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AN ABSTRACT LQ-OPTIMAL CONTROL PROBLEM AND ITS APPLICATIONS

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In our recent paper [1], we discussed an LQ-optimal control problem in the framework of the L^2-space. A further analysis of the topic reveals the possibility of an abstract formulation of that LQ-optimal control problem. In turn, the abstract statement permits other particularization, such as the case of discrete dynamical systems.

This is the formulation of the abstract LQ-optimal control problem we have in mind: Let $\mathcal{H}$ and H be two real Hilbert spaces, and assume $T: H \supset K \to \mathcal{H}$ is a linear continuous operator, with K closed in H and convex; find $\bar{u} \in K$, such that

(1) $$\min_{u \in K} \{ < Px, x > + < Qu, u > ; x = Tu \}$$

is attained at $\bar{u}$, where $P = P^* \geq 0$ and $Q = Q^* > 0$ are given linear operators on $\mathcal{H}$ and H, respectively.

The problem formulated above has an immediate answer, which is contained in the following statement.

Theorem. Under the assumptions considered above on $\mathcal{H}, H, T, K, P$ and Q, there exists a unique $\bar{u} \in K$ providing the solution in (1).

Proof. Let us denote

(2) $$C(x, u) = < Px, x > + < Qu, u >$$

for $u \in H$, $x = Tu$, and notice that our assumptions imply

$$C(x,u) \geq 0. \tag{3}$$

The only possibility for having $C(x,u) = 0$ occurs when $u = \theta$. Hence, $x = \theta \in \mathcal{H}$.

We shall introduce on the space H a new scalar product, which generates a norm equivalent to the initial one.

Indeed, we can set

$$\ll u,v \gg \; = \; < Px,y > + < Qu,v > , \tag{4}$$

where $u,v \in H$, and $x = Tu$, $y = Tv$. It is obvious from (2) and (4) that

$$\ll u,u \gg \; = C(x,u), \quad u \in H, \quad x = Tu. \tag{5}$$

It is an elementary exercise to check that (4) is a new scalar product on H, and we leave this task to the reader.

Since (1) means finding $\min \ll u,u \gg$ when $u \in K$, we realize that our *LQ*-optimal problem can be reduced to a well-known result from Hilbert space theory: Any convex closed set in a Hilbert space contains a unique element with minimal norm.

We need only prove that under the new norm (5), H is also complete. Hence, the set $K \subset H$ which has been assumed closed in the initial topology, will be also closed in the topology induced by the norm (5). It is obviously sufficient to show that the norm (5) is equivalent to the norm $\|\cdot\|$ on H, derived from the scalar product $< \cdot,\cdot >$ initially defined on H.

From our assumptions one easily derives

(6) $$\ll u,u \gg \; \geq \; < Qu,u > \; \geq \lambda \| u \|^2,$$

where $\lambda > 0$ exists according to the hypothesis $Q = Q^* > 0$. On the other hand,

(7) $$\ll u,u \gg \; \leq (\| P \| \, \| T \| + \| Q \|) \| u \|^2 .$$

The inequalities (6) and (7) prove that the initial norm $\| \cdot \|$ on H and the norm derived from the new scalar product $\ll \cdot , \cdot \gg$ are equivalent, which ends the proof of the Theorem.

As mentioned, the above abstract scheme is inspired by the L^2-space setting adopted in [1]. Briefly, $\mathcal{H} = L^2([0,T],R^n)$, $H = L^2([0,T],R^m)$, while the system leading to the operator T is

(8) $$\dot{x}(t) = (Ax)(t) + (Bu)(t), \; t \in [0,T],$$

where A is a linear continuous Volterra operator on $\mathcal{H}$, and B is a similar operator on H. The initial condition $x(0) = \theta \in R^n$ is attached. The cost functional is

(9) $$C(x,u) = \int_0^T \{ < (Px)(t), x(t) > + < (Qu)(t), u(t) > \} dt$$

with $P(t)$ and $Q(t)$ satisfying conditions of the same nature as those specified above.

It has been shown in [1] that the relationship between x and u can be represented in the form

(10) $$x(t) = \int_0^t X(t,s) u(s) ds,$$

under convenient conditions for $X(t,s)$.

The conclusion is that there exists a unique controller $\bar{u} \in K \subset L^2([0,T], R^m)$, such that $C(x,\bar{u})$ is minimal with respect to all controllers in K, K being closed and convex.

Another possibility for applying the above theorem is related to discrete control processes. More precisely, let us assume that the linear plant is described by the system of difference equations involving abstract Volterra operators

(11) $$x(k+1) = (Ax)(k) + (Bu)(k), \quad k \geq 0,$$

where $x \in R^n$ and $u \in R^m$. An initial condition of the form $x(0) = x^0 \in R^n$ must be associated with (11), in order to determine a unique solution, as soon as the controller $u \in (\ell^2)^m$ is assigned. Of course, we need to assure that the solution $x = Tu$ belongs to a given Hilbert space, such as $(\ell^2)^n$, a condition which is not necessarily satisfied for arbitrary A and B, even though $u \in (\ell^2)^m$. It is known that such kind of properties take place, under supplementary conditions on the operators A, B [2]. These authors term these kinds of properties as admissibility properties for linear systems of difference equations. These properties are complex enough to be detailed here. It is useful noticing that special solutions of systems like (11) may be uniquely determined by an initial conditions of the form $P_0 x(0) = \xi_0 \in X_0$, when X_0 is a subspace of the space R^n, and P_0 is the projector on X_0.

Let us consider now a special case of (11), namely,

(12) $$x(k+1) = Ax(k) + Bu(k), \quad k \geq 0,$$

where A is a square matrix of order n, while B is a matrix of type n by m. As usual, the initial condition $x(0) = x^0 \in R^n$ will be attached to the system (12), in order to determine a unique solution $\{x(k);\ k \geq 0\}$. For simplicity we will consider the initial condition $x(0) = \theta \in R^n$.

In order to assure the existence of convenient solutions to (12), for any $u \in (\ell^2)^m$, we will impose the following restriction on the matrix A:

$$(13) \qquad \det(\lambda I - A) = 0 \Rightarrow |\lambda| < 1.$$

In other words, the spectrum of A must lie inside the disc $|\lambda| < 1$.

Condition (13) is termed as the stability condition for the matrix A, and it is known that, under this condition, all solutions of $x(k+1) = Ax(k)$ tend to zero as $k \to \infty$.

It is useful to notice that condition (13) is also (necessary and) sufficient for the system

$$(14) \qquad x(k+1) = Ax(k) + b(k),\ k \geq 0,$$

to possess all the solutions in $(\ell^\infty)^n$, where ℓ^∞ is the space of all bounded sequences on $k \geq 0$, for any $b \in (\ell^\infty)^m$. See, for instance, [3].

In order to formulate our problem in regard to the system (12), we shall define now the space $\mathcal{H}$ of the theorem. This space will be a sequence space, the scalar product/norm being defined by means of a weight $g = (g_1, g_2, \ldots, g_m, \ldots) \in \ell^1$, with $g_k > 0$, $k \geq 1$. Namely, the space $\mathcal{H}$ will be the space $(\ell^2_g)^n$, where ℓ^2_g is the space of all sequences $x = (x_k;\ k \geq 1)$, such that

$$(15) \qquad \sum_{k=1}^{\infty} g_k x_k^2 < \infty,$$

with the inner product given by

(16) $$< x, y >_g = \sum_{=1}^{\infty} g_k x_k y_k .$$

It is obvious that

(17) $$\ell^2 \subset \ell^\infty \subset \ell^2_g ,$$

with $g \in \ell^1$ chosen as above.

The next step is to choose the space H of controllers. We take $H = (\ell^2)^m$.

Returning to the system (12), one sees that for any $u \in (\ell^2)^m$, the solution of (12) satisfying $x(0) = \theta \in R^n$ is in $\mathcal{H}$, bec se it is bounded and (17) holds true. Hence, the operator $T: H \rightarrow \mathcal{H}$ is well defined. Actually, one can get easily from (12), and $x(0) = \theta \in R^n$,

(18) $$x(k+1) = Bu(k) + ABu(k-1) + \ldots + A^k Bu(0),$$

which stands for $x = Tu$.

The cost functional will have the form

(19) $$C(x, u) = < Px, x >_g + < Qu, u > ,$$

where $P = P^* \geq 0$ is a linear continuous operator on $\mathcal{H} = (\ell^2_g)^n$ and $Q = Q^* > 0$ is linear and continuous on $H = (\ell^2)^m$.

The conclusion is the existence and uniqueness of the optimal controller $\bar{u} \in K$, where $K \subset H$ denotes any closed convex set, i.e.,

$$C(\bar{x},\bar{u}) = \inf C(x,u), \quad u \in K,$$

with $\bar{x} = T\bar{u}$.

As shown in [1], approximation schemes can be devised to find $\bar{u}$, in the class of Ritz-Galerkin procedures.

If instead of the init l condition $x(0) = \theta \in R^n$ one chooses the general case $x(0) = x^0 \in R^n$, the problem can be reduced to the case treated above by means of the substitution $x = y + \xi$, where ξ is the solution of $\xi(k+1) = A\xi(k)$, with $\xi(0) = x^0 \in R^n$. The set $K \subset H$ is then subject to a translation, but it remains convex and closed.

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