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Impulsive Integral Inequalities of Bihari Type

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1. The first impulsive integral inequalities of Gronwall type appeared in Perestyuk and Samoilenko [6], Borisenko [1] and Chernikova and Perestyuk [5]. Pirapikaran [7] begin the study on nonlinear integral inequalities of Bihari-Type. Of a great interest are the works [1-3] of Bainov and Simeonov who consider impulsive differential inequalities for the first time.

In this paper we study several integral inequalities as

$$u(t) \leq h(t) + \int_{t_0}^{t} \lambda(s)\omega_1(u(s))ds + \sum_{t_0<t_k<t} b_k\omega_2(u(t_k)), \quad t \geq t_0$$

where $\omega_i : [0,\infty) \to [0,\infty)$ $i = 1,2$ are continuous and nondecreasing functions and either ω_1/ω_2 or ω_2/ω_1 is nondecreasing. Our approach and proofs are different.

The results are faithful extension of the ordinary Bihari's inequality not only by the estimates but for the properties of the mayorant function. The mayorant functions can be computed and have the useful qualitative properties of the Bihari's mayorant. (See [10, 13])

2. Let $\mathbf{F} = \mathbf{F}(\mathbf{N}, \mathbf{C})$ the vectorial space of the complex sequence x : $\mathbf{N} \to \mathbf{C}$. Define on $\mathbf{F}$ the four operators Δ, σ, E_0 and π given by

$$\Delta x_{k+1} = x(k+1) - x(k), \quad \Delta x_0 = 0$$

$$(\sigma x)_k = \sum_{i=0}^{k} x_i$$

$$(E_0 x)_k = x(0)$$

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$$(\pi.x)_k = \pi_{i=0}^{k} x_i$$

The main properties of Δ, σ, E_0 and π are

$(\sigma,1)\sigma, E_0$ and π are monotone operators.

$(\sigma,2)\Delta\sigma = Id$ (the identity operator).

$(\sigma,3)\sigma(\Delta) = Id - E_0$

We must adopt the following conventions

$$\sum_{i=0}^{-1} x_i = 0 \qquad \Pi_{i=0}^{-1} x_i = 1.$$

Notations and Definitions.

1) The fixed impulse times

$$\nu = \{t_i\}_{i=1}^{\infty} \subset (t_0, \infty),\ t_i < t_{i+1},\ t_i \to \infty.$$

For $t \in [t_0, \infty)$, let $k = k(t)$ the integer such that $t_k \le t < t_{k+1}$. We define

$$\sum_{t_0<t_k<t} b_k = \sum_{(t_0,t)} b_k = \sum_{i=1}^{k(t)} b_i$$

$$\Pi_{(t_0,t)} b_k = \Pi_{t_0<t_k<t} b_k = \Pi_{i=1}^{k(t)} b_i$$

2) $C_\nu^+([t_0,\infty)) = \{u : [t_0,\infty) \to [0,\infty)/u$ is a continuous function except in $t_i (1 \le i)$ where u is left continuous and $u(t_i^+)$ exists $\}$.

3) $\Omega = \{\omega : [0,\infty) \to [0,\infty)/\omega$ is continuous and nondecreasing, $\omega(u) > 0$ for $u > 0\}$ and

$$\Omega_r = \{\omega \in \Omega/\omega(\alpha u) \le r(\alpha)\omega(u), \alpha > 0, u \ge 0\},$$

where $r : (0,\infty) \to (0,\infty)$ is a function.

4) $\omega_1 \propto \omega_2$ if ω_2/ω_1 is nondecreasing on $(0,\infty)$.

5) $W_i(u) = \int_{a_i}^{u} \frac{ds}{\omega_i(s)}$, $u > 0$, $a_i > 0$, $i = 1,2$ and W_i^{-1} its inverse function.

6) $\psi_i(u) = W_i^{-1}[W_i(u) + \alpha_i]$, $\alpha_i \ge 0$, $i = 1,2$ and

$$\varphi_1 = \psi_1 \circ \psi_2, \quad \varphi_2 = \psi_2 \circ \psi_1.$$

The functions ψ_i are defined for u so that

$$\alpha_i \leq \int_u^\infty \frac{ds}{\omega_i(s)}. \tag{1}$$

Thus if

$$\int_1^\infty \frac{ds}{\omega_i(s)} = \infty \quad i = 1,2 \tag{2}$$

then for every $\alpha_i \geq 0$ $(i = 1,2)$ the functions ψ_i are defined for all $u \geq 0$. If the dual condition

$$\int_{0+}^1 \frac{ds}{\omega_i(s)} = \infty \quad i = 1,2 \tag{3}$$

is fulfilled then for every $\alpha_i \geq 0$, $(i = 1,2)$ the functions ψ_i are defined for all $u \leq u_0$ small enough. Finally, if $1/\omega_i \in L_1$ $i = 1$ or $i = 2$ then ψ_j is not defined if

$$\alpha_j > \int^\infty \frac{ds}{\omega_i(s)}. \tag{4}$$

Other important Properties of the functions ψ_i are:

1) ψ_i are non-negative, non-decreasing and continuous functions on its domain and they are independent on α_i.

2) When ψ_i is considered as function on α_i the above is also true

3) Perhaps the most important property is the stability property:

$$\psi_i(0^+) = 0,$$

which is implied by (3).

The functions φ_i heritate these properties and they can be computed.

Example 1. Calculus of ψ_i and φ for $\omega_i(u) = u^{n_i}$.

We have that $\psi_i(u) = W_i^{-1}[W_i(u) + \alpha_i]$ is given by

$$\psi_i(u) = \begin{cases} [u^{1-n_i} + \alpha_i(1 - n_i)]^{\frac{-1}{n_i - 1}} & \text{if } n_i \neq 1 \\ u \cdot exp\alpha_i & \text{if } n_i = 1 \end{cases}$$

Thus for $n_i \neq 1$, $\varphi = \psi_2 \circ \psi_1$ takes the form

$$\varphi(u) = \{[u^{1-n_1} + \alpha_1(1-n_1)]^{\frac{n_2-1}{n_1-1}} + \alpha_2(1-n_2)\}^{\frac{-1}{n_2-1}}. \tag{5}$$

Moreover, since $\varphi_i^{-1}(u) = W_i^{-1}[W_i(u) - \alpha_i]$ then $\varphi^{-1}(u)$ is obtained from (5) replacing any α_k by $-\alpha_k$. Thus for $n_i > 1$, $\varphi(u)$ is defined for all $\varphi^{-1}(\infty)$, where

$$\varphi^{-1}(\infty) = \{[\alpha_1(n_1-1)]^{\frac{n_2-1}{n_1-1}} + \alpha_2(n_2-1)\}^{\frac{-1}{n_2-1}}.$$

For the case $n_1 = 1 < n_2$ we get

$$\begin{aligned}\varphi(u) &= [(ue^{\alpha_1})^{1-n_2} + \alpha_2(1-n_2)]^{\frac{-1}{n_2-1}}\\ &= ue^{\alpha_1}[1 + \alpha_2(1-n_2)(ue^{\alpha_1})^{n_2-1}]^{\frac{-1}{n_2-1}},\end{aligned}$$

defined for all $u < \varphi^{-1}(\infty) = [\alpha_2(n_2-1)]^{\frac{-1}{n_2-1}}$ and if $n_1 < 1 = n_2$ then

$$\varphi(u) = e^{\alpha_2}[u^{1-n_1} + \alpha_1(1-n_1)]^{\frac{-1}{n_1-1}}$$

defined for all $u < \varphi^{-1}(\infty) = \infty$.

Now, for $0 \le u$, $\lambda \in C_\nu^+([t_0,\infty))$ and $b : \mathbf{N} \to [0,\infty)$ we consider the inequality:

$$u(t) \le c + \int_{t_0}^{t} \lambda(s)\omega_1(u(s))ds + \sum_{t_0<t_k<t} b_k\omega_2(u(t_k)), c > 0 \text{ constant} \tag{6}$$

Denoting

$$\|u\|_{[a,d]} = sup_{a\le s\le d}|u(s)|,$$

we get

Lemma 1 *Assume that inequality (6) holds. For $\omega_i \in \Omega$ $(i = 1,2)$ and $t \in [t_k, t_{k+1}]$, we have*

$$v_{k+1} \le c + \sum_{i=0}^{k} \Lambda_i\omega_1(v_i) + \sum_{i=0}^{k} b_i\omega_2(v_i),$$

where $b_0 = 0$,

$$\Lambda_i = \int_{t_i}^{t_{i+1}} \lambda(s)ds(0 \le i \le k-1), \quad \Lambda_k = \int_{t_k}^{t} \lambda(s)ds$$

and

$$v_i = \|u\|_{[t_i,t_{i+1}]}(0 \le i \le k-1),$$

$$v_k = \|u\|_{[t_k,t]} \text{ and } v_{k+1} = \|u\|_{[t,t_{k+1}]}.$$

In fact,

$$\int_{t_0}^{t} \lambda(s)\omega_1(u(s))ds = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \lambda(s)\omega_1(u(s))ds + \int_{t_i}^{t} \lambda(s)\omega_1(u(s))ds \leq \sum_{i=0}^{k} \Lambda_i\omega_1(v_i).$$

Lemma 2 *If $\{z_k\}$ is a nondecreasing sequence and $w \in \Omega$, then*

$$\Delta W(z)(k+1) \leq \frac{\Delta z(k+1)}{\omega(z_k)}$$

Proof: Since ω is nondecreasing

$$\Delta W(z)(k+1) = \int_{z_k}^{z_{k+1}} \frac{ds}{\omega(s)} \leq \frac{\Delta z(k+1)}{\omega(z_k)}.$$

3. In this section, we obtain the first inequalities of Gronwall and Bihari type. In the following b is a nonnegative sequence, $0 \leq u$, $\lambda \in C_\nu^+([t_0, \infty))$ and $c \geq 0$ is a constant.

Theorem 1 *(Bihari's inequality) Suppose that $u \in C_\nu^+$ satisfies*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s)\omega(u(s))ds + \sum_{t_0<t_k<t} b_k\omega(u(t_k)) \tag{7}$$

then

$$u(t) \leq W^{-1}[W(c) + \int_{t_0}^{t} \lambda(s)ds + \sum_{t_0<t_k<t} b_k] \tag{8}$$

for any $t \geq t_0$ such that

$$\int_{t_0}^{t} \lambda(s)ds + \sum_{t_0<t_k<t} b_k < \int_{c}^{\infty} \frac{ds}{\omega(s)} \tag{9}$$

Proof: Let be $t \in [t_k, t_{k+1}]$. By Lemma 1, we have

$$v_{k+1} \leq c + \sum_{i=0}^{k}(\Lambda_i + b_i)\omega(v_i) := z_{k+1}$$

where $b_0 = 0$. Denoting z_{k+1} the right member of this last inequality since $v_k \leq z_k$ we get

$$\Delta z_{k+1} \leq \hat{\Lambda}_k\omega(z_k),\ \hat{\Lambda} = \Lambda + b$$

or

$$\frac{\Delta z_{k+1}}{\omega(z_k)} \leq \hat{\Lambda}_k$$

From Lemma 2, we obtain

$$\Delta W(z)(k+1) \leq \tilde{\Lambda}_k$$

and by $(\sigma.3)$ the summation gives

$$W(z_{k+1}) - W(z_0) \leq \sum_0^k \tilde{\Lambda}_i$$

or, since $z_0 = c$

$$z_{k+1} \leq W^{-1}[W(c) + \sum_{i=0}^{k}(\Lambda_i + b_i)]$$

and the result follows since $v_{k+1} \leq z_{k+1}$.

Remark: 1) If (2) holds then (8) is valid for every $t \geq t_0$. 2) If (3) holds then (8) is valid for every $t \geq t_0$ if c is small enough. In general (8) is valid for every $t \geq t_0$ if

$$\int_{t_0}^{\infty} \lambda(s)ds + \sum_{t_0<t_k} b_k < \int_c^{\infty} \frac{ds}{\omega(s)} \tag{10}$$

for $\lambda \in L_1$ and $b \in \ell_1$. 3) In general, if (2) and (10) do not hold then under condition (3) the result (8) is true only for c small enough.

Theorem 2 *Let be* $0 \leq u$, $\lambda \in C_v^+$, $b_k : \mathbf{N} \to [0,\infty)$ *a sequence and* $c \geq 0$. *The inequality*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s)u(s)ds + \sum_{t<t_k<t} b_k u(t_k)$$

implies that for every $t \in [t_k, t_{k+1})$

$$u(t) \leq c\Pi_{i=0}^{k-1}(1 + b_i + \int_{t_i}^{t_{i+1}} \lambda(s)ds) \cdot (1 + b_k + \int_{t_k}^{t} \lambda(s)ds),$$

where $b_0 = 0$. *In particular,*

$$u(t) \leq c\Pi_{(t_0,t)}(1 + b_k) \cdot exp \int_{t_0}^{t} \lambda(s)ds.$$

Proof: Fix $t \in [t_k, t_{k+1}]$. From lemma 1 we get

$$v_{k+1} \leq c + \sum_{i=0}^{k}(\Lambda_i + b_i)v_i := z_{k+1}$$

where $b_0 = 0$. Thus if z denotes the right member of the last inequality, we obtain

$$\Delta z_{k+1} \leq (\Lambda_k + b_k)z_k$$

and then

$$z_{k+1} \leq \Pi_{i=0}^{k}(1 + \Lambda_i + b_i) \cdot c.$$

Since $v \leq z$ the result follows.

As usual, we can deduce:

Theorem 3 *Suppose*

$$u(t) \leq h(t) + \int_{t_0}^{t} \lambda(s)u(s)ds + \sum_{(t_0,t)} b_k u(t_k), \quad t \geq t_0$$

Then for $t \geq t_0$

$$u(t) \leq h(t) \quad + \quad \sum_{t_k \in (t_0,t)} [\Pi_{t_j \in (t_k,t)} (1 + b_j) e^{\int_{t_k}^{t} \lambda(s)ds}] b_k h(t_k)$$

$$+ \quad \int_{t_0}^{t} [\Pi_{t_k \in (s,t)} (1 + b_k) e^{\int_s^t \lambda(\tau)d\tau} \cdot \lambda(s) \cdot h(s)ds]$$

Moreover when h *is a nondecreasing function we have*

$$u(t) \leq h(t)\Pi_{(t_0,t)}(1 + b_k) \cdot e^{\int_{t_0}^{t} \lambda(s)ds}$$

To establish the main result, denote:

$$\alpha_1(t) = \int_{t_0}^{t} \lambda(s)ds, \quad \alpha_2(t) = \sum_{(t_0,t)} b_k$$

and

$$\psi_i(c) = W_i^{-1}[W_i(c) + \alpha_i(T)], \quad i = 1, 2.$$

Theorem 4 *Assume*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s)\omega_1(u(s))ds + \sum_{(t_0,t)} b_k \omega_2(u(t_k)), \quad t \geq t_0 \tag{11}$$

1) If $\omega_1 \propto \omega_2$, then on $[t_0, T)$

$$u(t) \leq W_2^{-1}[W_2(\psi_1(c)) + \sum_{(t_0,t)} b_k], \tag{12}$$

where

$$\text{i) } \alpha_1(T) \leq \int_c^{\infty} \frac{ds}{\omega_1(s)}, \quad \text{ii)} \alpha_2(T) \leq \int_{\psi_1(c)}^{\infty} \frac{ds}{\omega_2(s)}. \tag{13}$$

2) If $\omega_2 \propto \omega_1$, then on $[t_0, T)$

$$u(t) \leq W_1^{-1}[W_1(\psi_2(c)) + \int_{t_0}^{t} \lambda(s)ds], \tag{14}$$

where

$$\text{i) } \alpha_1(T) \leq \int_{\psi_2(c)}^{\infty} \frac{ds}{\omega_1(s)}, \quad \text{ii) } \quad \alpha_2(T) \leq \int_c^{\infty} \frac{ds}{\omega_2(s)}. \tag{15}$$

Proof: Let be $t \in [t_k, t_{k+1}]$. From Lemma 1 we get

$$v_{k+1} \leq c + \sum_{i=0}^{k} \Lambda_i \omega_1(v_i) + \sum_{i=0}^{k} b_i \omega_2(v_i),$$

where $b_0 = 0$. Define

$$z_{k+1} = c + \sum_{i=0}^{k} A_i\omega_1(v_i) + \sum_{i=0}^{k} b_i\omega_2(v_i).$$

Thus $v \leq z$ and

$$\Delta z(k+1) \leq A_k\omega_1(z_k) + b_k\omega_2(z_k)$$

and hence

$$\frac{\Delta z(k+1)}{\omega_1(z_k)} \leq A_k + b_k\hat{\omega}(z_k), \quad \hat{\omega} = \omega_2/\omega_1$$

where $\hat{\omega} = \omega_2/\omega_1$. Since $\omega_1 \propto \omega_2$ means $\hat{\omega} = \omega_2/\omega_1$ is a nondecreasing function on $(0,\infty)$, by Lemma 2, we have

$$\Delta W_1(z)(k+1) \leq A_k + b_k\hat{\omega}(z_k)$$

and by $(\sigma.3)$ the summation yields

$$\begin{aligned} W_1(z_{k+1}) - W_1(z_0) &\leq \textstyle\sum_{i=0}^{k} A_i + \sum_{i=0}^{k} b_i\hat{\omega}(z_i) \\ &\leq \alpha_1(T) + \textstyle\sum_{i=1}^{k} b_i\hat{\omega}(z_i), \end{aligned}$$

where T satisfies (15). Then

$$W_1(z_{k+1}) \leq W_1(c) + \alpha_1(T) + \sum_{i=0}^{k} b_i\hat{\omega}(z_i).$$

Now, we denote $y = W_1(z)$. Thus y verifies

$$y_{k+1} \leq \check{c} + \sum_{i=0}^{k} b_i(\hat{\omega} \circ W_1^{-1})(y_i), \check{c} = W_1(c) + \alpha_1(T). \tag{16}$$

The function $\check{\omega} = \hat{\omega} \circ W_1^{-1}$ is again nondecreasing. By proceeding as in the proof of Theorem 1 we are able of solving inequality (16). In fact, defining

$$\check{W}(u) = \int_{\check{a}}^{u} \frac{ds}{\check{\omega}(s)}, \quad \check{a} = W_1(a_2)$$

we get

$$y_{k+1} \leq \check{W}^{-1}[(\check{W}(\check{c}) + \sum_{i=0}^{k} b_i]$$

for $k \leq \check{m}$ such that

$$\sum_{i=0}^{\check{m}} b_i \leq \int_{\check{c}}^{\infty} \frac{ds}{\check{\omega}(s)}. \tag{17}$$

But, it is easily seen that

$$\check{W} = W_2 \circ W_1^{-1}.$$

Then (17) is reduced to (15) ii) and

$$y_{k+1} \leq W_1 \circ W_2^{-1}[W_2(W_1^{-1}(\tilde{c})) + \sum_{i=0}^{k} b_i]$$

or, since $z = W_1^{-1}(y)$

$$z_{k+1} \leq W_2^{-1}[W_2(W_1^{-1}(\tilde{c})) + \sum_{i=0}^{k} b_i]$$

and hence

$$z_{k+1} \leq W_2^{-1}[W_2(\psi_1(c)) + \sum_{i=0}^{k} b_i],$$

from where (12) follows. The second part follows similarly to (12).

As an immediate consequence, we get

Theorem 5 *If $\omega(u)/u$ is nondecreasing and*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s)\omega(u(s))ds + \sum_{(t_0,t)} b_k u(t_k),$$

then for $t \in [t_0, T)$

$$u(t) \leq W^{-1}[W(c \sum_{(t_0,T)} b_k) + \int_{t_0}^{t} \lambda(s)ds],$$

where

$$\int_{t_0}^{T} \lambda(s)ds \leq \int_{c\sum_{(t_0,T)} b_k}^{\infty} \frac{ds}{\omega(s)}.$$

and

Theorem 6 *If $u/\omega(u)$ is nondecreasing and*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s)u(s) + \sum_{(t_0,t)} b_k\omega(u(t_k))$$

then

$$u(t) \leq \psi(c) \cdot exp(\int_{t_0}^{t} \lambda(s)ds),$$

where

$$\psi(c) = W^{-1}[W(c) + \Pi_{(t_0,t)}(1 + b_k)]$$

4. In this section, we study some semilinear inequalities. For that, denote

$$\tilde{b}_k = b_k \cdot \Pi_0^k(1 + \Lambda_i^0)^{-1} \cdot r_2(\Pi_{i=0}^{k-1}(1 + \Lambda_i^0)), \quad \tilde{\alpha}_2(t) = \textstyle\sum_{(t_0,t)} \tilde{b}_k$$

$$\tilde{\Lambda}_k = \Lambda_k \cdot \Pi_0^k(1 + \Lambda_i^0)^{-1} \cdot r_1(\Pi_{i=0}^{k-1}(1 + \Lambda_i^0)), \quad \tilde{\alpha}_1(t) = \textstyle\sum_{i=0}^{k} \tilde{\Lambda}_i$$

and let be $\tilde{T}$ defined by

$$\tilde{\alpha}_1(\tilde{T}) \leq \int_c^{\infty} \frac{ds}{\omega_1(s)}; \quad \tilde{\alpha}_2(\tilde{T}) \leq \int_{\tilde{\psi}_1(c)}^{\infty} \frac{ds}{\omega_2(s)},$$

where

$$\tilde{\psi}_1(c) = W_1^{-1}[W_1(c) + \tilde{\alpha}_1(T)].$$

So, we can state

Theorem 7 *Assume that $\omega_i \in \Omega_{r_i}$ $(i = 1, 2)$, $\omega_1 \propto \omega_2$ and that*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_0(s)u(s)ds + \int_{t_0}^{t} \lambda(s)\omega_1(u(s))ds + \sum_{(t_0,t)} b_k\omega_2(u(t_k)).$$

Then for $t \in [t_0, \tilde{T})$

$$u(t) \leq exp(\int_{t_0}^{t} \lambda_0(s)ds) \cdot W_2^{-1}[W_2(\tilde{\psi}_1(c)) + \sum_{(t_0,t)} \tilde{b}_k]$$

Proof: First, we have $k(t) \leq k(T)$. If $t \in [t_k, t_{k+1}[$, then from Lemma 1, we get

$$v_{k+1} \leq c + \sum_{i=0}^{k} \Lambda_i^0 v_i + \sum_{i=0}^{k} (\Lambda_i\omega_1(v_i) + b_i\omega_2(v_i)) := z_{k+1},$$

where Λ_i^0 are the integral defined in Lemma 1 corresponding to λ_0.

Denoting z_{k+1} the right member in the last inequality, we get

$$\Delta z_{k+1} \leq \Lambda_k^0 z_k + \Lambda_k\omega_1(z_k) + b_k\omega_2(z_k).$$

Since $\omega_i \in \Omega_{r_i}$ $(i = 1, 2)$

$$\frac{z_{k+1}}{\Pi_0^k(1 + \Lambda_i^0)} \leq \frac{z_k}{\Pi_0^{k-1}(1 + \Lambda_i^0)} + \tilde{\Lambda}_k\omega_1(\frac{z_k}{\Pi_0^{k-1}(1 + \Lambda_i^0)}) + \tilde{b}_k\omega_2(\frac{z_k}{\Pi_{i=0}^{k-1}(1 + \Lambda_i^0)}).$$

So

$$y_{k+1} = z_{k+1}/\Pi_0^k(1 + \Lambda_i^0)$$

satisfies after summation

$$y_{k+1} \leq c + \sum_{n=1}^{k} (\tilde{\Lambda}_n\omega_1(y_n) + \tilde{b}_n \cdot \omega_2(y_n)).$$

Thus, the application of Theorem 4 yields

$$y_{k+1} \leq W_2^{-1}[W_2(\tilde{\psi}_1(c)) + \sum_{n=1}^{k} \tilde{b}_n],$$

from where the result follows.

Next, denote

$$\tilde{\Lambda}_k = \Lambda_k \cdot \Pi_0^k(1+b_i^0)^{-1} \cdot r_1(\Pi_0^{k-1}(1+b_i^0)),\ \tilde{\alpha}_1(t) = \textstyle\sum_{i=0}^{k(t)} \tilde{\Lambda}_i$$

$$\tilde{b}_k = b_k \cdot \Pi_0^k(1+b_i^0)^{-1} \cdot r_2(\Pi_0^{k-1}(1+b_i^0)),\ \tilde{\alpha}_2(t) = \textstyle\sum_{(t_0,t)} \tilde{b}_k$$

and let be T defined by

$$\tilde{\alpha}_1(T) \le \int_c^\infty \frac{ds}{\omega_1(s)};\ \tilde{\alpha}_2(T) \le \int_{\tilde{\psi}_1(c)}^\infty \frac{ds}{\omega_2(s)},$$

where

$$\tilde{\psi}_1(c) = W_1^{-1}[W_1(c) + \tilde{\alpha}_1(T)].$$

So, we can state

Theorem 8 *Assume that* $\omega_1 \propto \omega_2$ *and that*

$$u(t) \le c + \sum_{(t_0,t)} b_k^0 u(t_k) + \int_{t_0}^t \lambda(s)\omega_1(u(s))ds + \sum_{(t_0,t)} b_k\omega_2(u(t_k)).$$

Then for $t \in [t_0, \tilde{T}]$ *we have*

$$u(t) \le \Pi_{i=1}^{k(t)}(b_i^0+1)W_2^{-1}[W_2(\tilde{\psi}_1(c)) + \sum_{n=1}^{k(t)} \tilde{b}_n]$$

Proof: If $t \in [t_k, t_{k+1}]$, then from Lemma 1, we obtain

$$v_{k+1} \le c + \sum_{i=0}^k b_i^0 v_i + \sum_{i=0}^k (\Lambda_i^1\omega_1(v_i) + b_i\omega_2(v_i)) := z_{k+1}.$$

Denoting z_{k+1} the right member in the last inequality we get

$$\Delta z_{k+1} \le b_k^0 z_k + \Lambda_k^1\omega_1(z_k) + b_k\omega_2(z_k).$$

So, proceeding as in the above proof, we have that $y_{k+1} = z_{k+1}/\Pi^k(1+b_i^0)$ verifies

$$y_{k+1} \le W_2^{-1}[W_2(\tilde{\psi}_1(c)) + \sum_{n=0}^k \tilde{b}_n]$$

and hence the results follows at once.

The following results are variants to Theorem 5 and 6. The main difference is based in that the condition $\omega \in \Omega_r$ replace $u \propto \omega$ or $\omega \propto u$.

Theorem 9 *If* $u \in C_v^+$, $\omega \in \Omega_r$ *and*

$$u(t) \le c + \int_{t_0}^t \lambda(s)u(s)ds + \sum_{(t_0,t)} b_k\omega(u(t_k))$$

then

$$u(t) \le exp(\int_{t_0}^t \lambda(s)ds)W^{-1}[W(c) + \sum_{i=0}^k \frac{b_i r(\Pi_{j=0}^{i-1}(1+\Lambda_j))}{\Pi_{j=0}^i(1+\Lambda_j)}]$$

Proof: Putting $b_0 = 0$, from Lemma 1,

$$v_{k+1} \leq c + \sum_{i=0}^{k} \Lambda_i v_i + \sum_{i=0}^{k} b_i \omega(v_i) = z_{k+1}.$$

Thus z_k satisfies

$$\Delta z_{k+1} \leq \Lambda_k z_k + b_k \omega(z_k).$$

So

$$\begin{aligned} z_{k+1} &\leq (1+\Lambda_k) z_k + b_k \omega(z_k) \\ \frac{z_{k+1}}{\Pi_{i=0}^{k}(1+\Lambda_i)} &\leq \frac{z_k}{\Pi_{i=0}^{k-1}(1+\Lambda_i)} + \frac{b_k \cdot r(\Pi_{i=0}^{k-1}(1+\Lambda_i))}{\Pi_{i=0}^{k}(1+\Lambda_i)} \omega(z_k/\Pi_{i=0}^{k-1}(1+\Lambda_i)). \end{aligned}$$

By Theorem

$$z_{k+1} \leq \Pi_{i=0}^{k}(1+\Lambda_i) W^{-1}[W(z_0) + \sum_{j=0}^{k} \frac{b_j r(\Pi^{j-1}(1+\Lambda_i))}{\Pi_{i=0}^{j}(1+\Lambda_i)}].$$

In a similar way if

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) \omega(u(s)) ds + \sum_{(t_0,t)} b_k u(t_k)$$

then

$$u(t) \leq \Pi_{(t_0,t)}(1+b_i) W^{-1}[W(c) + \int_{t_0}^{t} \frac{\lambda(s) r(\Pi_{i=0}^{k-1}(1+b_i)) ds}{\Pi_{i=0}^{k}(1+b_i)}]. \qquad (18)$$

5. Next we show an example comparing the use of our estimates with two nonlinearities ω_1, ω_2 with respect to an estimate obtained with the non linearity $\omega = \omega_1 + \omega_2$

Let $\omega_1(u) = u^2, \omega_2(u) = 2u$ and $\omega = \omega_1 + \omega_2, \lambda = \lambda_1 + \lambda_2$. It is not difficult to verify that

$$\psi(u) = W^{-1}[W(u) + \alpha] = \frac{ue^{2\alpha}}{1 - u(e^{2\alpha} - 1)/2}, \quad \alpha = \alpha_1 + \alpha_2,$$

where $W' = 1/\omega$ and

$$\begin{aligned} \psi_1(u) &= W_1^{-1}[W_1(u) + \alpha_1] = u(1 - u\alpha_1)^{-1}, \\ \psi_2(u) &= W_2^{-1}[W_2(u) + \alpha_2] = ue^{2\alpha_2}, \\ \varphi(u) &= \psi_1 \circ \psi_2(u) = ue^{2\alpha_2}(1 - ue^{2\alpha_2}\alpha_1)^{-1}. \end{aligned}$$

Thus if for instance we estimate the inequality

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) \omega_1(u(s)) ds + \sum_{t_k \in (t_0,t)} \lambda_2(t_k) \omega_2(u(t_k)), \qquad (19)$$

by the simpler and more restrictive Bihari's inequality

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)\omega(u(s))ds + \sum \lambda_2(t_k)\omega(u(t_k)),$$

we obtain the estimate

$$u(t) \leq \psi(c) = \frac{c\,exp2(\int_{t_0}^{t} \lambda_1(s)ds + \sum_{(t_0,t)} \lambda_2(s))}{1 + \frac{c}{2}(1 - exp[2(\int_{t_0}^{t} \lambda_1(s)ds + \sum_{(t_0,t)} \lambda_2(s))])}. \tag{20}$$

On the other hand, our estimates (14) gives for the original inequality (19)

$$u(t) \leq \varphi(c) = \frac{c\,exp2\sum_{(t_0,t)} \lambda_2(s)}{1 - c\alpha_1 exp2\sum_{(t_0,t)} \lambda_2(s)}, \quad \alpha_1 = \int_0^T \lambda_1(s)ds. \tag{21}$$

The differences are big:

1) For $\alpha_1 = 0\,(\lambda_1 \equiv 0)$, (21) gives the corresponding linear estimate while (20) does not.

2) For $\alpha_2 = 0(\lambda_2 \equiv 0)$ the global estimate obtained from (20) is worse than (21) which gives again the estimate corresponding to the non linear Bihari's estimate.

3) As it must be the case, the estimates are valid only for c small enough. However in (20) c must be smaller than in (21). In (21), for

$$c < 1/\alpha_1 e^{2\alpha_2}, \quad \alpha_2 = \sum_{t_k > t_0} \lambda_2(t_k)$$

the estimate (21) is valid for all $t \in [t_0, \infty)$, while in (20) it is necessary

$$c < \frac{2}{e^{2(\alpha_1+\alpha_2)} - 1}$$

which is very restrictive.

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