If we select K sufficiently large, the number of "large" intervals of $E(x)$, i.e., of intervals with $\frac{\log x}{K} < p_{j+1} - p_j \leq \log^2 p_j \leq \log^2$ is at least $c_1\, x/\log x$, with some positive c_1. It follows that $|E(x)| \geq$

$$\sum_{\substack{p_j \in E(x) \\ p_{j+1}-p_j > (\log x)/K}} (p_{j+1}-p_j) > c_1 \frac{x}{\log x} \cdot \frac{\log x}{K},$$ which is not $o(x)$.

This proves that the density of the p_j's among the primes is zero. It follows that $S \leq \log^2 x \cdot o(x/\log x) = o(x \log x)$, as claimed.

So far, we have shown that $x \geq \sum_{p_{j+1}\leq x} (p_{j+1}-p_j) \geq \sum_{p_{j+1}\leq x} \log^2 p_j + o(x \log x)$.

By use of the Prime Number Theorem, we also obtain the assymptotic equalities

$$\sum_{p_{j+1}\leq x} \log^2 p_j \sim \int_2^{\pi(x)} \log^2\left(\frac{z}{\log z}\right) dz \simeq \frac{x}{\log x} \log^2\left(\frac{x}{\log x}\right) \simeq x \log x .$$

It follows that, if we deny Theorem B, we obtain the false result $x \geq x \log x$.

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Lemma (Filaseta [4]). Let E stand for the set of integers $n \leq x$, such that the largest prime divisor of n is at most equal to $\log^2 x$ and denote $\sum_{n \in E} 1$ by $E(x)$.

Then $E(x) = 0\left(\frac{x \log\log x}{\log x}\right)$; in particular, $E(x) = o(x)$.

Proof. For convenience, set $\frac{x \log\log x}{\log x} = g(x)$. If we deny the lemma, then, for sufficiently large x, $E(x) > cg(x)$, for any, arbitrarily large (but fixed) constant c. It follows that $\sum_{n \in E} \log n > \sum_{n \leq cg(x)} \log n$,

because, by assumption, there are at most $cg(x)$ terms on the left and we only reduce the sum, if we replace $\log n$ with $n \in E$, by the smaller sum of $\log n$, taken over the first consecutive $cg(x)$ integers n. The right hand side equals

$$\sum_{n \leq \sqrt{x}} \log n + \sum_{\sqrt{x} n \leq cg(x)} \log n > \sum_{\sqrt{x} n \leq cg(x)} \log n \geq \tfrac{1}{2}(\log x)\Big(\sum_{n \leq cg(x)} 1\Big) + 0(\sqrt{x} \log x)$$

$$= \frac{c}{2} \log x \frac{x \log\log x}{\log x} = \frac{c}{2} x \log\log x$$

by replacing if necessary, c by a smaller c. However, the left hand side satisfies

$$\sum_{n \in E} \log n = \log \prod_{n \in E} n = \log \prod_{\substack{n \leq x \\ p|n \Rightarrow p \geq \log^2 x}} = \log \prod_{\substack{n < x \\ p \leq \log^2 x p^r || n}} \Big(\prod p^r\Big) = \sum_{\substack{p \leq \log^2 x \\ n < x \\ p^r || n}} \log \Big(\prod p^r\Big)$$

$$= \sum_{p \leq \log^2 x} \sum_{\substack{n < x \\ p^r || n}} \log p \leq 2 \sum_{p \leq \log^2 x} \Big[\frac{x}{p}\Big] \log p \leq 2 \sum_{p \leq \log^2 x} \frac{x \log p}{p} = 2 \sum_{p \leq \log^2 x} \frac{\log p}{p}.$$

By a classical result (see, e.g., Theorem 425 in [10]), $\sum_{p \leq y} \frac{\log p}{p} = \log y + 0(1)$.

Hence, $\sum_{n \in E} \log n < x \cdot 4 \log\log x$. If we combine the two inequalities, we obtain that, for fixed c, $\frac{c}{2} x \log\log x \leq \sum_{n \in E} \log n < 4x \log\log x$, which is obviously false for $c > 8$. This proves the lemma and, in particular, that, for almost all n, $P > \log^2 x \geq \log^2 n$, because $n \leq x$.