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OSCILLATION AND NONOSCILLATION IN INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. Two classes of integrodifferential equations are considered. Necessary conditions, sufficient conditions, and necessary and sufficient conditions for the existence of positive solutions of these equations are established.

0. Introduction.

The purpose of this paper is to investigate the problem of the oscillation and the nonoscillation of the solutions of certain integrodifferential equations. Such equations can be regarded as differential equations with unbounded delays; for a survey on equations with unbounded delays we refer to the article by Corduneanu and Lakshmikantham [3]. See Burton [1] for the general theory of integrodifferential equations. In the literature, there are few results about the oscillation and nonoscillation of integrodifferential equations. We mention the recent papers by Gopalsamy [5],[6],[7],[8], [9],[10], Ladas, the present author and Sficas [13], and the present author [16].

Consider the delay differential equation

$$x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) = 0, \tag{E_0}$$

where p_n <u>and</u> τ_n $(n=0,1,\ldots)$ <u>are nonnegative constants, and</u>

$$\tau \equiv \sup_{n=0,1,\ldots} \tau_n < \infty.$$

Consider also the integrodifferential inequalities

$$y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_0^t K(t-s)y(s)ds \leqq 0, \tag{I_1}$$

$$y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_{-\infty}^t K(t-s)y(s)ds \leqq 0 \tag{I_2}$$

and the corresponding integrodifferential equations

$$x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_0^t K(t-s)x(s)ds = 0, \tag{E_1}$$

$$x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_{-\infty}^t K(t-s)x(s)ds = 0, \tag{E_2}$$

where K *is a nonnegative continuous function on the interval* $[0,\infty)$.

If $T \in \mathbb{R}$, by a *solution on* $[T,\infty)$ *of* (E_0) we mean a continuous real-valued function x on $[T-\tau,\infty)$, which is differentiable on $[T,\infty)$ and satisfies (E_0) for every $t \geqq T$. A solution on an interval $[T,\infty)$, $T \in \mathbb{R}$, of (E_0) is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

Let T be a nonnegative number. A *solution on* $[T,\infty)$ *of* (I_1) [resp. *of* (E_1)] is a continuous real-valued function y [resp. x] on the interval $[\min\{0,T-\tau\},\infty)$, which is continuously differentiable on $[T,\infty)$ and satisfies (I_1) [resp. (E_1)] for every $t \geqq T$.

Let T be a real number. A continuous real-valued function y [resp. x] on the real line $\mathbb{R}$, which is continuously differentiable on $[T,\infty)$ and satisfies (I_2) [resp. (E_2)] for all $t \geqq T$, is called a *solution on* $[T,\infty)$ *of* (I_2) [resp. *of* (E_2)]. Also, by a *solution on* $\mathbb{R}$ *of* (I_2) we mean a continuously differentiable function y on $\mathbb{R}$, which satisfies (I_2) for every $t \in \mathbb{R}$.

Equations of the form (E_1) have been considered by Corduneanu [2], Corduneanu and Luca [4], and Luca [15]. In [15], Luca deals also with equations of the form (E_2). We refer to the recent papers by Gopalsamy [11],[12] for equations related to the equations considered here. The case of inequalities with unbounded delay has been investigated by many authors. For a detailed list of references to such inequalities see the survey article by Corduneanu and Lakshmikantham [3] and the references therein.

In this paper, we give a necessary and sufficient condition for all solutions of (E_0) to be oscillatory (Section 1). We also obtain necessary conditions for inequality (I_1) [Section 2] or inequality (I_2) [Section 3] to have a positive solution. Moreover, we establish sufficient conditions, and necessary and sufficient conditions for the existence of a positive solution of equation (E_1) [Section 2] or of equation (E_2) [Section 3].

If $p_i > 0$ $(i = 0,1,\ldots,m)$ and $p_i = \tau_i = 0$ $(i = m+1,m+2,\ldots)$, then the criterion given in Section 1 (Theorem 1.1) leads to a result due to Tramov [17] (see also Ladas, Sficas and Stavroulakis [14]). The method used in proving Theorem 1.1 paterns after that in [14]. The results of Section 2 extend some recent ones obtained by Ladas, the present author and Sficas [13] for the special case where $p_n = \tau_n = 0$ $(n = 0,1,\ldots)$. Our results in Section 3 are new even in the special case where $p_n = \tau_n = 0$ $(n = 0,1,\ldots)$. The techniques applied in Sections 2 and 3 are motivated by those of [13].

1. Equation (E_0).

Our aim in this section is to obtain a necessary and sufficient condition for all solutions of (E_0) to be oscillatory.

THEOREM 1.1. <u>Every solution of</u> (E_0) <u>is oscillatory if and only if</u>

$$(C_0) \qquad +\infty \geqq -\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda\tau_n} > 0 \ \underline{\text{for all}}\ \lambda > 0.$$

PROOF. If $p_n = 0$ for all $n \in \{0,1,\ldots\}$, then (E_0) has the positive solution $x(t) = 1$. So, assume that $p_m > 0$ for some $m \in \{0,1,\ldots\}$. Moreover, suppose that (C_0) is not satisfied, which means that there exists a $\lambda_0 > 0$ so that

$$-\lambda_0 + \sum_{n=0}^{\infty} p_n e^{\lambda_0\tau_n} \leqq 0.$$

Set

$$F(\lambda) = -\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda\tau_n} \text{ for } \lambda \in (0,\lambda_0].$$

Clearly, F is a continuous real-valued function on the interval $(0,\lambda_0]$. Since $F(\lambda_0) \leqq 0$ and $F(0+0) \geqq p_m > 0$, we always have $F(\lambda_0^*) = 0$ for some $\lambda_0^* \in (0,\lambda_0]$. Hence

$$-\lambda_0^* + \sum_{n=0}^{\infty} p_n e^{\lambda_0^*\tau_n} = 0$$

and consequently (E_0) has the positive solution $x(t) = e^{\lambda_0^* t}$.

Conversely, let (C_0) be satisfied and assume, for the sake of contradiction, that (E_0) has a nonoscillatory solution x on an interval $[T,\infty)$, $T \in \mathbb{R}$. Without loss of generality, we suppose that $x(t) \neq 0$ for all $t \geqq T-\tau$. Furthermore, since $-x$ is also a solution on $[T,\infty)$ of (E_0), we restrict our discussion to the case where x is positive on $[T-\tau,\infty)$. From (E_0) it follows that

$$x'(t) = -\sum_{n=0}^{\infty} p_n x(t-\tau_n) \leqq 0 \text{ for every } t \geqq T$$

and consequently x is decreasing on $[T,\infty)$.

Consider the set Λ of all numbers $\lambda > 0$ for which there exists a $T_\lambda \geqq T$ such that

$$x'(t)+\lambda x(t) \leqq 0 \text{ for all } t \geqq T_\lambda.$$

From condition (C_0) it follows that there exists a $m \in \{0,1,\dots\}$ such that $p_m > 0$. But, by using the fact that x is decreasing on $[T,\infty)$, form (E_0) we obtain for $t \geqq T+\tau_m$

$$0 = x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) \geqq x'(t)+p_m x(t-\tau_m) \geqq x'(t)+p_m x(t).$$

Hence, $p_m \in \Lambda$ and consequently Λ is nonempty. The set Λ is a subinterval of $(0,\infty)$ with $0 = \inf \Lambda$.

Next, we will prove that Λ is bounded from above. To this end, we first observe that condition (C_0) ensures that $M \equiv \left\{n \in \{0,1,\dots\} : p_n > 0\right\} \neq \emptyset$. If

$\tau_n = 0$ for all $n \in M$, then from (E_0) it follows that $\sum_{n\in M} p_n < \infty$ and hence condition (C_0) leads to the contradiction

$$-\lambda + \sum_{n\in M} p_n > 0 \text{ for all } \lambda > 0.$$

Thus, there exists a $m \in M$ with $\tau_m > 0$. By the definition of M, we always have $p_m > 0$. Now, (E_0) gives

$$x'(t) + p_m x(t-\tau_m) \leqq 0 \text{ for } t \geqq T$$

and consequently from [14] it follows that

$$(*) \qquad x(t) > Lx(t-\tau_m) \text{ for all large } t,$$

where $L = (\frac{1}{2} p_m \tau_m)^2 > 0$. Since x is decreasing on $[T,\infty)$, we must have $L < 1$. We have

$$\sup \Lambda \leqq \vartheta \equiv -\frac{1}{\tau_m} \ln L.$$

Indeed, in the opposite case $\vartheta \in \Lambda$ and consequently for some $T_\vartheta \geqq T$

$$x'(t) + \vartheta x(t) \leqq 0 \text{ for every } t \geqq T_\vartheta.$$

This means that the function

$$\varphi_\vartheta(t) = e^{\vartheta t} x(t), \quad t \geqq T_\vartheta$$

is decreasing. Hence, for $t \geqq T_\vartheta + \tau_m$

$$e^{\vartheta t} x(t) = \varphi_\vartheta(t) \leqq \varphi_\vartheta(t-\tau_m) = e^{\vartheta(t-\tau_m)} x(t-\tau_m).$$

Therefore,

$$x(t) \leqq e^{-\vartheta\tau_m} x(t-\tau_m) = Lx(t-\tau_m) \text{ for all } t \geqq T_\vartheta + \tau_m,$$

which contradicts (*).

Now, set $\lambda^* = \sup \Lambda$, $0 < \lambda^* < \infty$. Also, consider an arbitrary number μ with $0 < \mu < \lambda^*$. Then $\lambda^* - \mu \equiv r \in \Lambda$ and hence there exists a $T_r \geqq T$ such that

$$x'(t) + rx(t) \leqq 0 \text{ for all } t \geqq T_r.$$

For any $n \in \{0,1,\dots\}$ and every $t \geqq T_r+\tau$, we get

$$\frac{x(t-\tau_n)}{x(t)} = e^{-\ln \frac{x(t)}{x(t-\tau_n)}} = e^{-\int_{t-\tau_n}^{t} \frac{x'(s)}{x(s)}\,ds} \geqq e^{\int_{t-\tau_n}^{t} r\,ds} = e^{r\tau_n}.$$

Hence, from (E_0) it follows that for all $t \geqq T_r+\tau$

$$0 = x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) \geqq x'(t) + \left(\sum_{n=0}^{\infty} p_n e^{r\tau_n}\right) x(t)$$

and consequently

$$\sum_{n=0}^{\infty} p_n e^{r\tau_n} \equiv \hat{\lambda} \in \Lambda$$

(note that $\hat{\lambda} > 0$, since $p_m > 0$ for some $m \in \{0,1,\dots\}$). Thus, we have

$$\lambda^* \geqq \sum_{n=0}^{\infty} p_n e^{r\tau_n} = \sum_{n=0}^{\infty} p_n e^{(\lambda^*-\mu)\tau_n} \geqq e^{-\mu\tau} \sum_{n=0}^{\infty} p_n e^{\lambda^*\tau_n}.$$

As $\mu \in (0,\lambda^*)$ is arbitrary, we can obtain

$$\lambda^* \geqq \sum_{n=0}^{\infty} p_n e^{\lambda^*\tau_n},$$

which contradicts condition (C_0). The proof is complete.

2. Inequality (I_1) - Equation (E_1). The following theorem provides necessary conditions for inequality (I_1) to have a positive solution.

THEOREM 2.1. *Let* K *be not identically zero on* $[0,\infty)$ *and suppose that*

$$\text{(C)} \qquad +\infty \geqq -\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda\tau_n} + \int_0^{\infty} e^{\lambda s} K(s)\,ds > 0 \ \text{for all} \ \lambda > 0.$$

Moreover, let $T \geqq 0$. *Then there is no solution on* $[T,\infty)$ *of* (I_1) *which is positive on* $[\min\{0,T-\tau\},\infty)$.

PROOF. Assume, for the sake of contradiction, that there exists a solution y on $[T,\infty)$ of (I_1) which is positive on $[\min\{0,T-\tau\},\infty)$. From (I_1) we obtain for every $t \geqq T$

$$y'(t) \leqq -\sum_{n=0}^{\infty} p_n y(t-\tau_n) - \int_0^t K(t-s)y(s)ds \leqq 0,$$

which means that y is decreasing on $[T,\infty)$.

Let Λ be the set of all numbers $\lambda > 0$ for which there exists a $T_\lambda \geqq T$ such that

$$y'(t)+\lambda y(t) \leqq 0 \text{ for every } t \geqq T_\lambda .$$

The set Λ is nonempty. Indeed, since K is not identically zero on $[0,\infty)$, we can consider a $\tau^* > 0$ so that

$$\lambda_0 \equiv \int_0^{\tau^*} K(s)ds > 0.$$

Then, by the fact that y is decreasing on $[T,\infty)$, from (I_1) we obtain for $t \geqq T+\tau^*$

$$\begin{aligned} 0 &\geqq y'(t) + \int_0^t K(t-s)y(s)ds = y'(t) + \int_0^t K(s)y(t-s)ds \\ &\geqq y'(t) + \int_0^{t-T} K(s)y(t-s)ds \geqq y'(t)+\left[\int_0^{t-T} K(s)ds\right]y(t) \\ &\geqq y'(t)+\lambda_0 y(t) \end{aligned}$$

and consequently $\lambda_0 \in \Lambda$. Thus $\Lambda \neq \emptyset$. Clearly, Λ is a subinterval of $(0,\infty)$ with $0 = \inf \Lambda$.

Furthermore, we have $\sup \Lambda < \infty$. To prove this assertion, we observe that, since K is not identically zero on $[0,\infty)$, there exist τ_* and ε with $\tau_* > \varepsilon > 0$ so that

$$A \equiv \int_\varepsilon^{\tau_*} K(s)ds > 0.$$

Using (I_1) and the fact that y decreases on $[T,\infty)$, we obtain for $t \geqq T+\tau_*$

$$\begin{aligned} 0 &\geqq y'(t) + \int_0^t K(s)y(t-s)ds \geqq y'(t) + \int_\varepsilon^{t-T} K(s)y(t-s)ds \\ &\geqq y'(t) + \left[\int_\varepsilon^{t-T} K(s)ds\right]y(t-\varepsilon) \geqq y'(t)+\left[\int_\varepsilon^{\tau_*} K(s)ds\right]y(t-\varepsilon). \end{aligned}$$

Therefore,

$$y'(t)+Ay(t-\varepsilon) \leqq 0 \text{ for every } t \geqq T+\tau_*$$

and hence it follows from [14] that

$$(*) \qquad y(t) > Ly(t-\varepsilon) \text{ for all large } t,$$

where $L = (\varepsilon A)^2/4$. Clearly, we always have $0 < L < 1$. Then $\vartheta \equiv -\frac{1}{\varepsilon} \ln L$ is an upper bound of Λ. Indeed, in the opposite case we have $\vartheta \in \Lambda$ and hence $y'+\vartheta y \leqq 0$ on $[T_\vartheta, \infty)$ for some $T_\vartheta \geqq T$. Thus, the function $\varphi_\vartheta(t) = e^{\vartheta t} y(t)$, $t \geqq T_\vartheta$ is decreasing and consequently $\varphi_\vartheta(t) \leqq \varphi_\vartheta(t-\varepsilon)$ for $t \geqq T_\vartheta+\varepsilon$. So, we get

$$y(t) \leqq e^{-\vartheta\varepsilon} y(t-\varepsilon) = Ly(t-\varepsilon) \text{ for all } t \geqq T_\vartheta+\varepsilon,$$

which contradicts $(*)$.

Set now $\lambda^* = \sup \Lambda$, $0 < \lambda^* < \infty$, and consider an arbitrary number $\mu \in (0,\lambda^*)$. Then $\lambda^*-\mu \equiv r \in \Lambda$, which means that there exists a $T_r \geqq T$ so that

$$y'(t)+ry(t) \leqq 0 \text{ for every } t \geqq T_r.$$

For every $\xi \geqq 0$ and $t \geqq T_r+\xi$, we have

$$\frac{y(t-\xi)}{y(t)} = e^{-\ln \frac{y(t)}{y(t-\xi)}} = e^{-\int_{t-\xi}^{t} \frac{y'(u)}{y(u)}\, du} \geqq e^{r\int_{t-\xi}^{t} du} = e^{r\xi}.$$

Therefore,

$$y(t-\tau_n) \geqq e^{r\tau_n} y(t) \text{ for all } t \geqq T_r+\tau \quad (n = 0,1,\ldots)$$

and

$$y(t-s) \geqq e^{rs} y(t) \text{ for } t \geqq T_r \text{ and } 0 \leqq s \leqq t-T_r.$$

Thus, from (I_1) we obtain for every $t \geqq T_r+\tau$

$$\begin{aligned} 0 &\geqq y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_0^t K(s)y(t-s)ds \\ &\geqq y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_0^{t-T_r} K(s)y(t-s)ds \\ &\geqq y'(t) + \left[\sum_{n=0}^{\infty} p_n e^{r\tau_n} + \int_0^{t-T_r} e^{rs}K(s)ds\right] y(t). \end{aligned}$$

We now claim that

$$(**) \qquad \sum_{n=0}^{\infty} p_n e^{r\tau_n} + \int_0^{t-T_r} e^{rs}K(s)ds \leqq \lambda^* \quad \text{for all } t \geqq T_r+\tau .$$

Otherwise, for some $\hat{T}_r \geqq T_r+\tau$

$$\hat{\lambda} \equiv \sum_{n=0}^{\infty} p_n e^{r\tau_n} + \int_0^{\hat{T}_r - T_r} e^{rs}K(s)ds > \lambda^*$$

and consequently we have for $t \geqq \hat{T}_r$

$$0 \geqq y'(t) + \left[\sum_{n=0}^{\infty} p_n e^{r\tau_n} + \int_0^{\hat{T}_r - T_r} e^{rs}K(s)ds\right] y(t) = y'(t)+\hat{\lambda}y(t).$$

This means that $\hat{\lambda} \in \Lambda$, which contradicts the hypothesis that $\hat{\lambda} > \lambda$. So, (**) is true. This gives

$$\sum_{n=0}^{\infty} p_n e^{r\tau_n} + \int_0^{\infty} e^{rs}K(s)ds \leqq \lambda^*$$

or

$$\sum_{n=0}^{\infty} p_n e^{(\lambda^*-\mu)\tau_n} + \int_0^{\infty} e^{(\lambda^*-\mu)s}K(s)ds \leqq \lambda^* .$$

The last inequality holds for all $\mu \in (0,\lambda^*)$. So, as $\mu \to 0$, we get

$$\sum_{n=0}^{\infty} p_n e^{\lambda^*\tau_n} + \int_0^{\infty} e^{\lambda^* s}K(s)ds \leqq \lambda^* ,$$

which contradicts condition (C) and completes our proof.

Note that *condition* (C) *is satisfied if*

$$(C^*) \qquad +\infty \geqq \sum_{n=0}^{\infty} p_n\tau_n + \int_0^{\infty} sK(s)ds > \frac{1}{e} .$$

Indeed, we have $\min\limits_{\lambda>0} (e^{\lambda\xi}/\lambda) = e\xi$ for $\xi > 0$. Hence, for any $\lambda > 0$

$$e^{\lambda\tau_n} \geqq \lambda e\tau_n \quad (n = 0,1,\ldots), \text{ and } e^{\lambda s} \geqq \lambda e s \text{ for } s \geqq 0$$

and consequently

$$-\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda\tau_n} + \int_0^{\infty} e^{\lambda s} K(s)ds \geqq \lambda\left[-1 + \sum_{n=0}^{\infty} p_n e\tau_n + \int_0^{\infty} esK(s)ds\right].$$

COROLLARY 2.1. *Let* K *be not identically zero on* $[0,\infty)$ *and suppose that* (C) *holds. Moreover, let* $T \geqq 0$. *Then there is no solution on* $[T,\infty)$ *of* (E_1) *which is positive* (*resp. negative*) *on* $[\min\{0,T-\tau\},\infty)$.

The next result gives a sufficient condition for the existence of a positive solution of (E_1).

THEOREM 2.2. *Let* y *be a solution on* $[0,\infty)$ *of* (I_1) *which is positive and decreasing on* $[-\tau,\infty)$. *Moreover, let* $T > 0$ *and suppose that* K *is not identically zero on* $[0,T]$. *Then there exists a solution* x *on* $[T,\infty)$ *of* (E_1), *which is positive on* $[\min\{0,T-\tau\},\infty)$ *and such that*

$$x(t) \leqq y(t) \text{ for every } t \geqq \min\{0,T-\tau\}$$

and

$$x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_0^t K(t-s)x(s)ds \leqq 0 \text{ for } 0 \leqq t < T.$$

PROOF. From (I_1) we obtain for $t^* \geqq t \geqq 0$

$$y(t) \geqq y(t^*) + \int_t^{t^*}\left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \int_0^u K(u-s)y(s)ds\right]du$$

$$> \int_t^{t^*}\left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \int_0^u K(u-s)y(s)ds\right]du.$$

Hence, we have

$$y(t) \geqq \int_t^{\infty}\left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \int_0^u K(u-s)y(s)ds\right]du \text{ for all } t \geqq 0.$$

Let X be the set of all continuous real-valued functions x on the interval $[\min\{0,T-\tau\},\infty)$ with

$$0 \leq x(t) \leq y(t) \text{ for every } t \geq \min\{0, T-\tau\},$$

For any function x in X we define

$$(Sx)(t) = \begin{cases} \displaystyle\int_t^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_0^u K(u-s)x(s)ds \right] du, & \text{if } t \geq T \\ \displaystyle\int_T^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_0^u K(u-s)x(s)ds \right] du + \int_t^T \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \right. & \\ \qquad\qquad \displaystyle\left. + \int_0^u K(u-s)y(s)ds \right] du, & \text{if } 0 \leq t \leq T \\ \displaystyle\int_T^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_0^u K(u-s)x(s)ds \right] du + \int_0^T \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \right. & \\ \qquad\qquad \displaystyle\left. + \int_0^u K(u-s)y(s)ds \right] du, & \text{if } \min\{0, T-\tau\} \leq t \leq 0. \end{cases}$$

Then it is easy to verify that the above formula defines an increasing operator S of X into itself. Note that the increasing character of S will be considered with respect to the usual pointwise ordering in X. Furthermore, set

$$x_0 = y, \text{ and } x_m = Sx_{m-1} \quad (m = 1, 2, \ldots)$$

and observe that $(x_m)_{m=0,1,\ldots}$ is a decreasing sequence of functions in X. Thus, by defining

$$x = \lim_{m \to \infty} x_m \text{ pointwise on } [\min\{0, T-\tau\}, \infty)$$

and applying the Lebesgue dominated convergence theorem, we obtain $x = Sx$, i.e.

$$x(t) = \begin{cases} \displaystyle\int_t^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_0^u K(u-s)x(s)ds \right] du, & \text{if } t \geq T \\ \displaystyle\int_T^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_0^u K(u-s)x(s)ds \right] du + \int_t^T \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \right. & \\ \qquad\qquad \displaystyle\left. + \int_0^u K(u-s)y(s)ds \right] du, & \text{if } 0 \leq t \leq T \\ \displaystyle\int_T^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_0^u K(u-s)x(s)ds \right] du + \int_0^T \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \right. & \\ \qquad\qquad \displaystyle\left. + \int_0^u K(u-s)y(s)ds \right] du, & \text{if } \min\{0, T-\tau\} \leq t \leq 0. \end{cases}$$

Obviously, x is a continuous nonnegative function on $[\min\{0,T-\tau\},\infty)$ with $x \leq y$. Moreover, we have

$$x'(t) = -\left[\sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_0^t K(t-s)x(s)ds\right] \text{ for all } t \geq T,$$

which means that x is a solution on $[T,\infty)$ of (E_1). Also, we obtain for $0 \leq t < T$

$$x'(t) = -\left[\sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_0^t K(t-s)y(s)ds\right]$$

$$\leq -\left[\sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_0^t K(t-s)x(s)ds\right].$$

Finally, we will prove that x is positive on $[\min\{0,T-\tau\},\infty)$. Since y is positive on $[0,T]$ and K is not identically zero on this interval, we have

$$\int_0^T K(T-s)y(s)ds = \int_0^T K(s)y(T-s)ds > 0$$

and consequently

$$\int_t^T \int_0^u K(u-s)y(s)ds\,du > 0 \quad \text{for } 0 \leq t < T.$$

This ensures that $x(t) > 0$ for $\min\{0,T-\tau\} \leq t < T$. So, it remains to show that x is also positive on $[T,\infty)$. To this end, we assume that $Q = \{\xi \geq T : x(\xi) = 0\} \neq \emptyset$ and we put $T^* = \inf Q$. Then $T^* \geq T$, $x(T^*) = 0$ and $x(t) > 0$ for $\min\{0,T-\tau\} \leq t < T^*$. Thus, we have

$$\int_0^{T^*} K(T^*-s)x(s)ds = \int_0^{T^*} K(s)x(T^*-s)ds > 0$$

and hence

$$\int_{T^*}^{\infty} \int_0^u K(u-s)x(s)ds > 0.$$

Therefore, we obtain

$$0 = x(T^*) = \int_{T^*}^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_0^u K(u-s)x(s)ds\right]du$$

$$\geq \int_{T^*}^{\infty} \int_0^u K(u-s)x(s)ds\,du > 0,$$

which is a contradiction and the proof is complete.

COROLLARY 2.2. *Let the following condition be satisfied:*

$$\text{(H)} \qquad -\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda\tau_n} + \int_0^{\infty} e^{\lambda s} K(s)ds \leqq 0 \text{ for some } \lambda > 0.$$

Moreover, let $T > 0$ *and suppose that* K *is not identically zero on* $[0,T]$. *Then there exists a solution* x *on* $[T,\infty)$ *of* (E_1), *which is positive on* $[\min\{0,T-\tau\},\infty)$ *and such that*

$$x(t) \leqq e^{-\lambda t} \text{ for every } t \geqq \min\{0,T-\tau\}$$

and

$$x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_0^{t} K(t-s)x(s)ds \leqq 0 \text{ for } 0 \leqq t < T.$$

PROOF. Set $y(t) = e^{-\lambda t}$, $t \geqq -\tau$. Then for every $t \geqq 0$

$$y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_0^{t} K(t-s)y(s)ds = y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_0^{t} K(s)y(t-s)ds$$

$$= e^{-\lambda t}\left[-\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda\tau_n} + \int_0^{t} e^{\lambda s}K(s)ds\right] \leqq e^{-\lambda t}\left[-\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda\tau_n} + \int_0^{\infty} e^{\lambda s}K(s)ds\right]$$

and hence, by condition (H), y is a solution on $[0,\infty)$ of (I_1). Clearly, y is positive and decreasing on $[-\tau,\infty)$. So, it suffices to apply Theorem 2.2.

The following result is a consequence of Corollaries 2.1 and 2.2.

COROLLARY 2.3. *Let* $T > 0$ *and suppose that* K *is not identically zero on* $[0,T]$. *Then* (H) *is a necessary and sufficient condition for* (E_1) *to have a solution on* $[T,\infty)$ *which is positive on* $[\min\{0,T-\tau\},\infty)$.

3. Inequality (I_2)-Equation (E_2).

The following theorem states that (C) is also a necessary condition for inequality (I_2) to have a positive solution.

THEOREM 3.1. *Let* K *be not identically zero on* $[0,\infty)$ *and suppose that* (C) *holds. Moreover, let* $T\in\mathbb{R}$. *Then there is no solution on* $[T,\infty)$ *of* (I_2) *which is positive on* $\mathbb{R}$.

PROOF. Let y be a solution on $[T,\infty)$ of (I_2) which is positive on $\mathbb{R}$. Set $\tilde{T}=\max\{0,T\}\geqq 0$. For every $t\geqq\tilde{T}$, we obtain

$$\begin{aligned} 0 &\geqq y'(t)+\sum_{n=0}^{\infty} p_n y(t-\tau_n)+\int_{-\infty}^{t} K(t-s)y(s)ds \\ &= y'(t)+\sum_{n=0}^{\infty} p_n y(t-\tau_n)+\int_{-\infty}^{0} K(t-s)y(s)ds+\int_{0}^{t} K(t-s)y(s)ds \\ &\geqq y'(t)+\sum_{n=0}^{\infty} p_n y(t-\tau_n)+\int_{0}^{t} K(t-s)y(s)ds. \end{aligned}$$

Hence, the function $\tilde{y}=y\big|[\min\{0,\tilde{T}-\tau\},\infty)$ is a solution on $[\tilde{T},\infty)$ of (I_1). But, $\tilde{y}$ is positive on $[\min\{0,\tilde{T}-\tau\},\infty)$ and so Theorem 2.1 gives a contradiction.

COROLLARY 3.1. *Let* K *be not identically zero on* $[0,\infty)$ *and suppose that* (C) *holds. Moreover, let* $T\in\mathbb{R}$. *Then there is no solution on* $[T,\infty)$ *of* (E_2) *which is positive on* $\mathbb{R}$.

Now, we will give sufficient conditions for the existence of a positive solution of (E_2).

THEOREM 3.2. *Suppose that* K *is not identically zero on* $[0,\infty)$. *Let* y *be a positive solution on* $\mathbb{R}$ *of* (I_2) *and let* $T\in\mathbb{R}$. *Then there exists a solution* x *on* $[T,\infty)$ *of* (E_2), *which is positive on* $\mathbb{R}$ *and such that*

$$x(t)\leqq y(t) \text{ for every } t\in\mathbb{R}$$

and

$$x'(t)+\sum_{n=0}^{\infty} p_n x(t-\tau_n)+\int_{-\infty}^{t} K(t-s)x(s)ds\leqq 0 \text{ for } t<T.$$

PROOF. For every t^*, t with $t^* \geqq t$, we get

$$y(t) \geqq y(t^*) + \int_t^{t^*} \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \int_{-\infty}^{u} K(u-s)y(s)ds \right] du$$

$$> \int_t^{t^*} \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \int_{-\infty}^{u} K(u-s)y(s)ds \right] du$$

and consequently

$$y(t) \geqq \int_t^{\infty} \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \int_{-\infty}^{u} K(u-s)y(s)ds \right] du \text{ for all } t \in \mathbb{R}.$$

Let X be the set of all nonnegative continuous functions x on $\mathbb{R}$ with

$$x(t) \leqq y(t) \text{ for every } t \in \mathbb{R}.$$

The formula

$$(Sx)(t) = \begin{cases} \int_t^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_{-\infty}^{u} K(u-s)x(s)ds \right] du, & \text{if } t \geqq T \\ \int_T^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_{-\infty}^{u} K(u-s)x(s)ds \right] du + \int_t^{T} \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \right. \\ \qquad \left. + \int_{-\infty}^{u} K(u-s)y(s)ds \right] du, & \text{if } t \leqq T \end{cases}$$

defines an increasing mapping $S : X \to X$. Define the decreasing sequence $(x_m)_{m=0,1,\ldots}$ of functions in X, where

$$x_0 = y, \text{ and } x_m = Sx_{m-1} \quad (m = 1,2,\ldots).$$

Furthermore, set

$$x = \lim_{m\to\infty} x_m \text{ pointwise on } \mathbb{R}.$$

By the Lebesgue dominated convergence theorem, we obtain $x = Sx$, that is

$$x(t) = \begin{cases} \int_t^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_{-\infty}^{u} K(u-s)x(s)ds \right] du, & \text{if } t \geqq T \\ \int_T^{\infty} \left[\sum_{n=0}^{\infty} p_n x(u-\tau_n) + \int_{-\infty}^{u} K(u-s)x(s)ds \right] du + \int_t^{T} \left[\sum_{n=0}^{\infty} p_n y(u-\tau_n) + \right. \\ \qquad \left. + \int_{-\infty}^{u} K(u-s)y(s)ds \right] du, & \text{if } t \leqq T. \end{cases}$$

Then

$$x'(t) = -\left[\sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_{-\infty}^{t} K(t-s)x(s)ds\right] \text{ for every } t \geqq T,$$

which means that x is a solution on $[T,\infty)$ of (E_2). Also, it is obvious that

$$x(t) \leqq y(t) \text{ for all } t \in \mathbb{R}.$$

Moreover, for $t < T$

$$x'(t) = -\left[\sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_{-\infty}^{t} K(t-s)y(s)ds\right] \leqq -\left[\sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_{-\infty}^{t} K(t-s)x(s)ds\right].$$

Finally, we will establish that x is positive on $\mathbb{R}$. Since y is positive on $\mathbb{R}$ and K is not identically zero on $[0,\infty)$, we have

$$\int_{-\infty}^{u} K(u-s)y(s)ds > 0 \text{ for } u \in \mathbb{R}$$

and consequently

$$\int_{t}^{T}\int_{-\infty}^{u} K(u-s)y(s)ds\,du > 0, \text{ if } t < T.$$

Hence, x is positive on $(-\infty,T)$. We shall prove that x is also positive on $[T,\infty)$. Let T^* be the first zero on $[T,\infty)$ of x. Then $x(T^*) = 0$ and $x(t) > 0$ for $t < T^*$. We have

$$0 = x(T^*) \geqq \int_{T^*}^{\infty}\int_{-\infty}^{u} K(u-s)x(s)ds\,du$$

and therefore

$$\int_{-\infty}^{u} K(u-s)x(s)ds = 0 \text{ for all } u \geqq T^*.$$

In particular, we get

$$\int_{-\infty}^{T^*} K(T^*-s)x(s)ds = 0,$$

which contradicts the fact that K is not identically zero on $[0,\infty)$ and x is positive on $(-\infty,T^*)$. The proof is complete.

COROLLARY 3.2. _Suppose that_ K _is not identically zero on_ $[0,\infty)$ _and let_ *condition* (H) *be satisfied. Moreover, let* $T \in \mathbb{R}$. *Then there exists a solution* x *on* $[T,\infty)$ *of* (E_2), *which is positive on* $\mathbb{R}$ *and such that*

$$x(t) \leq e^{-\lambda t} \quad \text{for every } t \in \mathbb{R}$$

and

$$x'(t) + \sum_{n=0}^{\infty} p_n x(t-\tau_n) + \int_{-\infty}^{t} K(t-s)x(s)ds \leq 0 \quad \text{for } t < T$$

PROOF. If we put $y(t) = e^{-\lambda t}$ for $t \in \mathbb{R}$, then we obtain for every $t \in \mathbb{R}$

$$y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_{-\infty}^{t} K(t-s)y(s)ds = y'(t) + \sum_{n=0}^{\infty} p_n y(t-\tau_n) + \int_{0}^{\infty} K(s)y(t-s)ds$$

$$= e^{-\lambda t}\left[-\lambda + \sum_{n=0}^{\infty} p_n e^{\lambda \tau_n} + \int_{0}^{\infty} e^{\lambda s} K(s)ds\right].$$

Thus, by condition (H), y is a positive solution on $\mathbb{R}$ of (I_2) and so it is enough to apply Theorem 3.2.

From Corollaries 3.1 and 3.2 we obtain the following result.

COROLLARY 3.3. *Suppose that* K *is not identically zero on* $[0,\infty)$ *and let* $T \in \mathbb{R}$. *Then* (H) *is a necessary and sufficient condition for* (E_2) *to have a solution* x *on* $[T,\infty)$ *which is positive on* $\mathbb{R}$.

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