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ON THE SMOOTHNESS OF REGULARIZATION IN NEWTONIAN GRAVITATIONAL SYSTEMS*

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ABSTRACT. We prove that, for any choice of the masses, every solution of the three-body problem leading to the total collapse can be C^1-regularized with respect to time. If the motion is rectilinear the extension of the solution is unique, otherwise there exist two possible continuations and consequently, the smoothness of the regularization can not be improved. Using symbolic dynamics the chaotic character of the set of bounded solutions with multiple extensions is pointed out. Generalizations of these results to the n-body problem are considered. The main theorem extends thus a classical result of Siegel, by showing not only that nonrectilinear collision orbits can not be, in general, analytically regularized, but also the fact that a better regularization than that of C^1-type can not be found.

1. INTRODUCTION

We consider in this note the old question of extending solutions in the n-body problem beyond a collision singularity, process called ***regularization.*** This was first done by Sundman [Su] at the beginning of this century in order to give a series expansion for the position vectors in the three-body problem, convergent for all real time variable. He succeeded to obtain an ***analytic regularization*** of binary collision orbits by expressing the solution as a con-

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vergent power series in $(t^* - t)^{2/3}$, where t^* is the singularity. Consequently, the solution makes sense (and remains analytic) not only on the interval $[0, t^*]$ but also for $t \geq t^*$. The problem of giving an analytic regularization of triple collision orbits is more difficult and was solved by Siegel [Si] in 1941. He proved that, excepting eventually a negligible set of masses, series expansions of triple collision solutions involve powers of $(t^* - t)^\alpha$, where α is irrational, thus an extension of the orbit across t^* makes no sense anymore. Sperling [Sp] and Saari [S2] proved later that solutions of the n-body problem involving multiple binary collisions can also be analytically extended with respect to the time variable. Thus, simple or multiple binary collision orbits can be analytically regularized but solutions involving a collision of at least three particles are not analytically extendable, except for a negligible set of masses.

From the point of view of the modern theory of dynamical systems the following natural question was posed: does this regularization process carry with it the property of continuity of the solution with respect to initial data (i.e. relative to nearby solutions)? It is worth to mention that this question appeared in an embrionary form by Levi-Civita in 1920 [L].

The recent papers of Easton [E] and McGehee [M1], [M2], [M3] showed that, in general, there is no relation between a regularization with respect to initial conditions (defined and studied in [E]) and a regularization with respect to the time variable. Accidentally in the n-body problem the regularization of simple binary collisions is possible in both ways.

Besides these directions of work it is still an interesting problem to see what happens beyond a triple collision, how do particles move across an elastic bounce. Since analytic continuations of the motion are known to be impossible, one may ask whether a C^k-regularization, $k \in \mathbb{N} \cup \{\infty\}$, of the solutions, would work without to leave the confined energy level. The remarkable paper [M1] of McGehee allows the study of orbits passing close to triple collisions, by studying a fictitious motion on the so-called *collision manifold*. These geometrical facts led us to the idea that a total collision orbit and a total ejection one might be pasted together in a smooth fashion with respect to the time variable, giving a picture of how the motion can be continued across such a collision. How smooth can such a prolongation be, in case it exists? The results of this note are extensions and completions to some previous ideas of the author [D1], [D2], [D3]. Remark also that a C^1-regularization of double binary collisions in the rectilinear four-body problem, for a set of masses of codimension two, was given by Belbruno [B].

For expository reasons we first present the main results in the three-body case by proving that, for any choice of the masses, any triple collision solution can be C^1-regularized with respect to time. This is done by introducing, with the help of suitable coordinate and time transformations, some new, equivalent equations of motion and by showing that the corresponding vector field yields a C^1-prolongation of the solution with respect to the new (fictitious) time variable, across the total collapse singularity. Excepting the rectilinear case, the extension is not unique and, consequently, the smoothness of the regularization cannot be improved. We have two ways to prolong the orbit in a C^1 fashion and an infinity of extensions in case the continuation is only differentiable.

As a consequence, bounded triple collision solutions can be extended each time a collision appears, the global motion having a dilatation-contraction structure, with a total collision at periodic intervals of time. We see that symbolic dynamics can be created for this class of solutions, pointing out the analogy with Smale horseshoe and consequently its chaotic character. The possibility of a weaker regularization is also discussed.

Finally we show that these facts can be generalized in certain situations to the planar n-body problem (for almost all masses) and to the spatial case (for an open set of masses).

2. EQUATIONS OF MOTION

The equations of motion of the n-body problem in an arbitrary fixed frame in $\mathbb{R}^3$ are

$$\begin{cases} \dot{q} = M^{-1}p \\ \dot{p} = \nabla U(q) \end{cases} \tag{2.1}$$

where $\boldsymbol{q}_i = (q_i^1, q_i^2, q_i^3) \in \mathbb{R}^3$, $\boldsymbol{p}_i = m_i \dot{\boldsymbol{q}}_i$, $i = \overline{1,n}$, are the *position vector* and *momentum* of the i-th particle, $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$, $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ represent the *configuration*, respectively the *momentum* of the system,

$$U : \mathbb{R}^{3n} - \Delta \longrightarrow \mathbb{R}_+, \quad U(\boldsymbol{q}) = \sum_{1 \leq i < j \leq n} m_i m_j \, |\boldsymbol{q}_i - \boldsymbol{q}_j|^{-1}$$

is the *potential function* ($-U$ being the *potential energy*),

$$\Delta = \bigcup_{1 \leq i < j \leq n} \{\boldsymbol{q} \mid \boldsymbol{q}_i = \boldsymbol{q}_j\}$$

denotes the collision set, $|\cdot|$ is the Euclidean norm, ∇ represents the gradient and $\boldsymbol{M}$ is the $3n \times 3n$ matrix:

$$\boldsymbol{M} = \operatorname{diag}(m_1, m_1, m_1, m_2, m_2, m_2, \ldots, m_n, m_n, m_n),$$

the constants $m_i > 0$, $i = \overline{1,n}$, denoting the masses of the particles.

Standard results of the theory of differential equations ensure, for given initial conditions $(\boldsymbol{q},\boldsymbol{p})(0) \in (\mathbb{R}^{3n} - \Delta) \times \mathbb{R}^{3n}$, the existence and uniqueness of an analytic solution $(\boldsymbol{q},\boldsymbol{p})$ of the Eqs. (2.1), defined locally on some interval (t^-, t^+), $-\infty \leq t^- < 0 < t^+ \leq +\infty$. Because of the symmetry we may restrict our study to the interval $[0, t^*)$ and extend it analytically to a maximal one $[0, t^*)$, $0 < t^+ \leq t^* \leq +\infty$. If t^* is finite we say that the solution experiences a *singularity* at this moment, which physically corresponds to a *collision* or to a *pseudo-collision (non-collision singularity)*, i.e. a motion becoming unbounded in finite time (see e.g. [M4], [MM], [X]).

Without loss of generality we may restrict the Eqs. (2.1) to the invariant set $\boldsymbol{Q} \times \boldsymbol{P}$, where

$$\boldsymbol{Q} = \left\{\boldsymbol{q} \in \mathbb{R}^{3n} \mid \sum_{i=1}^{n} m_i \boldsymbol{q}_i = \boldsymbol{0}\right\} \quad \text{and} \quad \boldsymbol{P} = \left\{\boldsymbol{p} \in \mathbb{R}^{3n} \mid \sum_{i=1}^{n} \boldsymbol{p}_i = \boldsymbol{0}\right\},$$

which physically means that the origin of the frame is considered in the center of mass of the particle system.

In order to make our presentation more simple we first work in the three-body case. We consider therefore the system:

$$\begin{cases} \dot{q}_i = m_i^{-1} p_i, & i = 1, 2, 3, \\ \dot{p}_i = \partial_i U(q), & i = 1, 2, 3, \end{cases} \tag{2.2}$$

where the operators ∂_i are defined by $\nabla = (\partial_1, \partial_2, \partial$

3. PRELIMINARY FACTS

In order to have a self-contained presentation we need a preliminary development. Let's begin with a property due to Pollard and Saari [PS], [S2]:

THEOREM 3.1. *Consider (q,p) to be a solution of the Eqs. (2.1) leading to a collision, when $t \to t^*$; denote this by $q(t) \to q^* = (q_1^*, \dots, q_n^*) \in \Delta$, $t \to t^*$. Then*

$$\sum_{i=1}^{n} m_i |q_i(t) - q_i^*|^2 \sim A(t^* - t)^{4/3} \quad \text{and} \quad \dot{r}(t)(t^* - t) = o(1),$$

where $A > 0$ is a constant and

$$r(t) := (q(t) - q^*)(t^* - t)^{-2/3}.$$

Siegel solved the so-called *infinite spin* problem of Painlevé-Wintner for the three-body case (see [Si], [S-M]) showing that a total collapse orbit approaches a limiting orientation. In order to give a precise formulation of it define the set

$$\mathcal{R}^3 = \mathbb{R}^3 / \sim,$$

where $\sim$ is the symmetry relation with respect to the origin in $\mathbb{R}^3$. Obviously $\mathcal{R}^3$ is a smooth manifold.

THEOREM 3.2. *If (q,p) is a total collapse solution of the Eqs. (2.2) then*

$$(q_k(t) - q_s(t)) \sim A_{ks}(t^* - t)^{2/3}, \qquad k, s \in \{1, 2, 3, \},\ k \neq s,\ t \to t^*,$$

where $A_{ks} \in \mathcal{R}^3$. More than this

$$p_k(t) \sim A_k(t^* - t)^{-1/3}, \qquad k \in \{1, 2, 3\},\ t \to t^*,$$

where $A_k \in \mathbb{R}^3$.

Remark that A_{ks} and A_k may be $\mathbf{0}$ too. We will use these to prove the following:

LEMMA 3.3. *Let (q,p) be a solution of the Eqs. (2.2) leading to the total collision when $t \to t^*$. Then there exist $i,j \in \{1,2,3\}$ and a constant $A_{ij} \in \mathbb{R}^3$ with $A_{ij} = |A_{ij}| > 0$, such that*

$$\text{(i)} \quad |q_i(t) - q_j(t)| \sim A_{ij}(t^* - t)^{2/3}, \qquad t \to t^*,$$

$$\text{(ii)} \quad (m_i^{-1}p_i(t) - m_j^{-1}p_j(t)) \sim (2/3)A_{ij}(t^* - t)^{-1/3}, \qquad t \to t^*.$$

Renumbering eventually the bodies we may suppose that $j = 1$ and $i = 2$.

Proof: In case of the total collision, $q(t) \to q^* = \mathbf{0} \in \Delta$, thus, by Th. 3.1 and a direct computation we obtain

$$m^{-1} \sum_{1 \le i < j \le 3} m_i m_j |q_i(t) - q_j(t)|^2 = \sum_{i=1}^{3} m_i |q_i(t)|^2 \sim A(t^* - t)^{4/3},$$

with $A > 0$ and $m = m_1 + m_2 + m_3$. It means, by Th. 3.2, that if $A_{ks} = \mathbf{0}$, $\forall k, s \in \{1,2,3\}$, then $A = 0$, a contradiction. Thus the first part of the theorem follows. Observe further that $(t^* - t)\dot{r}(t) = o\,(1)$ implies

$$[\,\dot{q}(t)(t^* - t)^{1/3} - (2/3)q(t)\,(t^* - t)^{-2/3}] \to \mathbf{0}, \quad t \to t^*,$$

thus

$$(\,\dot{q}_i(t) - \dot{q}_j(t))(t^* - t)^{1/3} \to (2/3)\,A_{ij}, \qquad t \to t^*,$$

yielding the final conclusion. □

An important notion in the study of the n-body problem is that of *central configuration.*

DEFINITION 3.4. *The n-bodies $m_1, \ldots, m_n$ are said to form a central configuration at time t_0 if*

$$\nabla U(q(t_0)) = \sigma M q(t_0), \tag{3.1}$$

where $\sigma > 0$ is a constant.

Sundman [Su] proved that the configuration of the system fulfills, asymptotically, relation (3.1) when $t \to t^*$. Saari and Hulkower [SH] showed that total collapse solutions always tend to a central configuration, a result which will be mainly used in the followings. Observe that this doesn't necessarily follow from the result of Sundman because of the SO(3)-invariance (SO(2)-invariance in the three-body case since triple collision solutions are known to be planar). Consequently the motion might enter into an infinite spin without to reach a definite limiting configuration.

For the three-body case it is known that exactly five central configurations exist, two equilateral and three collinear (for more information see [S1], [Mo], [W]). If $n \geq 4$ it is not known whether the number of central configurations (for n given masses) is finite or not.

We can use these facts in order to state the following, very useful, lemma:

LEMMA 3.5. *Consider a solution $(\boldsymbol{q},\boldsymbol{p})$ of the Eqs. (2.2), leading to the total collision when $t \to t^*$. Then there exist two real analytic, positive valued functions u and v such that*

$$|q_2(t) - q_3(t)| = u(t)\,|q_1(t) - q_2(t)| \tag{3.2}$$

$$|q_1(t) - q_3(t)| = v(t)\,|q_1(t) - q_2(t)| \tag{3.3}$$

and $u(t) \to u^* > 0, \quad v(t) \to v^* > 0, \quad t \to t^*$.

Proof: As we have seen before, the particles tend to one of the five central configurations described above, when $t \to t^*$. Consequently the ratios of the mutual distances tend to a constant, the existence and the analyticity of u and v following obviously. Observe that in the equilateral cases, $u^* = v^* = 1$ and in the collinear cases u^* and v^* depend on the values of the masses (which give, by a complicated formula, the distances between particles) being, however, strictly positive. □

4. THE REGULARIZING TRANSFORMATIONS

Let's first give a precise definition of what we understand by regularizing a solution.

DEFINITION 4.1. *Consider an analytic solution (q,p) of the Eqs. (2.1), defined on $[0,t^*)$, with t^* finite, leading to a collision when $t \to t^*$. We say that this solution is C^k-regularizable with respect to time, $k \in \mathbb{N}^* \cup \{\infty\} \cup \{\omega\}$ (where C^ω means analytic), if it can be extended, as a function of class C^k, to the interval $[0,t^*]$.*

Using certain coordinate and time transformations we will give an equivalent form of the Eqs. (2.2) and then show that triple collision solutions of the new equations of motion are C^1-regularizable.

Consider first the coordinate transformation:

$$\begin{aligned} x_1 &= q_1 - q_2, \\ x_2 &= q_2, \\ x_3 &= q_3, \end{aligned} \tag{4.1a}$$

$$\begin{aligned} y_1 &= m_1^{-1} p_1 |q_1 - q_2|^2, \\ y_2 &= (m_2^{-1} p_2 - m_1^{-1} p_1)|q_1 - q_2|^2, \\ y_3 &= p_3 |q_1 - q_2|^2, \end{aligned} \tag{4.1b}$$

having the inverse

$$\begin{aligned} q_1 &= x_1 + x_2, \\ q_2 &= x_2, \\ q_3 &= x_3, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} p_1 &= m_1 y_1 |x_1|^{-2}, \\ p_2 &= m_2 (y_2 + y_1)|x_1|^{-2}, \\ p_3 &= y_3 |x_1|^{-2}. \end{aligned} \tag{4.2b}$$

By these transformations and Lemma 3.5, the equations of motion (2.2) become:

$$\begin{cases} \dot{\boldsymbol{x}}_1 = -\boldsymbol{y}_2\,|\boldsymbol{x}_1|^{-2} \\ \dot{\boldsymbol{x}}_2 = (\boldsymbol{y}_2 + \boldsymbol{y}_1)\,|\boldsymbol{x}_1|^{-2} \\ \dot{\boldsymbol{x}}_3 = m_3^{-1}\boldsymbol{y}_3\,|\boldsymbol{x}_1|^{-2} \\ \dot{\boldsymbol{y}}_1 = [-m_2\boldsymbol{x}_1 + m_3(\boldsymbol{x}_3 - \boldsymbol{x}_2 - \boldsymbol{x}_1)v^{-3}]\,|\boldsymbol{x}_1|^{-1} - 2m_2^{-1}\boldsymbol{y}_1(\boldsymbol{x}_1^T\boldsymbol{y}_2)\,|\boldsymbol{x}_1|^{-4} \\ \dot{\boldsymbol{y}}_2 = [(m_1 + m_2)\boldsymbol{x}_1 + m_3(\boldsymbol{x}_3 - \boldsymbol{x}_2)u^{-3} - \\ \qquad - m_3(\boldsymbol{x}_3 - \boldsymbol{x}_2 - \boldsymbol{x}_1)v^{-3}]\,|\boldsymbol{x}_1|^{-1} + 2m_2^{-2}\boldsymbol{y}_2(\boldsymbol{x}_1^T\boldsymbol{y}_2)\,|\boldsymbol{x}_1|^{-4} \\ \dot{\boldsymbol{y}}_3 = [m_1m_3(\boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3)u^{-3} + m_2m_3(\boldsymbol{x}_2 - \boldsymbol{x}_3)v^{-3}]\,|\boldsymbol{x}_1|^{-1} - \\ \qquad - 2m_2^{-1}\boldsymbol{y}_3(\boldsymbol{x}_1^T\boldsymbol{y}_2)\,|\boldsymbol{x}_1|^{-4}. \end{cases} \tag{4.3}$$

Observe that the set $\boldsymbol{X} \times \boldsymbol{Y}$ is invariant for the Eqs. (4.3), where

$$\boldsymbol{X} = \{\boldsymbol{x} \mid m_1\boldsymbol{x}_1 + (m_1 + m_2)\boldsymbol{x}_2 + m_3\boldsymbol{x}_3 = 0\},$$

$$\boldsymbol{Y} = \{\boldsymbol{y} \mid (m_1 + m_2)\boldsymbol{y}_1 + m_2\boldsymbol{y}_2 + \boldsymbol{y}_3 = 0\}$$

correspond to $\boldsymbol{Q}$ respectively $\boldsymbol{P}$.

Also note that in the new coordinates the relations (3.2), (3.3) become

$$|\boldsymbol{x}_2(t) - \boldsymbol{x}_3(t)| = u(t)\,|\boldsymbol{x}_1(t)|, \tag{4.4}$$

$$|\boldsymbol{x}_1(t) + \boldsymbol{x}_2(t) - \boldsymbol{x}_3(t)| = v(t)\,|\boldsymbol{x}_1(t)|, \tag{4.5}$$

and the condition for the total collision is

$$|\boldsymbol{x}_1(t)| \to 0, \quad t \to t^*.$$

Consider further a time transformation which can be formally written as

$$d\tau = |\boldsymbol{x}_1(t)|^{-1/2}\,dt$$

Using this the Eqs. (4.3) become (by abuse of notation):

$$\begin{cases} \boldsymbol{x}_1' = -\boldsymbol{y}_2|\boldsymbol{x}_1|^{-3/2} \\ \boldsymbol{x}_2' = (\boldsymbol{y}_2 + \boldsymbol{y}_1)|\boldsymbol{x}_1|^{-3/2} \\ \boldsymbol{x}_3' = m_3^{-1}\boldsymbol{y}_3|\boldsymbol{x}_1|^{-3/2} \\ \boldsymbol{y}_1' = [-m_2\boldsymbol{x}_1 + m_3(\boldsymbol{x}_3 - \boldsymbol{x}_2 - \boldsymbol{x}_1)v^{-3}]\,|\boldsymbol{x}_1|^{-1/2} - \\ \qquad\qquad - 2m_2^{-1}\boldsymbol{y}_1(\boldsymbol{x}_1^{\mathsf{T}}\boldsymbol{y}_2)|\boldsymbol{x}_1|^{-7/2} \\ \boldsymbol{y}_2' = [(m_1 + m_2)\boldsymbol{x}_1 + m_3(\boldsymbol{x}_3 - \boldsymbol{x}_2)u^{-3} - \\ \qquad - m_3(\boldsymbol{x}_3 - \boldsymbol{x}_2 - \boldsymbol{x}_1)v^{-3}]\,|\boldsymbol{x}_1|^{-1/2} + 2m_2^{-2}y_2(\boldsymbol{x}_1^{\mathsf{T}}\boldsymbol{y}_2)|\boldsymbol{x}_1|^{-7/2} \\ \boldsymbol{y}_3' = [m_1m_3(\boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3)u^{-3} + \\ \qquad + m_2m_3(\boldsymbol{x}_2 - \boldsymbol{x}_3)v^{-3}]\,|\boldsymbol{x}_1|^{-1/2} - 2m_2^{-1}\boldsymbol{y}_3(\boldsymbol{x}_1^{\mathsf{T}}\boldsymbol{y}_2)\,|\boldsymbol{x}_1|^{-7/2} \end{cases} \tag{4.6}$$

where prime denotes differentiation with respect to the time variable τ. The singularity t^* is thus transformed into

$$\tau^* := \lim_{t\to t^*} \int_0^t |\boldsymbol{x}_1(s)|^{-1/2} ds,$$

which obviously exists by the monotonicity of the function of t defining the integral.

5. C^1 AND $C^{1/2}$-REGULARIZATIONS

We are able to prove now that the Eqs. (4.6) yield a C^1- regularization of the total collision solutions.

THEOREM 5.1. *Let $(\boldsymbol{x},\boldsymbol{y})$ be a total collapse solution of the Eqs. (4.6), defined on $[0,\tau^*)$. Then*

(i) *τ^* is finite;*

(ii) *$(\boldsymbol{x},\boldsymbol{y})$ can be extended in a C^1 fashion across τ^*;*

(iii) *excepting the rectilinear solutions for which the prolongation is unique, there exist two distinct extensions of $(\boldsymbol{x},\boldsymbol{y})$ beyond τ^*.*

Proof:

(i) In order to prove that τ^* is finite observe that, by Lemma 3.3 and the coordinate-time transformations, we obtain:

$$\begin{aligned}\lim_{\tau\to\tau^*} y_2(\tau)|x_1(\tau)|^{-3/2} &= \lim_{t\to t^*} y_2(t)|x_1(t)|^{-3/2}\\ &= \lim_{t\to t^*}(m_2^{-1}p_2(t) - m_1^{-1}p_1(t))|q_1(t)-q_2(t)|^{1/2}\\ &= (2/3)A_{21}A_{21}^{1/2} \neq 0.\end{aligned}$$

This means, by the Eqs. (4.6) that $\lim_{\tau\to\tau^*} x'(\tau) \neq 0$, yielding the desired conclusion.

(ii) Using further Th. 3.2 we obtain that $p_i(t^*-t)^{1/3}$ and $q_i(t)(t^*-t)^{-2/3}$ tend to definite limits in $\mathbb{R}^3$. Analogously we obtain that $y_1'(\tau) \to 0$, $i = 1,2,3$, when $\tau \to \tau^*$. All these imply that the vector field given by (4.6) can be made continuous at τ^*, therefore (x,y) is a continuously differentiable function of τ when $\tau = \tau^*$ and the C^1-extension of the solution beyond the total collapse is possible.

(iii) We know that the three bodies tend to form an equilateral or a rectilinear central configuration. In the first case we have two possibilities to continue the motion:

(1) the bodies move on the same orbits like before the collision but having the opposite velocities (FIG. 1).

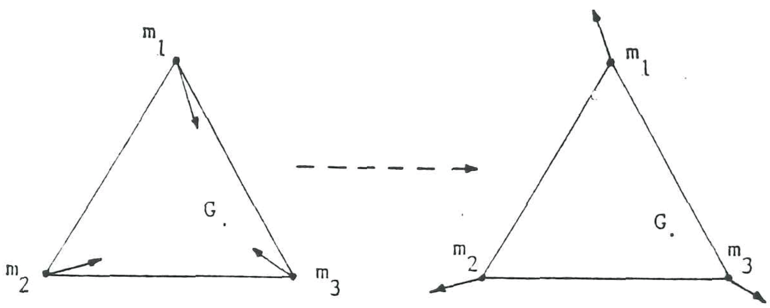

FIGURE 1. The first type of equilateral extension.

(2) the bodies pass through the collision, continuing to move on orbits symmetric with respect to the collision point and maintaining the sense of the velocities (FIG. 2).

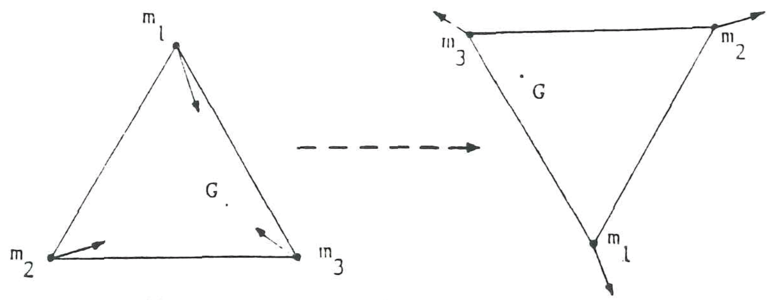

FIGURE 2. The second type of equilateral extension.

In the sec ıd case, since the motion takes place on a fixed line, the particles cannot pass through each other and there is a single possible extension which can be described exactly like in case (1) but thinking to a rectilinear collision solution instead of an equilateral one (FIG. 3).

FIGURE 3. The rectilinear extension.

In all pictures G denotes the center of mass of the particle system.

This geometrical description of the extensions is easy to make clear analytically by observing that both, the collision and ejection solutions (defined for $\tau < \tau^*$ respectively for $\tau > \tau^*$) can be pasted together such that to get, at $\tau = \tau^*$, the limits which appear in (i) and (ii). Consequently, the extended solutions are C^1 at τ^*. This proves the result completely.

REMARK. The existence of two extensions for the equilateral case implies that a smoother regularization is not possible. If the vector field in

(4.6) would be C^1 (or at least locally Lipschitz) then the standard uniqueness results for ordinary differential equations would work, contradicting our conclusion. Thus the result of Siegel [Si],[SM], stating that there are no analytic regularizations for triple collision solutions, excepting a negligible set of masses, can be improved by saying that, for almost all values of the masses the best possible regularization is of class C^1.

A weaker regularization can be obtained in the class $C^{1/2}$ of differentiable functions which are not C^1. In this case we only ask the solution to remain on the same energy level without to require the continuity of the vector field (4.6) at $\tau = \tau^*$. Consequently we obtain a $C^{1/2}$-regularization.

In the case of an equilateral collapse we have an infinity of continuations of the solution. Indeed, due to the SO(3)-invariance (actually SO(2)-invariance in case of the triple collision) any ejection solution like that in FIG. 1, which has been rotated by an arbitrary angle around G, can be differentiably pasted with the collision solution in FIG. 1. This follows obviously by the Eqs. (4.6). More than this, the ejection solution can have the opposite orientation of the triangle m_1, m_2, m_3 in FIG. 1. Thus, we actually have two different classes of extensions beyond the collision, in the set of continuations factorized to the SO(3) symmetry group.

In the rectilinear case the $C^{1/2}$ extension is unique.

6. BOUNDED REGULARIZED SOLUTIONS

Consider bounded triple collision solutions of the Eqs. (4.6) and let $(\boldsymbol{x}(0), \boldsymbol{y}(0))$ be an initial condition of this kind. A solution of this type leads to the collision, after regularization the bodies eject, then lead to a new collision etc., the motion being defined for all $\tau \in \mathbb{R}$. Since there are two possible continuations beyond the collision in the class C^1, respectively two classes of continuations in the class $C^{1/2}$, to every initial condition $(\boldsymbol{x}(0), \boldsymbol{y}(0))$ corresponds an uncountable set $S = S(\boldsymbol{x}(0), \boldsymbol{y}(0))$ of piecewise analytic solutions, every solution alternating contractions and expansions, following at every critical point one of the two existing possibilities.

We will see further that symbolic dynamics can be created for the sets S, proving the complicated structure of the regularized solutions.

Let's therefore attach to each element of S a sequence of symbols.

$$\mathbf{s} = (s_o, s_1, \ldots, s_k, \ldots), \qquad s_i \in \{0, 1\},$$

where 0 is attached to the first type of prolongation and 1 to the second one. Note that this correspondence between S and the set

$$\Sigma = \{\mathbf{s} = (s_o, s_1, \ldots, s_k, \ldots) \mid s_i \in \{0, 1\}, i \in \mathbf{N}\}$$

of symbols is one-to-one and onto.

It becomes clear now that S is uncountable. Indeed, it is known that every number in $[0, 1]$ can be expressed in base two, periodic sequences corresponding to rationals and nonperiodic to irrationals in this interval.

Define the shift automorphism

$$\sigma : \Sigma \to \Sigma, \, \sigma(s) = (\sigma_1(s), \sigma_2(s), \ldots, \sigma_k(s), \ldots)$$
$$\sigma_k(s) = s_{k+1}, \quad k \in \mathbf{N}^*,$$

and attach to each element $s \epsilon \Sigma$ the corresponding orbit

$$\sigma s = (\sigma(s), \sigma^2(s), \ldots, \sigma^k(s), \ldots)$$

of the shift automorphism, where $\sigma^k = \underbrace{\sigma \circ \sigma \circ \ldots \circ \sigma}_{k \ \ times}$.

Denote by $\boldsymbol{\sigma\Sigma}$ the set of orbits and observe that the correspondence between $\boldsymbol{\Sigma}$ and $\boldsymbol{\sigma\Sigma}$ is one-to-one and onto. Endowing $\boldsymbol{\Sigma}$ with a natural metric (see e.g. [De], [Wi]) and translating its topological properties in terms of the set S we obtain

THEOREM 6.1. *The dynamics of S is of Smale horseshoe type, i.e.:*

(i) *the set of periodic orbits in S consisting of orbits of all periods is countable and dense in S.*

(ii) *the set of non-periodic orbits in S has the cardinality of the continuum.*

(iii) *there exists a dense orbit in* $\boldsymbol{S}$.

Also note that since the choice of every next type of prolongation is unpredictable, there is a kind of sensitive dependence on initial data, namely: solutions having the same s_1 symbol may be far from one another.

All these properties characterize chaotic behavior, thus the complicated structure of $\boldsymbol{S}$ becomes clear.

7. THE N-BODY PROBLEM

The extension of Th. 5.1 to the total collapse solutions of the n-body problem is now easy to understand and not dificult to prove. Since Th. 3.2 works in the general case (see [SH]) we have an analog of Lemma 3.3 for the Eqs. (2.1).

Using the coordinate transformations:

$$\begin{aligned}
\boldsymbol{q}_1 &= \boldsymbol{x}_1 + \boldsymbol{x}_2, \\
\boldsymbol{q}_i &= \boldsymbol{x}_i, & i &= \overline{2,n}, \\
\boldsymbol{p}_1 &= m_1 \boldsymbol{y}_1 |\boldsymbol{x}_1|^{-2}, \\
\boldsymbol{p}_2 &= (\boldsymbol{y}_2 + m_2 \boldsymbol{y}_1)|\boldsymbol{x}_1|^{-2}, \\
\boldsymbol{p}_i &= \boldsymbol{y}_i |\boldsymbol{x}_1|^{-2}, & i &= \overline{3,n},
\end{aligned} \tag{7.1}$$

having the inverse

$$\begin{aligned}
\boldsymbol{x}_1 &= \boldsymbol{q}_1 - \boldsymbol{q}_2, \\
\boldsymbol{x}_i &= \boldsymbol{q}_i, & i &= \overline{2,n}, \\
\boldsymbol{y}_1 &= m_1^{-1} \boldsymbol{p}_1 |\boldsymbol{q}_1 - \boldsymbol{q}_2|^2, \\
\boldsymbol{y}_2 &= (m_2^{-1} \boldsymbol{p}_2 - m_1^{-1} \boldsymbol{p}_1 |\boldsymbol{q}_1 - \boldsymbol{q}_2|^2, \\
\boldsymbol{y}_i &= \boldsymbol{p}_i |\boldsymbol{q}_1 - \boldsymbol{q}_2|^2, & i &= \overline{3,n},
\end{aligned} \tag{7.2}$$

we can associate in the same way the time transformation in Section 4. The new equations of motion are:

$$\begin{cases} \boldsymbol{x}_1' = -\boldsymbol{y}_2|\boldsymbol{x}_1|^{-3/2}, & \\ \boldsymbol{x}_2' = (\boldsymbol{y}_2+\boldsymbol{y}_1)|\boldsymbol{x}_1|^{-3/2}, & \\ \boldsymbol{x}_i' = m_i^{-1}\boldsymbol{y}_1|\boldsymbol{x}_1|^{-3/2}, & i=\overline{3,n}, \\ \boldsymbol{y}_i' = \boldsymbol{F}_i(x)|\boldsymbol{x}_1|^{-1/2} - \boldsymbol{G}_i(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}_1|^{-7/2}, & i=\overline{1,n}. \end{cases} \tag{7.3}$$

where

$$\boldsymbol{F}_i(\boldsymbol{x}(\tau(t))) = O((t^*-t)^{2/3})$$

and

$$\boldsymbol{G}_i((\boldsymbol{x},\boldsymbol{y})(\tau(t))) = O((t^*-t)^{8/3}), \qquad i=\overline{1,n}.$$

We can see now that the following extension of Th. 5.1 becomes true:

THEOREM 7.1. *Let $(\boldsymbol{x},\boldsymbol{y})$ be a total collapse solution of the Eqs. (4.6), defined on $[0,\tau^*)$. Then*

(i) *τ^* is finite*

(ii) *$(\boldsymbol{x},\boldsymbol{y})$ can be extended in a C^1-fashion across τ^* as follows:*

   (1) for all choice of the masses, in the rectilinear case;

   (2) excepting eventually a set of masses of lower dimension, in the planar case;

   (3) for an open set of masses, in the spatial case.

(iii) *if one body lies at the center of mass (for all time where the solution is defined) or the orbits of two colliding particles have a common tangent at $\tau=\tau^*$ (i.e. there exist i,j such that $A_i = A_j$ in $\boldsymbol{S}$ for the asymptotic estimation of the momenta: $\boldsymbol{p}_k \sim \boldsymbol{A}_k(t^*-t)^{-1/3}$, $k\epsilon\{i,j\}$), then there is a unique C^1- prolongation of the solution across the collision; in all other cases there are exactly two C^1-extensions.*

Proof:

(i) The fact that τ^* is finite follows like in Th. 5.1 (i).

(ii) It is known from [SII] that in a total collapse solution the particles tend to the center of mass without to spin around similar central configurations, under the action of the SO(3) group. This means $x(\tau) \to \boldsymbol{CC}/\,\mathrm{SO}(3)$, where $\boldsymbol{CC}$ is the set of central configurations factorized to the equivalence relation given by homotheties. It is not generally known if $\boldsymbol{CC}/\,\mathrm{SO}(3)$ is finite and, in case it is not, it's not clear whether it contains a continuum or is formed by isolated points.

(1) In the rectilinear case there are $n!/2$ central configurations [Mou] and the conclusion follows like in Th. 5.1.

(2) & (3) In the planar case it is known [P] that excepting a set of lower dimension, $\boldsymbol{CC}/\,\mathrm{SO}(3)$ is formed by isolated points; in the spatial case the same is true for at least an open set of masses. The conclusion follows obviously like in Th. 5.1.

(iii) We have seen in Th. 5.1 that nonrectilinear triple collision solutions have two possible prolongations. It is obvious that this happens in the n-body case, too, with some exceptions. The exceptions are mainly due to the fact that if at a collision, the orbits of two bodies have the same common tangent, and momenta with opposite sign (this situation arises always for a binary collision in the n-body problem [W]) then the bodies cannot go through each other and only an elastic bounce can be taken into consideration. This fact actually allows the real analytic regularization of binary collisions.

In our case the possibility of having a unique or two prolongations is due to the type of central configuration at which the particles tend asymtotically, as described in the statement. Observe that the rectilinear case is included in the class of solutions with a unique extension.

Finally, we would like to remark that the above C^1-regularization can be extended to the most general case of multiple collisions. A very recent paper [E] solved the infinite spin problem by showing that the orbits in any type of multiple collision tend to the collision point without to follow the SO(3)-group action. Consequently, transformations similar to (7.1) - (7.2) can be used for each cluster of colliding bodies in order to obtain the desired result. However the study of the dynamics of bounded regularized solutions might be more complicated than in the total collapse case.

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