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ORDER-S-QUASIBARRELLED VECTOR LATTICES AND CLOSED GRAPH THEOREM

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ABSTRACT

We introduce a new class of ordered locally convex vector lattices called order-S-quasibarrelled vector lattices. We give some characterizations, study some permanence properties and prove a closed graph theorem and a Banach-Steinhaus Theorem for this class of locally convex vector lattices.

§1 INTRODUCTION: Barrelled spaces, introduced by Bourbaki [1] in 1950, form an important class of locally convex spaces; they are neither complete nor metrizable although they inherit some of the important properties of Banach and Fréchet spaces; for details, the reader is referred to [2]. In 1978, Snipes [5] introduced S-barrelled spaces similar to barrelled spaces, using a notion of sequential convergence.

A locally convex space equipped with the finest locally convex topology is barrelled, and the order-bound topology (the topology generated by the class of all order-bornivorous circled convex sets) on a vector lattice is the finest locally solid topology. But a locally convex vector lattice, equipped with the order-bound topology need not be barrelled ([8], page 187). This led Wong [7] (or See [8] or [2]) to introduce the concept of order-infrabarrelled Riesz space (henceforth called order-quasibarrelled vector lattice). Motivated by the concepts of Snipes [5] and Wong [7], we introduce and study, in the present paper, the concept of, what we call, order-S-quasi-barrelled vector lattice, similar to the concept of order-quasibarrelled vector lattice of Wong [7].

In Section 2, we collect some definitions and results which will be used in the sequel. In section 3, we introduce the concept of order-S-quasibarrelled vector lattice; we give some characterizations for order-S-quasi-barrelled vector lattices and study some of their permanence properties. In Section 4, we prove a closed graph theorem for order-S-quasibarrelled vector lattices. In Section 5, we present a Banach-Steinhaus theorem for order-S-quasi-barrelled vector lattices.

§2 PRELIMINARIES: All vector spaces are over the field $\mathbb{R}$ of real numbers. We abbreviate locally convex space and locally convex vector lattice to l.c. space and l.c. vector lattice respectively. We write (E, u) to denote an l.c. space with topology u. A sequentially closed, circled, convex and absorbing subset B of (E, u) is called a sequential barrel (abbreviated to S-barrel). We say that V is a sequential neighbourhood of O in (E, u) if every sequence in (E, u) which converges to 0 is ultimately in V. An l.c. space (E, u) is called an S-barrelled space if every S-barrel in (E,u) is a neighbourhood of 0.

We write (E, C) to denote a vector lattice with the positive cone. A subset B of (E,C) is called order-bornivorous if B absorbs all order-bounded subsets of E. We write (E,C,u) to denote an l.c. vector lattice. In an l.c. vector lattice, the positive cone C is normal (that is, there is a neighbourhood basis at 0 consisting of full sets - A set A in (E,C) is said to be full if $A = |A|$ where $|A| = \{z \in E : x \leq z \leq y, \text{ for } x \in A, y \in A\}$). This implies that every order-bounded subset of E is u-bounded ([3], page 62). An l.c. vector lattice (E,C,u) is called order-quasi-barrelled if each order-bornivorous barrel in (E,C,u) is a neighbourhood of 0.

§3 ORDER-S-QUASIBARRELLED VECTOR LATTICES

3.1 DEFINITION: An l.c. vector lattice (E,C,u) is called an order-S-quasibarrelled vector lattice (henceforth abbreviated to O-S-Q vector lattice) if every order-bornivorous S-barrel in (E,C,u) is a sequential neighbourhood of 0.

3.2 REMARK: Every S-barrelled l.c. vector lattice is an O-S-Q vector lattice. In particular, Banach and Fréchet lattices are O-S-Q vector lattices.

Let B be a subset of a vector lattice (E,C). The set K(B) defined by

$$K(B) = \{x \in E : [-|x|, |x|] \subseteq B\}$$

is called the solid kernel of B; it is the largest solid set contained in B and is convex if B is so. Let B be circled. Then K(B) is order-bornivorous <==> B is so ([7]; or [2], page 103).

3.3 THEOREM: An l.c. vector lattice (E,C,u) is an O-S-Q vector lattice if and only if each solid S-barrel in (E,C,u) is a sequential neighbourhood of O.

Proof: Assume that (E,C,u) is an O-S-Q vector lattice. Let V be a solid S-barrel in (E,C,u). Since V is solid and absorbing, it follows that V is order-bornivorous and hence a sequential neighbourhood of O in (E,C,u). To prove the converse, let U be an order-bornivorous S-barrel in (E,C,u) and K(U) the solid kernel of U. Since U is circled and order-bornivorous, K(U) is order-bornivorous as remarked before 3.3. Also K(U) is convex. Using $\overline{K(U)}^s$ for the sequential closure of K(U), we have

$$K(U) \subseteq \overline{K(U)}^s \subseteq \bar{U}^s = U$$

so that $K(U) = \overline{K(U)}^s$. Hence K(U) is sequentially closed. Thus, K(U) is a solid S-barrel in (E,C,u) and hence a sequential neighbourhood of O. Since K(U) $\subseteq$ U, it follows that U is a sequential neighbourhood of O in (E,C,u).

Let X be a topological space, $a \varepsilon X$ and f a real-valued function on X. Then f is called sequentially lower semi-continuous at a if given $\varepsilon > 0$, there exists a sequential neighbourhood U of a such that $x \varepsilon U \Longrightarrow f(x) > f(a) - \varepsilon$; equivalently, given any interval $(c, +\infty)$ in $\mathbb{R}$ such that $f(a) \varepsilon (c, +\infty)$, the set $f^{-1}[(c, +\infty)]$ is a sequential neighbourhood of a. If f is sequentially lower semi-continuous at each element of X, we say that f is sequentially lower semi-continuous, [5].

3.4 THEOREM: Let (E,C,u) be an l.c. vector lattice. The following statements are equivalent:

(a) (E,C,u) is an O-S-Q vector lattice.

(b) Each order-bounded sequentially lower semi-continuous semi-norm on E is sequentially continuous.

(c) Each sequentially lower semi-continuous lattice semi-norm on E is sequentially continuous.

Proof: (a) $\Longleftrightarrow$ (b): A semi-norm p on E is order-bounded and sequentially lower semi-continuous if and only if the set $V = \{x \varepsilon E: p(x) \leq 1\}$ is an order-bornivorous S-barrel in (E,C,u). (a) $\Longleftrightarrow$ (c): A semi-norm q on E is a sequentially lower semi-continuous lattice-semi-norm if and only if the set $U = \{x \varepsilon E : q(x) \leq 1\}$ is a solid S-barrel in (E,C,u).

An l.c. space E is said to be C-sequential if every convex sequentially open subset of E is open. E is C-sequential if and only if every convex, circled and sequential neighbourhood of 0 in E is a neighbourhood of 0 ([4]). In view of this, the following result is obvious.

3.5 PROPOSITION: A C-sequential O-S-Q vector lattice is an order-quasibarrelled vector lattice.

An l.c. space E is said to be sequential if every sequentially open subset of E is open [6]. In view of this, the following result is obvious.

3.6 PROPOSITION: A sequential order-quasibarrelled vector lattice is an O-S-Q vector lattice.

Let (E,C,u) be an l.c. vector lattice. For any given $x \in C$, define

$$p_x(f) = |f|(x) = \sup\{|\langle y,f\rangle| : y \in [-x,x]\}, \quad f \in E'.$$

Then $\{p_x : x \in C\}$ defines a locally solid topology on E' denoted by $\sigma_s(E', E)$. Similarly we have the topology $\sigma_s(E,E')$ on E. It is easy to see that

$$\sigma(E', E) \subset \sigma_s(E', E) \subset \beta(E', E).$$

For more details, see [7] or [8].

Now we present an example of an O-S-Q vector lattice which is not order-quasibarrelled.

3.7 EXAMPLE: Consider the space $(\ell^1, \sigma_s(\ell^1, \ell^\infty))$. Let u denote the lattice-norm topology on ℓ^1. First we note that the $\sigma_s(\ell^1, \ell^\infty)$ - convergence and u-convergence of sequences in ℓ^1 are the same. If B is a solid S-barrel in $(\ell^1, \sigma_s(\ell^1, \ell^\infty))$, then B is sequentially closed in (ℓ^1, u) and hence closed in (ℓ^1, u). Since (ℓ^1, u) is order-quasibarrelled, B is a neighbourhood of 0 in (ℓ^1, u) and $\sigma_s(\ell^1, \ell^\infty)$ - sequential neighbourhood of 0 in ℓ^1. Thus, $(\ell^1, \sigma_s(\ell^1, \ell^\infty))$ is an O-S-Q vector lattice. Since $\sigma_s(\ell^1, \ell^\infty)$ is strictly weaker than u, and since (ℓ^1, u) and $(\ell^1, \sigma_s(\ell^1, \ell^\infty))$ have the same topological duals, it follows that $(\ell^1, \sigma_s(\ell^1, \ell^\infty))$ is not order-quasibarrelled.

Now we present an example to show that the converse in 3.2 need not be true.

3.8 EXAMPLE: Let $m = \ell^\infty$ be the Banach lattice of all bounded real sequences with the usual pointwise ordering and the supremum norm $||\cdot||$. If E_0 is the subspace of m consisting of finite sequences and if e is the sequence having 1 in every coordinate, then

$$E = \{x + \lambda e \; : \; x \varepsilon E_0, \lambda \varepsilon \mathbb{R} \},$$

with the supremum norm $||\cdot||$ induced from m, is an order-unit-normed vector lattice with e as order-unit; hence E is an order-quasibarrelled vector lattice. But E is not barrelled ([8]; or see [2], page 106; or [9], chapter 5, # 26). Since E is metrizable and hence sequential, and since a sequential barrelled space is S-barrelled ([5], page 222), it follows that E is not S-barrelled. But, by 3.6, it follows that E is an O-S-Q vector lattice.

Now we consider conditions under which the converse of 3.2 is true.

Let (E,C,u) be an l.c. vector lattice and $t\in C$. Then $E_t = \bigcup_n n[-t, t]$ is the lattice ideal in E generated by t. If p_t denotes the gauge of $[-t, t]$ on E_t and $C_t = C \cap E_t$, then (E_t, C_t, p_t) is a normed vector lattice for which the relative topology on E induced by u is coarser than the norm topology on (E,u) given by p_t.

3.9 PROPOSITION: Let (E,C,u) be an O-S-Q vector lattice. For any $t\in C$, if E_t is complete for the norm p_t, then (E,C,u) is S-barrelled.

Proof: Let V be an S-barrel in (E,C,u) and $t\in C$. Since E_t is complete for the norm p_t, (E_t, C_t, p_t) is S-barrelled. Clearly $V \cap E_t$ is an S-barrel in (E_t, C_t, p_t) because the relative topology on E_t is coarser than the norm topology given by p_t; consequently, $V \cap E_t$ absorbs $[-t, t]$. Since $\{[-t, t] : t\in C\}$ is a fundamental system of order-bounded sets in E, it follows that V is an order-bornivorous S-barrel in (E,C,u) and hence a sequential neighbourhood of 0 in E. This proves that (E,C,u) is S-barrelled.

A vector lattice (E,C) is said to be σ-order-complete if every increasing sequence in E that is majorized in E has a supremum in E ([8], page 16).

3.10 COROLLARY: A σ-order-complete O-S-Q vector lattice is barrelled.

3.11 PROPOSITION: A sequentially complete C-sequential O-S-Q vector lattice (E,C,u) is barrelled.

Proof: By 3.5, (E,C,u) is order-quasibarrelled. But a sequentially complete order-quasibarrelled vector lattice is barrelled ([2], page 116).

Let (E,u) and (F,v) be l.c. spaces and $f : E \longrightarrow F$ a linear map. f is said to be almost sequentially open if, given a sequential neighbourhood U of 0 in E, the smallest v-sequentially closed set $\overline{f(U)}^{ss}$ containing f(U) is a v-sequential neighbourhood of 0 in F. f is said to be almost sequentially continuous if, given any neighbourhood V of 0 in F, the smallest u-sequentially closed set $\overline{f^{-1}(V)}^{ss}$ containing $f^{-1}(V)$ is a sequential neighbourhood of 0 in E ([5]).

3.12 PROPOSITION: Let (E,C,u) be an O-S-Q vector lattice and (F,K,v) an l.c. vector lattice. If f is a lattice homomorphism of E into F, then f is almost sequentially continuous.

Proof: If V is a convex and solid neighbourhood of 0 in F, then $\overline{f^{-1}(V)}^{ss}$ is an order-bornivorous S-barrel in (E,C,u) and hence a sequential neighbourhood of 0.

3.13 PROPOSITION: Let (E,C,u) and (F,K,v) be l.c. vector lattices and $f:E \longrightarrow F$ a sequentially continuous and almost sequentially open lattice homomorphism. If (E,C,u) is an O-S-Q vector lattice, so is (F,K,v).

Proof: Let B be an order-bornivorous S-barrel in (F,K,u). Then $f^{-1}(B)$ is an order-bornivorous S-barrel in (E,C,u) and hence a sequential neighbourhood of 0 in E. Since f is almost open, $\overline{ff^{-1}(B)}^{ss}$ is a sequential neighbourhood of 0 in F. But

$$\overline{f(f^{-1}(B)}^{ss} \subset \bar{B}^{ss} = B$$

so that B is a sequential neighbourhood of 0 in F. This proves that (F,K,v) is an O-S-Q vector lattice.

3.14 COROLLARY: Let (E,C,u) be an O-S-Q vector lattice and M a sequentially closed lattice ideal in E. Then E/M is an O-S-Q vector lattice.

3.15 THEOREM: The l.c. direct sum of a family of O-S-Q vector lattices is an O-S-Q vector lattice.

Proof: Let $\{(E_\alpha, C_\alpha, u_\alpha) : \alpha\varepsilon I\}$ be a family of O-S-Q vector lattices. Let $E = \oplus_\alpha E_\alpha$, $C = \pi_\alpha C_\alpha$, $K = C \cap E$ and $u = \oplus_\alpha u_\alpha$ (the l.c. direct sum topology on E). Let $i_\alpha : E_\alpha \longrightarrow E$ be the injection map for each α. It is well-known that

(E,K,u) is an l.c. vector lattice. If V is a solid S-barrel in (E,K,u), then $i_\alpha^{-1}(V)$ is a solid S-barrel in $(E_\alpha, C_\alpha, u_\alpha)$ and hence a sequential neighbourhood of 0 in $(E_\alpha, C_\alpha, u_\alpha)$. This implies that V is a sequential neighbourhood of 0 in E. This proves that (E,K,u) is an O-S-Q vector lattice.

The following example shows that a lattice ideal of an O-S-Q vector lattice need not be of the same sort.

3.16 EXAMPLE: Let E = C[0,1], the space of continuous real-valued functions on [0,1], equipped with the supremum norm $||\cdot||$ and ordered by the positive cone

$$K = \{f \in E : f(x) \geq 0 \text{ for all } x \in [0,1]\} .$$

E is a Banach lattice and hence an O-S-Q vector lattice by 3.2. Let M be the subspace of E consisting of all elements $f \in E$ which vanish in a neighbourhood (depending on f) of x = 0. M is a lattice ideal in E. To show that M is not an O-S-Q vector lattice under the relative topology on M induced by the norm $||\cdot||$ of E, let

$$V = \{f \in M : n|f(\tfrac{1}{n})| \leq 1 \text{ for all } n \in \mathbb{N}\} .$$

Then V is a solid barrel in M, but not a neighbourhood of 0 in M ([7] or [8]; or see [2], page 120; or [9], Chapter 5, # 23). Hence M is not an order-quasibarrelled vector lattice. Since M is a sequential l.c. space, it follows, by 3.6, that M is not an O-S-Q vector lattice.

Under certain conditions, we show that a lattice ideal of an O-S-Q vector lattice is of the same sort.

A lattice ideal M in a vector lattice (E, C) is a *σ-normal* subspace if it follows from $\sup_n x_n = s \in E$ with $0 \leq x_n \in M$ for all $n \geq 1$ that $s \in M$ ([8]).

3.17 THEOREM: Let (E,C,u) be a σ-order-complete O-S-Q vector lattice and M a σ-normal subspace of E. Then M is an O-S-Q vector lattice under the induced topology.

Proof: Let V be a solid S-barrel in M and let

$$U = \{x \in E : y \in V \text{ whenever } 0 \leq y \leq |x| \text{ and } y \in M\}.$$

Then U is a sequentially closed, convex and solid set in E such that $U \cap M = V$. We show that U is absorbing. Suppose not. Then there exists an element $x \in C$ which is not absorbed by U. Hence, for each positive integer n, there exists $y_n \in M$ such that $0 \leq y_n \leq \frac{1}{n} x$ and $y_n \notin V$. Since E is σ-order-complete, $y = \sup\{n\, y_n : n \geq 1\}$ exists in E. Since M is a σ-normal subspace of E, $y \in M$. Thus, $\{n\, y\}_{n=1}^{\infty}$ is contained in the order-interval $[0,y]$ in M and is not absorbed by V. But this contradicts the assumption that V is absorbing. Thus U is a solid S-barrel in E and so a sequential neighbourhood of 0 in E. Hence $V = U \cap M$ is a sequential neighbourhood of 0 in M. This proves that M is an O-S-Q vector lattice.

§4 CLOSED GRAPH THEOREM

In this section we obtain an analogue of closed graph theorem for order-S-quasibarrelled vector lattices.

4.1 THEOREM: Let (E,C,u) be an order-S-quasibarrelled vector lattice. For any Fréchet space (F,v), a linear map f of E into F is sequentially continuous if it satisfies the following conditions:

(i) f sends each order-bounded subset of E into an u-bounded subset of F.

(ii) The graph of f is sequentially closed.

<u>Proof</u>: Let V be a circled and convex neighbourhood of 0 in (F,v). Then $f^{-1}(V)$ is an order-bornivorous S-barrel in E and hence a sequential neighbourhood of 0 in E. This implies that f is almost sequentially continuous.

Let $\{V_i\}$ be a countable basis of neighbourhoods of 0 in F.

We may assume that

$$V_{i+1} + V_{i+1} \subset V_i \quad \text{for each } i$$

and that each V_i is closed. Since f is almost sequentially continuous, there is a sequence $\{U_i\}$ of sequential neighbourhoods of 0 in E such that

$$U_i \subset \overline{f^{-1}(V_i)}^s \quad , \text{ for all } i \in \mathbb{N} \quad (1).$$

We assume that

$$U_{i+1} + U_{i+1} \subset U_i \quad \text{for each } i \in \mathbb{N}.$$

It follows from (1) that

$$U_{k_o} \subset f^{-1}(V_{k_o}) + U_{k_o+1}, \text{for each } k_o.$$

Let $x_{k_o} \in U_{k_o}$. then there exist $y_{k_o} \in f^{-1}(V_{k_o})$ and $x_{k_o+1} \in U_{k_o+1}$ such that

$$x_{k_o} = y_{k_o} + x_{k_o+1}.$$

So we can find inductively a sequence x_k $(k>k_o)$ and y_k $(k \geq k_o)$ such that

$$x_k = y_k + x_{k+1}; x_k \in U_k; f(y_k) \in V_k, \text{ for each } k \geq k_o.$$

Summing up both sides of

$$x_k = y_k + x_{k+1} \text{ for } k_o \leq k \leq n,$$

we get

$$x_{k_o} = \sum_{k=k_o}^{n} y_k + x_{n+1} .$$

Thus

$$\sum_{k=k_0}^{n} y_k \longrightarrow x_{k_0}.$$

On the other hand,

$$f\left(\sum_{k=k_0}^{n} y_k\right) = \sum_{k=k_0}^{n} f(y_k).$$

Since $f(y_k) \in V_k$ for each k, the series $\sum_{k \geq k_0} f(y_k)$ is a Cauchy sequence. Hence, since F is complete, it converges to some point z. Moreover

$$\sum_{k=k_0}^{n} f(y_k) \in \sum_{k=k_0}^{n} V_k \subset V_{k_0-1}, \text{ for all } n.$$

Thus

$$z \in V_{k_0-1}.$$

Moreover

$$x_{k_0} = \sum_{k=k_0}^{n} y_k + x_{n+1}, \text{ for all } n \qquad (2).$$

Let U be an arbitrary sequential neighbourhood of 0 in E and fix $n \in \mathbb{N}$.

By (2),

$$\sum_{k=k_0}^{n} y_k = x_{k_0} - x_{n+1} \in x_{k_0} + \overline{f^{-1}(V_{n+1})}^{s} \subset x_{k_0} + f^{-1}(V_{n+1}) + U.$$

Hence there exists z' such that

$$\sum_{k=k_o}^{n} y_k - z' \varepsilon x_{k_o} + U \qquad (3)$$

and $f(z') \varepsilon V_{n+1}$. On the other hand,

$$\sum_{k=k_o}^{n} f(y_k) - z = \sum_{k=n+1}^{\infty} f(y_k) \varepsilon V_{n+1} + V_{n+2} + \ldots \subset V_n.$$

Thus

$$f(\sum_{k=k_o}^{n} y_k - z') - z \,\varepsilon V_n + V_{n+1} \subset V_{n-1} \qquad (4) .$$

Combining (3) and (4) , we see that (x_{k_o}, z) is in the sequential closure of G_f in EXF and hence $f(x_{k_o}) = z$.

§5 BANACH-STEINHAUS THEOREM

Let H be a set of sequentially continuous linear maps from E to F, E and F any topological vector spaces. H is said to be equi-sequentially continuous if for each neighbourhood V of 0 in F, there is a sequential neighbourhood U of 0 in E such that $f(U) \subset V$ for all $f \varepsilon H$ (equivalently $\bigcap_{f\varepsilon H} f^{-1}(v)$ is a sequential neighbourhood of 0 in E) [10] .

5.1 THEOREM: Let (E,C,u) be an O-S-Ω vector lattice and (F,K,v) any l.c. vector lattice. Let H be a pointwise bounded set of sequentially continuous lattice homomorphisms of E into F. Then H is equi-sequentially continuous.

Proof: Let V be a closed, convex and solid neighbourhood of 0 in F, and let $W = \bigcap_{f \in H} f^{-1}(V)$. Then W is a solid S-barrel in E and hence a sequential neighbourhood of 0 in E. This proves that H is equi-sequentially continuous.

5.2 COROLLARY: Let (E,C,u) and (F,K,v) be as in 5.1. Let $\{f_n\}$ be a sequence of sequentially continuous homomorphisms of E into F, converging pointwise to $f:E \longrightarrow F$. Then f is a sequentially continuous lattice homomorphism.

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