

ZEROS OF BOULIGAND-NAGUMO FIELDS, FLOW-INVARIANCE AND THE
BOUWER FIXED POINT THEOREM

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Abstract. Mainly, in this paper we prove that if D is a convex compact of \mathbb{R}^n , then the Brouwer fixed point property of D is equivalent to the fact that every Bouligand-Nagumo vector field on D , has a zero in D . Using a version of this result on a normed space, as well as the Day [9] and Dugundji [10] theorems, we give a new proof to the fact that in every infinite dimensional Banach space X , there exists a continuous function from the closed unit ball B (of X) into B , without fixed points in B . We also show that our results include several classical results. Some applications to Flight Mechanics are given, too.

§1. Introduction. Throughout this paper X denotes a normed space (of norm $\|\cdot\|$) and D a nonempty closed subset of X , unless otherwise specified. $\overset{\circ}{D}$ and ∂D denote the interior and the boundary of D , respectively. Finally, B and $S=\partial B$ denote the closed unit ball and the unit sphere of X (respectively), unless otherwise specified. In Section 2, we are introducing and characterizing the tangent cones to D , as subsets of X . The relationship of our approach with the standard way (of introducing the tangent cones to D as a subset of X^*) is discussed in terms of the duality mapping J of X . We define the notions "A vector field f on D which is nowhere normal to D (actually to ∂D)" and " f is nowhere (never) radial to S ".

Section 3 is devoted to the introduction and characterization of Bouligand cones, Clarke cones and Bouligand-Nagumo (in short (B-N)) vector fields on D . It is known (see [6, p.53]) that if D is convex then f is a (B-N) field on D iff f is a Clarke field on D . We give a simple proof of the fact that the same result holds without convexity assumption on D , provided D is compact (Appendix). By the end of this section (Theorem 3.1) we present a first extension of a theorem of Miranda (see [24, p.214]). However, the main results of this paper are concentrated in Sections 4 and 5. Section 4 deals with the equivalence of the following three fundamental conditions on D : 1) The (Brouwer) fixed-point

property of D (in short:(f.p.)), 2) The normal property of D (n.p.) and 3) The tangential property of D (t.p.) - see Theorem 4.1 and 4.2. The famous fixed point theorem of Brouwer (1912)- perhaps the most famous of all fixed-point theorems- says that if X is finite dimensional, then B has

the fixed-point property. It is well-known that in some infinite dimensional Banach spaces X (e.g. c_0) C^1 , separable Hilbert spaces, function spaces), B has no the fixed-point property. For instance, in c_0 , the function $T:B \rightarrow B$ given by

$Tx = (1-||x||, x_1, x_2, \dots, x_n, \dots)$ for $x=(x_1, \dots, x_n, \dots) \in c_0$, is continuous and has no fixed points in B . We now give a new proof (via Theorem 4.3) to the fact that for no infinite dimensional Banach space does B have the fixed-point property. The proof of Theorem 4.3 uses Theorem 4.2', the theorem of Day [9] (on the existence of an infinite dimensional closed subspace L of X , L -with a Schander basis), the theorem of Dugundji [10] and the continuous radial selection P of the (possible multivalued) projection operator P_B on B (see (2.15)). To my best knowledge, the proof presented here to Theorem 4.3 is new. In particular, it implies that: "it is not possible to define a satisfactorily concept of degree for the class of all continuous functions from B into X , in no infinite dimensional Banach space X ". This is because such a concept of degree would imply the fixed point property of B (which is in conflict with Theorem 4.3).

For $X=C([0,1])$, the above remark on the degree was emphasized by Leray (1936). We refer to Lloyd [12, p. 53] for details. Theorem 4.3 is not new (a first proof can be found in Dugundji [10, p. 362]).

A. Verjowski (in a private communication) has sketched a proof to Theorem 4.3 by using a sequence in B without any Cauchy subsequence. Independently, the author has followed a similar way by considering such a sequence on S . Another proof of Theorem 4.3 can be derived from the fact that (if $\dim X=\infty$) S is a retract of B . Conversely, from Theorem 4.3 it follows that if $\dim X=\infty$, then S is a retract of B . Let us point out a first open problem: Suppose that X is an infinite dimensional Banach space. Set

$$\tilde{M}=\{\tilde{T}; \tilde{T}:B \rightarrow B, \tilde{T} \text{ continuous}, \tilde{T}x \neq x, \forall x \in B\}$$

$$M=\{T; T:B \rightarrow B, T \text{ continuous}\}.$$

According to Theorem 4.3, \tilde{M} is nonempty so \tilde{M} is a proper subset of M and M is a subset of $C(X)$ (the set of all continuous bounded functions from X into X , endowed with supremum norm).

(1) Is \tilde{M} dense in M ?

By Corollary 4.1, a convex compact D of X which is invariant with respect to a vector field f on D , contains a zero of f .

A second open problem (see P(4.1) after Remark 4.4)

(2) In what conditions does a (B-N) field on $D \subset \mathbb{R}^n$ have a zero on ∂D ? (This is the case if $n=3$ and $B=D$ [11, p.149])

A third open problem

(3) Does the conclusion of (ITF) (i.e. (5.5)) remain valid if Y is an infinite dimensional Banach space?

A fourth open problem

(4) Does the conclusion of Corollary 5.2 remain valid if D_g^- is a Jordan domain [29, p.102] in \mathbb{R}^n , $n > 2$?

Finally, a fifth open problem is that raised in Remark 5.4.

In Section 5 we prove that in \mathbb{R}^2 the conclusion of Corollary 4.1 remains valid without the convexity assumption on D (provided ∂D is a Jordan curve). We also prove that the index of ∂D with respect to f is one, provided that f is a (B-N) field on D without zeros on ∂D (Theorem 5.2). This is done by using a characterization of K_D via an inverse Taylor formula (see Theorem 5.1) in Banach spaces, given in [23] and then included in [20]). This in conjunction with an optimum principle (Theorem 5.4, which includes Euler's equation from the Calculus of Variations, and Lagrange multipliers [17]) allows us to prove Theorem 5.5 on the uniform motion on a sphere in a real Hilbert space.

§ 2. Normal cones.

The distance $d(v; D)$ from the element v of X to D is defined as usual

$$d(v; D) = \inf \{ \|v-z\|; z \in D \} \quad (2.1)$$

Let us introduce the multivalued mapping $x \rightarrow N_D(x)$ from D into 2^X by

$$N_D(x) = \{ u \in X; d(x+u; D) = \|u\| \}, \quad x \in D \quad (2.2)$$

$N_D(x)$ will be called - the normal cone to D at x or the cone of normals to D at x .

In order to characterize $N_D(x)$ in terms of duality mapping J of X , we recall several basic notations and results

$$J(x) = \{ x^* \in X^*, x^*(x) = \|x\|^2 = \|x^*\|^2 \}, \quad x \in X \quad (2.3)$$

$$\langle y, x \rangle_s = \lim_{h \downarrow 0} \frac{\|x+hy\|^2 - \|x\|^2}{2h}; \quad \langle y, x \rangle_+ = \lim_{h \downarrow 0} \frac{\|x+hy\| - \|x\|}{h} \quad (2.4)$$

$\langle y, x \rangle_i$ and $\langle y, x \rangle$ are defined as in (2.4) with “ \lim ” respectively $\lim_{h \uparrow 0}$

Clearly

$$\langle y, x \rangle_s = \|x\| \langle y, x \rangle_+ ; \langle y, x \rangle_i = \|x\| \langle y, x \rangle_- , \forall x, y \in X \quad (2.5)$$

If X is real, the following well-known result holds (see e.g.)

$$\begin{aligned} \langle y, x \rangle_s &= \sup \{x^*(y); x^* \in J(x)\} = x_s^*(y) \\ \langle y, x \rangle_i &= \inf \{x^*(y); x^* \in J(x)\} = x_i^*(y), \forall x, y \in X \end{aligned} \quad (2.6)$$

for some x_s^* and x_i^* in $J(x)$.

If X^* is strictly convex (see e.g. [20, p.1]) then $J(x)$ consists of a single element and therefore, in this case

$$\langle y, x \rangle_- = \langle y, x \rangle_+ = \lim_{h \rightarrow 0} \frac{\|x+hy\| - \|x\|}{h} = \|x\|^{-1}(J(x))(y), \quad x \neq 0, y \in X \quad (2.7)$$

Let us introduce the following two subsets of X

$$N_D^1(x) = \{u \in X; \langle z-x, u \rangle_- \leq 0, \forall z \in D\} \quad (2.8)$$

$$N_D^2(x) = \{u \in X; \exists x^* \in J(u) \text{ such that } x^*(z-x) \leq 0, \forall z \in D\}$$

for $x \in D$, where

$$\langle z-x, u \rangle_- = \lim_{h \uparrow 0} h^{-1}(\|u+h(z-x)\| - \|u\|) \quad (2.9)$$

Clearly, if $x \in \dot{D}$, then $N_D^p(x) = \{0\}$, $p = 1, 2$. Of course, $N_D^1(x) = N_D^2(x)$. In the theory of zeros of some vector fields on D (next sections) the following result is needed.

Lemma 2.1 Let D be a convex subset of X . Then for every $x \in D$,

$$N_D(x) = N_D^1(x) = N_D^2(x). \quad (2.10)$$

Proof. The only fact we have to prove is $N_D(x) = N_D^1(x)$. The inclusion “ $N_D^1(x) \subset N_D(x)$ ” holds without convexity assumption on D . Indeed, if $u \in N_D^1(x)$, then $h^{-1}(\|u+h(z-x)\| - \|u\|) \leq 0, \forall h < 0$ which yields (for $h = -1$) $\|u\| \leq \|x+u-z\|, \forall z \in D$, i.e. $\|u\| \leq d(x+u; D)$. As the inverse inequality is obvious, it follows $u \in N_D(x)$. We now prove “ $N_D(x) \subset N_D^1(x)$ ”. Take $u \in N_D(x)$. Then $\|u\| \leq \|x+u-v\|, \forall v \in D$. For $z \in D$, we have $v = v_h = (1-h)x + hz \in D, \forall h \in [0, 1]$. Substituting this

$v=v_h$ in the previous inequality, rearranging and then letting $h \downarrow 0$ we conclude that $u \in N_D^1(x)$. The proof is complete.

Definition 2.1. A vector field f on D is simply a continuous function from D into X . We say that F is nowhere normal to D if for every $x \in D$, " $f(x) \in N_D(x)$ if and only if $f(x)=0$ " (i.e. if $f(x) \neq 0$, then $f(x) \notin N_D(x)$). A point x is a zero of f if $f(x)=0$.

The standard way [6, p.52] is to introduce the normal cone to D at x as a subset of X^* , namely

$$N_D^*(x) = \{x^* \in X^*, x^*(z-x) \leq 0, \forall z \in D\} \quad (2.11)$$

In view of Definition 2.1 it is clear that for the goal of this paper, we need the normal cone $N_D(x)$ as a subset of X (and not of X^*). The relationship between $N_D(x)$ and $N_D^*(x)$ is given by

Proposition 2.1. Let D be a convex subset of X with X^* strictly convex. Then

$$N_D^*(x) = J(N_D(x)), \forall x \in D \quad (2.12)$$

Proof. On the basis of (2.10), $J(N_D(x)) \subset N_D^*(x)$. Conversely, take $x^* \in N_D^*(x)$. As the duality mapping J of X is surjective (onto), there is $u \in X$, such that $x^* = J(u)$. This means just $u \in N_D(x)$ so (2.12) follows. Note that the surjectivity of J is a consequence of the fact that J is the subdifferential ∂f of f with $f(x) = \frac{1}{2} \|x\|^2$ (so J is maximal monotone) and of the fact that J is coercive (see e.g. [21, pp. 238-242]). This completes the proof.

We associate with each $v \in X$

$$P_D(v) = \{w \in D; d(v; D) = \|v - w\|\} \quad (2.13)$$

(the set of all projections of v on D , if any). Suppose that $P_D(v)$ is nonempty (this is the case in either of the following situations: 1. $D=B$; 2. D is compact; 3. D is a closed convex set of a reflexive space X - see e.g. [20, p.5], [21, p.256]). If in addition D is convex, then $P_D(v)$ is convex and $P_D(v) \subset S(v, r)$ - the sphere of radius r about v , with $r = d(v; D)$. Thus, if X is strictly convex and $P_D(v)$ -nonempty, then it is single-valued. By "a selection" of the (possible) multivalued mapping P_D we mean a function $P: X \rightarrow D$ with the property: $P(v) \in P_D(v), \forall v \in X$, hence

$$d(v; D) = \|v - P(v)\|, \forall v \in X \quad (2.14)$$

In the case $D = B$ we can easily check that the function $P: X \rightarrow B$ defined by

$$P(v) = v, \text{ if } \|v\| \leq 1 \text{ and } P(v) = v/\|v\|, \text{ if } \|v\| > 1 \quad (2.15)$$

is a continuous selection of P_B . The fact that $P(v)$ as defined by (2.15) is a projection of v on B follows from

$$\|v - \frac{v}{\|v\|}\| \leq \|v - z\|, \forall z \in B \text{ (for } \|v\| \geq 1) \quad (2.16)$$

which is a consequence of the triangle inequality.

For each $x \in B$ set

$$\tilde{N}_B(x) = \{u \in X, P(x+u) = x\} \quad (2.17)$$

with P as in (2.15).

As P satisfies (2.14) with $D=B$ (according to (2.16)) it follows $\tilde{N}_B(x) \in N_B(x)$ (where $N_B(x)$ is given by (2.2) with $D=B$). If X is not strictly convex, then $N_B(x)$ is strictly larger than $\tilde{N}_B(x)$. We will see this fact, below (Remark 2.1).

Define

$$M(x) = \{0\}, \text{ if } \|x\| < 1 \text{ and } M(x) = \{\lambda x, \lambda \geq 0\}, \text{ if } \|x\| \geq 1 \quad (2.18)$$

Lemma 2.2. In every normed space X we have

$$M(x) = \tilde{N}_B(x), \forall x \in B$$

Proof. Take $u \in \tilde{N}_B(x)$. If $\|x\| < 1$, then $\|x+u\| < 1$ (otherwise we are led to the absurdity: $1 > \|x\| = \|P(x+u)\| = 1$) so $P(x+u) = x+u = x$, i.e. $u=0 \in M(x)$. If $\|x\| = 1$ and $u \neq 0$, then $\|x+u\| > 1$ (otherwise $P(x+u) = x+u \neq x$) so $P(x+u) = \|x+u\|^{-1}(x+u) = x$, which yields $u = \lambda x$ with $\lambda = \|x+u\| - 1 > 0$, i.e. $u \in M(x)$. Therefore $\tilde{N}_B(x) \subset M(x)$, $\forall x \in B$. The inverse inclusion is obvious so the proof is complete.

Remark 2.1. In the case $X = \mathbb{R}^2$ with the supremum norm i.e. $\|x\| = \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2)$ one can prove that

$$N_B(1, x_2) = \{u = (u_1, u_2); u_1 \geq |u_2|\} \text{ for } |x_2| < 1$$

and

(2.19)

$$N_B(1,1) = \{u = (u_1, u_2); u_1 + u_2 \geq 0\}$$

Definition 2.2 A vector field F on B is nowhere (never) radial to S if for every $x \in S$, $f(x) \neq \lambda x$ for all $\lambda > 0$ (i.e. $f(x)$ does not lie in the continuation of x). $\tilde{N}_B(x)$ is said to be the “radial subcone” of $N_B(x)$, $x \in B$. Of course, the property of f to be nowhere radial to S is equivalent to the following two conditions “ $f(x) \neq 0$ implies $f(x) \notin \tilde{N}_B(x)$, $x \in B$ ” or “ $f(x) \in \tilde{N}_B(x)$ iff $f(x) = 0$ ”. Finally, the result below will be used in Section 4.

Proposition 2.2.

I. Suppose that X is strictly convex and one of the conditions below is fulfilled:

1) $D = B$; 2) D is a convex compact of X ; 3) D is a closed convex subset of X and X is uniformly convex [20, p.1].

Then the projection operator $P_D: X \rightarrow D$ defined by (2.13) is (single valued and) continuous and for all $x \in D$,

$$\begin{aligned} N_D(x) &= \{u \in X; d(x+u; D) = \|u\|\} = \{u \in X; P_D(x+u) = x\} = N_D^1(x) \\ &= N_D^2(x) \text{ (see (2.8)). } N_B(x) = \tilde{N}_B(x), \forall x \in B \end{aligned} \quad (2.20)$$

II. If in addition to the above hypotheses, X^* is strictly convex, then we also have

$$\begin{aligned} N_D(x) &= \{u \in X; (J(u))(z-x) \leq 0, \forall z \in D\} = \\ &= \{u \in X; \langle z-x, u \rangle \leq 0, \forall z \in D\}, \forall x \in D \end{aligned} \quad (2.21)$$

where $\langle z-x, u \rangle = \lim_{h \rightarrow 0} h^{-1}(\|u+h(z-x)\| - \|u\|)$

III. If H is a Hilbert space of inner product $\langle \cdot, \cdot \rangle$ and D a closed convex subset of H , then

$$N_D(x) = \{u \in H; P_D(x+u) = x\} = \{u \in H; \operatorname{Re} \langle u, z-x \rangle \leq 0, \forall z \in D\} \quad (2.22)$$

for all $x \in D$.

Proof. One combines the previous results and remarks of this section. In connection with (2.20), one takes into account that if $u \in N_D(x)$ then we have

$\|u\| = d(x+u; D) = \|x+u-x\| \leq \|x+u-z\|, \forall z \in D$ so $x \in P_D(x+u)$, i.e. $x = P_D(x+u)$. We also refer to [20, p.4] and [21, p.256].

§ 3. Bouligand-Nagumo vector fields. Nowhere normal fields.

Let us recall the definition of the Bouligand [3] contingent cone $K_D(x)$ to D at $x \in D$ (see also [6, p.55]).

A vector $y \in X$ is said to be "tangent" to D at $x \in D$ if

$$\lim_{h \downarrow 0} h^{-1}d(x+hy; D) = 0 \quad (3.1)$$

Then

$$K_D(x) = \{y \in X; y \text{ satisfies (3.1)}\} \quad (3.2)$$

If D is convex, then $K_D(x)$ is also convex. If $x \in \overset{\circ}{D}$, then $K_D(x) = X$ (i.e. in this case (3.1) is trivially satisfied). Therefore, we are actually interested only in $x \in \partial D$. Recall that if D is a C^k -submanifold of X ($k \geq 1$) then the tangency of y to D at $x \in D$ in the sense of (3.1) coincides with the classical tangency $y \in T_x(D)$ (the tangent space to D at x , see [16]). It is readily seen that (3.1) is equivalent to:

For each $h > 0$, there is $p(h) = p(h, x, y) \in X$ such that:

$$p(h) \rightarrow 0 \text{ as } h \downarrow 0 \text{ and } x + hy + hp(h) \in D, \forall h > 0 \quad (3.3)$$

(the simplest proof is given in [16]).

The Clarke's tangent cone to D at $x \in D$ (call it $C_D(x)$) is defined by

$$C_D(x) = \{y \in X; \forall h_n \downarrow 0, \forall x_n \rightarrow x (x_n \in D), \exists y_n \rightarrow y \text{ such that } x_n + h_n y_n \in D, n = 1, 2, \dots\} \quad (3.4)$$

(see Clarke [6, p.53]).

Clearly $C_D(x) \subset K_D(x)$, $\forall x \in D$. Moreover, if D is convex, then $C_D(x) = K_D(x)$, $\forall x \in D$ (cf. [6, p. 55]). A vector field f on D is said to be tangent to D if for every $x \in D$, $f(x) \in K_D(x)$, i.e.

$$\lim_{h \downarrow 0} h^{-1}d(x+hf(x); D) = 0, \forall x \in D \quad (3.5)$$

In view of (3.3), (3.5) is equivalent to:

For each $x \in D$ and $h > 0$, there exists $p(h) = p(h, x) \in X$ such that

$$p(h, x) \rightarrow 0 \text{ as } h \downarrow 0 \text{ and } x + h(f(x) + p(h)) \in D, \forall h > 0 \quad (3.5)'$$

Note that the tangency of f to D in the sense of (3.5) has been first used by Nagumo [18] in the theory of flow-invariance of D with respect to the differential equation

$$u'(t)=f(u(t)), u(0)=x \in D, u(t) \in D, t \geq 0 \quad (3.6)$$

This is the reason for which we will call a vector field f satisfying (3.5) - a Bouligand-Nagumo field on D (in short - a (B-N) field on D). Namely the following important result holds.

Theorem 3.0. Suppose that D is a closed subset of the Banach space X , f is a vector field on D and one of the conditions below are fulfilled: (1) X is finite dimensional; (2) X is arbitrary and D is compact; (3) X is arbitrary and $f(D)$ is compact; (4) f is locally Lipschitz on D and X is arbitrary; (5) X is arbitrary and f is dissipative. Then (3.5) is a necessary and sufficient condition in order for D to be flow-invariant with respect to $u'=f(u)$ (or equivalently, D is flow-invariant with respect to $u'=f(u)$ iff f is a (B-N) field on D).

The proof of Theorem 3.1 can be found in [20, Ch. 2]. Note that its conclusion under (1)–(2)–(3), (4) and (5) was (independently) obtained by Nagumo [18]–Crandall [8]–Yorke [28], Brezis [4] and Martin [14] respectively. Theorem (3.1) was extended by the author in [19] and [20]. Clearly, the condition “ f is a (B-N) field on D ” is independent of the norm $\|\cdot\|$ on X (i.e. it holds in any other equivalent norm $\|\cdot\|$ on X). f is said to be a Clarke vector field on D (in short - a (C) field on D) if $f(x) \in C_D(x), \forall x \in D$. Of course if f is (C) on D then f is (B-N) on D . If D is convex, then f is (C) on D iff f is (B-N) on D . Therefore the following result is significant (for it holds without convexity assumption on D).

Proposition 3.1. Let D be a compact subset of the Banach space X . Then f is a (B-N) field on D iff f is a (C) field on D (the proof is given in the Appendix).

Proposition 3.2.

I. In every (real) normed space X the following conditions are equivalent:

- (1) $\lim_{h \downarrow 0} h^{-1}d(x+hy; B)=0, \|x\|=1, y \in X$
- (2) $\|x\|=1, \langle y, x \rangle_+ \leq 0; \quad (3) \|x\|=1, x^*(y) \leq 0, \forall x^* \in J(x)$

- II. $\lim_{h \downarrow 0} h^{-1}d(x+hy; S)=0$, iff $\langle y, x \rangle_+ = 0$ ($\|x\|=1$). (In this proposition, “ \lim ” can be replaced by “ \liminf ”).

Proof. In view of (2.5) and (2.6) we see that “(2) \Leftrightarrow (3)”. Now, assume that (1) holds. Then according to (3.3), we have $\|x+hy+hp(h)\| \leq \|x\|$ with some $p(h) \rightarrow 0$ as $h \downarrow 0$. It follows:

$\lim_{h \downarrow 0} h^{-1}(\|x+hy\| - \|x\|) = \lim_{h \downarrow 0} h^{-1}(\|x+hy+hp(h)\| - \|x\|) \leq 0$
 (so (1) \Leftrightarrow (2)). We now prove that (2) \Rightarrow (1). If $\langle y, x \rangle_+ < 0$ then $\|x+hy\| < \|x\|$ for all $h > 0$ sufficiently small (i.e. $x+hy \in B$) so (1) is trivially fulfilled. If $\langle y, x \rangle_+ = 0$, then $\|x+hy\| \geq \|x\|$, $\forall h > 0$, so by (2.14)–(2.16) we have

$$\lim_{h \downarrow 0} h^{-1}d(x+hy; B) = \lim_{h \downarrow 0} h^{-1} \left\| x+hy - \frac{x+hy}{\|x+hy\|} \right\| = \lim_{h \downarrow 0} h^{-1}(\|x+hy\| - \|x\|) = \langle y, x \rangle_+ = 0 \quad (3.7)$$

The proof of II proceeds similarly. Part II is actually known ([20, p.118], [21, p.59]). Proposition 3.2 yields

Corollary 3.1. Let X be a real normed space and let $f: B \rightarrow X$ be a field on B . Then

- I. f is a (B-N) field on B iff

$$\langle f(x), x \rangle_+ \leq 0, \quad \forall x \in S \quad (3.8)$$

or equivalently

$$x^*(f(x)) \leq 0, \quad \forall x \in S, \forall x^* \in J(x) \quad (3.8)'$$

- II. f is a (B-N) field on S iff $\langle f(x), x \rangle_+ = 0, \forall x \in S$ or equivalently,

$$x^*(f(x)) = 0, \quad \forall x \in S, \forall x^* \in J(x)$$

- III. If H is a Hilbert space (of inner product $\langle \cdot, \cdot \rangle$), then f is a (B-N) field on B (or S) iff

$$\operatorname{Re} \langle f(x), x \rangle \leq 0, \quad \forall x \in S \quad (\operatorname{Re} \langle f(x), x \rangle = 0, \quad \forall x \in S) \text{ respectively.}$$

Remark 3.1. 1) A (B-N) field f on B is nowhere radial to S . Indeed, if $\|x\|=1$, and $f(x) \neq 0$, then (3.8)' implies $f(x) \neq \lambda x, \forall \lambda > 0$ (otherwise, i.e. $f(x) = \lambda x$ for some $\lambda > 0$, we are led to $x^*(\lambda x) = \lambda \|x\|^2 = \lambda > 0$ which contradicts (3.8)'). 2) In the hypotheses of Proposition 2.2, a (B-N) field f on D is nowhere normal to D . Indeed, if $x \in D$ and $f(x) \neq 0$, then $f(x) \notin N_D(x)$ (otherwise, i.e. if $f(x) \in N_D(x)$ then (2.8) and (3.5)' would imply $x^*(f(x)) \leq 0$, for some $x^* \in J(f(x))$ i.e. $\|f(x)\|^2 \leq 0$).

Example 3.1. If f is a linear field from X into X and if it satisfies (3.8), then $\langle f(x), x \rangle \leq 0$ for all $x \in X$ (i.e. f is dissipative [21, p. 246]). Therefore, the only linear (B-N) fields on B are the dissipative linear bounded operators from X into X . If $X = \mathbb{R}^n$, then $f_A(x) = Ax$ ($A = (a_{ij})$ - a $n \times n$ matrix) is a (B-N) field on B iff A is negative semidefinite. f_A is a (B-N) field on S iff $a_{ii} = 0$, $a_{ij} = -a_{ji}$ ($i \neq j$) $i, j = 1, \dots, n$. Thus, the only linear (B-N) fields on S in \mathbb{R}^2 are given by $f(x) = (-x_2, x_1)$ and $f(x) = (x_2, -x_1)$ for $x = (x_1, x_2)$ (the velocity fields of rotation of S counterclockwise - clockwise, respectively). The geometric meaning of a (B-N) field f on D is: for every $x \in \partial D$, $f(x)$ points "in the interior of D " or in the direction of "the tangent to ∂D " at $x \in D$.

Example 3.2. Let us characterize the (B-N) tangency of $f(x) = (f_1(x), \dots, f_n(x))$ to the parallelepiped $D = D_n$ of \mathbb{R}^n given by

$$D_n = \{x = (x_i) \in \mathbb{R}^n; a_i \leq x_i \leq b_i, i = 1, \dots, n\} \text{ with } -\infty < a_i < b_i < \infty \tag{3.9}$$

Proposition 3.3. The following three conditions are equivalent:

I. f is a (B-N) field on D_n .

$$\text{II. } \begin{cases} \text{a) } f_i(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n) \geq 0 \\ \text{b) } f_i(x_1, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_n) \leq 0 \end{cases} \tag{3.10}$$

for all $i = 1, \dots, n$ and $x = (x_i) \in \partial D_n$ (with the obvious convention: $x_0 \equiv x_1$ and $x_{n+1} \equiv x_n$).

III. For all $h \in [0, h_0]$ and $x \in \partial D_n$ we have $x + hf(x) \in D_n$, where

$$h_0 = \min \left\{ (b_i - a_i) / M, i = 1, \dots, n \right\} \text{ and } M = \sup \left\{ \|f(x)\|; x \in \partial D_n \right\} \tag{3.10}'$$

Proof. "(I) \Rightarrow (II)". Indeed (I) means (3.5)' with $p(h, x) = (p_i(h, x)) \rightarrow 0$ as $h \downarrow 0$. Therefore (3.5)' is equivalent to

$$a_i \leq x_i + hf_i(x_1, \dots, x_n) + hp_i(h, x) \leq b_i \tag{3.11}$$

for all $h > 0$, $i = 1, \dots, n$ and $x \in D_n$.

Substituting $x_i = a_i$ ($x_i = b_i$), dividing by $h > 0$ and letting $h \downarrow 0$ we get II a) and II b), respectively. "(II) \Rightarrow (III)". Let $x \in \partial D_n$, e.g. $x = (x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n) \equiv x^{a_i}$.

Clearly, II a) implies $a_i \leq a_i + hf_i(x^{a_i}), \forall h > 0$. On the other hand, as $a_i < b_i$, we have $a_i \leq a_i + hf_i(x^{a_i}) \leq a_i + hM < b_i$, for all $0 < h < (b_i - a_i) / M$.

Similarly

$$a_i < b_i + hf_i(x_1, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_n) \leq b_i, \quad 0 < h < (b_i - a_i)/M.$$

for all $i=1, \dots, n$ so (II) \Rightarrow (III). Obviously, (III) \Rightarrow (I) with $p(h, x)=0$ (see (3.5)') so the proof is complete.

Recall that D_n has the (Brouwer) fixed-point property (i.e. every continuous function $T: D_n \rightarrow D_n$ has a fixed point in D_n). We are now in a position to give a simple proof to

Theorem 3.1. The following two conditions are equivalent

- (I) D_n has the fixed-point property
- (II) Every continuous function $f: D_n \rightarrow \mathbb{R}^n$ satisfying the boundary conditions (3.10) has a zero in D_n .

Proof. (I) \Rightarrow (II). Let $P_{D_n} \equiv P$ be the projection operator on D_n (see (2.13)). Set $Tx = P(x + h_0 f(x))$ with h_0 given by (3.10)'. Clearly, $T: D_n \rightarrow D_n$ and it is continuous (see Prop. 2.2). By (I), there is $\bar{x} \in D_n$ such that $\bar{x} = T\bar{x} = P(\bar{x} + h_0 f(\bar{x}))$ (*). If $\bar{x} \in \partial D_n$, then by Proposition (3.3) III, $\bar{x} + h_0 f(\bar{x}) \in D_n$ so (*) yields $\bar{x} = \bar{x} + h_0 f(\bar{x})$, i.e. $f(\bar{x}) = 0$. If $\bar{x} \in \dot{D}_n$ then (*) implies $\bar{x} + h_0 f(\bar{x}) \in \dot{D}_n$ too, so $f(\bar{x}) = 0$. Conversely "(II) \rightarrow (I)".

Indeed, take a continuous $T: D_n \rightarrow D_n$ and set $f = T - I$ (I - the identity on \mathbb{R}^n). Then $x + hf(x) = (1-h)x + hTx \in D_n$, $\forall x \in D_n$ and $\forall h \in [0,1]$ i.e. f is a (B-N) field on D_n so f satisfies (3.10). By (II), f has a zero \bar{x} in D_n i.e. $T\bar{x} = \bar{x}$. The proof is complete.

Remark 3.2. In the case $a_i = -L$ and $b_i = L$, $i=1, \dots, n$ (with $L > 0$), D_n is a hyper-cube of \mathbb{R}^n and Th. 3.1 is just a well-known result of Miranda (see [24, p. 214]). It will be also discussed and extended in the next section.

§ 4. Zeroes of (B-N) fields.

This section is mainly devoted to the equivalence of the following three conditions below, as well as to the fact that the closed unit ball B of a Banach space X has the Brouwer fixed point property, if and only if X is finite dimensional. In particular we are proving that a convex compact D (of \mathbb{R}^n) which is a flow-invariant set with respect to the vector field f on D (i.e. with respect to $u' = f(u)$) contains a zero of f .

C(4.1) D has the fixed-point property (f.p)

C(4.2) D has the normal property (n.p)

C(4.3) D has the tangential property (f.p.)

We say that D has the fixed-point property if every continuous function $T:D \rightarrow D$ has a fixed point x in D . D has the normal property if every field f on D which is nowhere normal to D (see Def. 2.1) has a zero in D . Finally, D has the tangential property if every (B-N) field f on D has a zero in D . It is well-known that the (f.p) of D is invariant (preserved) under homeomorphisms. However, very simple examples in \mathbb{R}^2 show that the (t.p.) of D may not be preserved by homeomorphisms. For instance, take $D = \overline{AB} \cup Bx_2$ and $D_1 = Ox_2$ where $A=(1,0)$, $B=(0,1)$, \overline{AB} is the line segment with the end points A and B (in a Cartesian system Ox_1x_2 in \mathbb{R}^2), Ox_2 is the positive x_2 -semi-axis and $Bx_2 = \{x \in Ox_2; x=(0,y), 1 \leq y\}$. Obviously D and D_1 are homeomorphic and D has the (t.p.) (as $B=(0,1)$ is a zero of every (B-N) field f on D). However, D_1 has no the (t.p.) (e.g. the constant field f given by $f(x)=(0,1), \forall x \in D_1$ is a (B-N) field on D_1 without any zero).

Remark 4.1. It follows from the above comments that the equivalence of C(4.1), C(4.2) and C(4.3) for a D may not imply their equivalence for a D_1 homeomorphic with D .

Theorem 4.1. Suppose that either: 1) X is strictly convex and D is a convex compact of X (a normed space) or; 2) X is uniformly convex and D is a closed convex subset of X . Then conditions C(4.1), C(4.2) and C(4.3) are equivalent.

Proof: "C(4.1) \Rightarrow C(4.2)". Let $f:D \rightarrow X$ be such that $f(x) \neq 0$ implies $f(x) \notin N_D(x)$. Set $Tx = P_D(x+f(x))$. In the hypotheses of the theorem, $P_D:X \rightarrow D$ is a continuous function (Prop.2.2). Clearly $T:D \rightarrow D$ so by C(4.1) there exists $\bar{x} \in D$ such that $P_D(\bar{x}+f(\bar{x})) = \bar{x}$ (i.e. $f(\bar{x}) \in N_D(\bar{x})$). This implies $f(\bar{x})=0$ (Prop. 2.2). "C(4.2) \Rightarrow C(4.3)". Let $f:D \rightarrow X$ be a (B-N) field on D . Then f is nowhere normal to D (see Remark 3.1) so it has a zero in D . "C(4.3) \Rightarrow C(4.1)". Let $T:D \rightarrow D$ be continuous. Set $f=T-I$. Then $x+hf(x)=(1-h)x+hTx \in D, \forall h \in [0,1]$, so f is a (B-N) field on D . By C(4.3), f has a zero in D and the proof is complete.

Remark 4.2. Theorem 3.1 follows from Theorem 4.1. This is because for $X=\mathbb{R}^n$ and $D=D_n$, D has the fixed-point property. Therefore, Theorem 3.1 can be completed with the fact that each of (I) and (II) are true and equivalent to the normal property of D_n . In the case $D=B$ (the closed unit ball of X) the conclusion of Theorem 4.1 holds without any additional assumption on X . Moreover, C(4.2) can be relaxed to:

C(4.2)' B has the radial property (i.e. every field f on B which is nowhere radial to S -see Def.

2.2) has a zero in B

Hence, we have

Theorem 4.2. In every normed space X , Conditions $C(4.1)$ (with $D=B$), $C(4.2)'$ and $C(4.3)$ (with $D=B$) are equivalent.

Obviously, theorem 4.2 can be restated as:

Theorem 4.2'. In every normed space, the following three conditions are equivalent:

Non C(4.1): There exists a continuous function $T:B \rightarrow B$ without fixed points in B .

Non C(4.2)': There exists a vector field f on B , that is nowhere radial to S , without zeros in B .

Non C(4.3): There exists a (B-N) field f on B without zeros in B .

Proof. ($\text{Non C(4.1)} \Rightarrow \text{Non C(4.2)'} \Rightarrow \text{Non C(4.3)}$). Indeed, let $T:B \rightarrow B$ be continuous, with $Tx \neq x$ for all $x \in B$. Then $f = T-I$ is nowhere radial to S and has no zeros in B . ($\text{Non C(4.2)'} \Rightarrow \text{Non C(4.3)}$). Indeed, let f be nowhere radial to S without zeros in B . Set $Tx = P(x+f(x))$ with P defined by (2.15). Then $T: B \rightarrow B$ is continuous without fixed points (as $f(x) \notin \tilde{N}_B(x)$ - see (2.17)). Therefore, F given by $F(x) = P(x+f(x)) - x$ is a (B-N) field on B (see the proof of Theorem 4.1) without zeros in B . Finally, $\text{Non C(4.3)} \Rightarrow \text{Non C(4.1)}$. Indeed, let f be a (B-N) field on B without zeros in B . Then f is nowhere radial to S (see Remark 3.1) so T given by $Tx = P(x+f(x))$ has no zeros in B , $T: B \rightarrow B$ and T is continuous on B . The proof is complete.

The next theorem shows that Non C(4.1) holds iff S is infinite dimensional.

Theorem 4.3 If X is an infinite dimensional Banach space, then there exists a continuous function $T:B \rightarrow B$ without fixed points in B .

Proof: According to the Day's theorem [9] there exist biorthogonal sequences $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that $\{e_n\}$ is a Schauder basis for the closed subspace $L \subset X$ spanned by the set of all $\{e_n\}$ and $\|e_n\| = \|e_n^*\| = 1$, $e_n^*(e_m) = 0$ for all $n \neq m$, $n, m = 1, 2, \dots$. It follows $|x_i| = |e_i^*(x)| \leq \|x\|$ for all i and for all $x \in L$, where x_i is the i^{th} coordinate of x in the basis $\{e_n\}$ (i.e. $x = (x_i)$). Now let us introduce the sequence $v_n(x) = (1 - \|x\|)e_1 + 2^{-1}x_1e_2 + \dots + 2^{-n}x_n e_{n+1}$. It is easy to check that $\|v_{n+p}(x) - v_n(x)\| \leq 2^{-n}\|x\|$ and $\|v_n(x) - v_n(x_0)\| \leq 2\|x - x_0\|$ for all $x, x_0 \in L$ and $n, p = 1, 2, \dots$. Set $f(x) = \lim_{n \rightarrow \infty} v_n(x)$, $x \in L$. This means that f is continuous on L and the coordinates of $f(x)$ in $\{e_n\}$ are: $(1 - \|x\|, 2^{-1}x_1, \dots)$ for $x = (x_1, x_2, \dots)$. It is clear that $f(x) \neq 0$ for all $x \in L$ and that the unit ball B_L of L is just $B \cap L(f: B_L \rightarrow B_L)$, but unfortunately it may have fixed points. This is the case if e.g. $X = L = c_0$. It is easy to check that f is nowhere normal to the boundary of $B_L = B \cap L$. Therefore, $\text{Non C(4.2)'} \Rightarrow \text{Non C(4.1)}$ holds in L (with B_L in place of B).

In view of Theorem 4.2' (with L and B_L in place of X and B respectively) it follows the existence of a continuous function $\tilde{T}:B_L \rightarrow B_L$ without fixed points (e.g. $\tilde{T}(x)=P(x+f(x))$). According to the Dugundji's theorem [10], there exists a continuous extension T of \tilde{T} from a closed subset B_L of X to the whole X with $T(X) \subset \text{co}\tilde{T}(B_L) \subset B_L$. It follows that $T:B \rightarrow B_L \subset B$ is continuous without any fixed point. Indeed, if we assume that $x=Tx$ for some $x \in B$, then $x \in B_L$ so $Tx=\tilde{T}x=x$ i.e. x is also a fixed point in B_L of \tilde{T} (which has no fixed points). The proof is complete.

In view of the Brouwer [5] fixed point theorem, Theorem 4.3 can be restated as:

Theorem 4.3'. The closed unit ball B of a Banach space X has the (Brouwer) fixed-point property if and only if X is finite dimensional (or equivalently, iff B is compact)[10].

Remark 4.3. The Miranda's theorem [24, p.214] follows also from Theorem 4.2 (see Remark 3.2). Indeed, for $a_i=-L$ and $b_i=L$, $i=1, \dots, n$, D_n given by (3.9) becomes the hyper-cube C_n of R^n (say $L=1$). Or, C_n is just the closed unit ball of R^n supplied with the supremum norm (which is of course equivalent to the Euclidean norm). The boundary conditions (3.10) are equivalent to the (B-N) tangency of f to C_n (Prop. 3.3). Therefore, in the case of $D=C_n$ and $X=R^n$, Theorem 4.2 is a completion of the Miranda's theorem. Finally, Miranda's theorem follows directly from Theorem 4.1.

It is straightforward that condition $C(4.2)'$ is equivalent to:

$C(4.2)''$: Every field \tilde{f} on B satisfying $\tilde{f}(x) \neq \mu x$ for all $x \in S$ and all $\mu > 1$, has a fixed point in B .

Indeed, " $\tilde{f}(x) \neq \mu x, \forall \|x\|=1$ and $\mu > 1$ " iff " $\tilde{f}(x) - x \neq \lambda x, \forall \|x\|=1$ and $\lambda > 0$ ".

Therefore, Theorem 4.2 can also be restated as

Theorem 4.2'' Conditions $C(4.1)$, $C(4.2)'$, $C(4.2)''$ and $C(4.3)$ on B are equivalent. They are true iff $\dim X < +\infty$.

Theorem 4.2'' is also a completion of a classical result derived via degree theory [see e.g. 12, Th. 3.1.4, p.36, with $D = \dot{B}$].

Recall that $D \subset X$ is said to be a "flow-invariant set" with respect to the differential equation (E) $u' = f(u)$ (or the vector field f on D) if for every $x \in D$, there exists a solution $u = u(t, 0, x)$ of (E) which remains in D as long as it exists (i.e. for every $x \in D$, D contains a maximal solution of (E) through X). Of course, when f guarantees the uniqueness of the solution to the Cauchy problem for (E), then we say: " D is flow-invariant with respect to (E) if every solution starting in D remains in D as long as it exists".

If $\dim X < \infty$ and D is a closed subset of X , then D is flow-invariant with respect to f iff and only if f is a (B-N) field on D (Th.3.0). Therefore, Theorem 4.1 yields

Corollary 4.1. Let D be a (nonempty) convex compact of \mathbb{R}^n and let f be a vector field on D . If D is a flow-invariant set with respect to $(E)u' = f(u)$, then D contains a zero of f (a constant equilibrium solution of (E)).

Remark 4.4. We will prove (in Section 5) that in \mathbb{R}^2 the conclusion of Corollary 4.1 remains valid without the convexity hypothesis on D , provided that ∂D is a Jordan curve.

In Topology, the standard notation for S and B in \mathbb{R}^n is S^{n-1} and B^n respectively (cf. [11]). A nice result of Topology says that every tangent field to S^2 has a zero $x \in S^2$ (cf. [11, p.149]). Another nice classical result of Topology says that there exist tangent vector fields F to S^{n-1} without zeros in S^{n-1} iff $n-1$ is odd [15, p.30]. However, according to Corollary 4.1 such F must have a zero inside S^{n-1} (i.e. in \mathring{B}^n). This is because a tangent field F to S^{n-1} is a (B-N) field on B^n (see the beginning of Section 3). Other classical conditions equivalent to the Brouwer fixed point theorem are: " S^{n-1} is not contractible" or equivalently, " S^{n-1} is not a retract of B^n " [11, p.167]. From Theorem 4.3 we can prove in a standard way that in the case: $\dim x = \infty$, S is a retract of B . From the above comments it is clear that the following problem is interesting.

P(4.1). In what conditions does a (B-N) field on a convex compact D of \mathbb{R}^n have a zero on ∂D ? Finally, it is important to note that if F is (only) a (continuous) (B-N) field on D , then may exist solutions starting in D which do not remain in D . An example in this direction is the following one:

$$f(x) = -2\sqrt{|x|}, \quad x \in D = [0, +\infty[, \quad u'(t) = -2\sqrt{|u(t)|}, \quad (4.1)$$

We have $f(0) = 0$ so f is a (B-N) field on D (as $x + hf(x) \in D$ for $x = 0$ and $h > 0$) so D is a flow-invariant set with respect to (4.1). However, the solution u (of (4.1)) defined by

$$u(t) = -t^2, \text{ if } t \geq 0 \text{ and } u(t) = t^2 \text{ if } t \leq 0 \quad (4.2)$$

starts in D and $u(t) = -t^2 \notin D$ for $t > 0$.

§ 5. A characterization of (B-N) tangency via the "Invers Taylor formula" on Banach spaces and the uniform motion.

Let g be a continuous function from an open subset A of a Banach space X into a finite dimensional space V . Denote

$$D_g^- = \{x \in A, g(x) \leq 0\}, \quad D_g = \{x \in A, g(x) = 0\} = g^{-1}(0) \\ \text{and } D_g^+ = \{x \in A, g(x) > 0\} \quad (5.1)$$

Of course, if $z \in Y$, then by definition, " $z \leq 0$ iff all coordinates of z (with respect to a given basis of

Y) are ≤ 0 ". The following result is interesting by itself and it will be essentially used throughout this section. Part (5.4) of it is known [23].

Theorem 5.1. Let $g:A \subset X \rightarrow Y$ be continuous and Fréchet differentiable at a point $x \in g^{-1}(0)$. If the Fréchet derivative $u \rightarrow \dot{g}(x)(u)$ from X into Y is surjective, then the conditions (5.2) and (5.3) below are equivalent

$$\lim_{h \downarrow 0} h^{-1} d(x+hy; D_g^-) = 0, \quad g(x)=0, \quad y \in X \quad (5.2)$$

$$g(x)=0, \quad \dot{g}(x)(y) \leq 0 \quad (\text{i.e. } K_{D_g^-}(x) = \{y \in X; \dot{g}(x)(y) \leq 0\}) \quad (5.3)$$

$$K_{D_g^-}(x) = \{y \in X; \dot{g}(x)(y) = 0\} = \text{Ker } \dot{g}(x), \quad g(x)=0 \quad (5.4)$$

Proof. "(5.2) \Rightarrow (5.3)" For this part we need only the existence of the Fréchet derivative $\dot{g}(x)$ of g at x . Indeed, (5.2) is equivalent to (3.3) with $D = D_g^-$. Consequently, there exists $p(h) \rightarrow 0$ as $h \downarrow 0$ such that $g(x+hy+hp(h)) \leq 0, \forall h > 0$. Since $g(x)=0$ we have $g(x+hy+hp(h)) - g(x) \leq 0, \forall h > 0$ which implies (5.3). The proof of "(5.3) \Rightarrow (5.2)" is delicate. It requires an "Inverse Taylor formula" (in short (ITF) based on the surjectivity of $u \rightarrow \dot{g}(x)(u)$ and the Brouwer fixed point property of the ball of Y (see the paper [23] by Ursescu and the author). It can also be found in the author's book [20, p.112, Lemma 2.4].

(ITF) Let X be a Banach space, Y a finite dimensional space and $g:A \subset X \rightarrow Y$ continuous and Fréchet differentiable at x with $u \rightarrow \dot{g}(x)(u)$ surjective. Then for every $\epsilon > 0$ and $y \in X$, there is $\delta > 0$ such that for every $h \in]0, \delta[$ and $v \in Y$ with $\|v\| < \delta$, there is $p(h) \in X$ satisfying

$$\|p(h)\| < \epsilon, \quad x+h(y+p(h)) \in A \quad \text{and} \quad (5.5)$$

$$g(x+hy+hp(h)) = g(x) + h(\dot{g}(x)(y) + v)$$

(We say that (5.5) is an "inverse Taylor formula", since it means that given the "Remainder" $h v$ we can find an increment $p(h)$ satisfying (5.5) (which is of Taylor type).

Proof of Theorem 5.1. (continuation) Part "(5.3) \Rightarrow (5.2)". In view of (5.3), for $v=0$ (5.5) yields the existence of $p(h) \in X$ with $p(h) \rightarrow 0$ as $h \downarrow 0$ such that $g(x+hy+hp(h)) \leq 0$, for all sufficiently small $h > 0$. Or this means $x+hy+hp(h) \in D_g^-$ which is equivalent to (5.2) (see also (3.3)). Part (5.4) is known, so the proof is complete.

Remark 5.1. For the conclusion of Theorem 5.1, the surjectivity of $u \rightarrow \dot{g}(x)u$ is not necessary. For instance, let $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the projection on x_1 -axis, i.e.

$g(x_1, x_2) = (x_1, 0), \forall x = (x_1, x_2) \in \mathbb{R}^2$. Hence $D_g = O_{x_2} = x_2$ -axis.

Clearly, the Jacobian matrix $\dot{g}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not surjective {precisely, the range of $\dot{g}(x)$ is

$R(\dot{g}(x)) = O_{x_1}$ (x_1 -axis). However, we have

$\text{Ker}(\dot{g}(x)) = O_{x_2} = K_{D_g}(x), \forall x \in D_g$, i.e. (5.4) still holds. On the other hand, without surjectivity of $u \rightarrow \dot{g}(x)(u)$, (5.4) may not be true.

Indeed, take $g(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2)^2, (x_1, x_2) \in \mathbb{R}^2$. Then $D_g = \{(1, 2)\}$ so $K_{D_g}(1, 2) = \{(0, 0)\}$.

Clearly, the gradient $\dot{g}(x)$ of g at $(1, 2)$ is zero - (i.e. $\dot{g}(1, 2) = (0, 0)$) so it is not surjective. Obviously $\text{Ker } \dot{g}(1, 2) = \mathbb{R}^2 \neq K_{D_g}(1, 2)$.

Note also that Th. 5.1 is useful in flow-invariance and orbital motions [23] and it has a unifying effect in optimization [16].

Corollary 5.1. Let g be as in Theorem 5.1. Then f is a (B-N) vector field on $D_g^-(D_g)$ iff

$$\dot{g}(x)(f(x)) \leq 0, \forall x \in D_g \quad (5.6)$$

and $\dot{g}(x)(f(x)) = 0, \forall x \in D_g \quad (5.7)$

Corollary 3.1 can be derived from Corollary 5.1 in the case X^* uniformly convex (when the norm of X is Fréchet diff. on S) by choosing $g(x) = \frac{1}{2}(\|x\| - r)^2$.

If g is a convex function on an open A , then D_g^- is convex and we can give results similar to Theorem 4.1. We will not do that here. What we will do, is to present results on zeros of (B-N) fields f on D_g^- , without the convexity of g . In the case $g: \mathbb{R}^2 \rightarrow \mathbb{R}$, the surjectivity of $u \rightarrow \dot{g}(x)(u)$ is equivalent to the fact that the gradient $\dot{g}(x)$ is different from zero ($\dot{g}(x) \neq 0$, for $x \in g^{-1}(0)$). Denote by $I_f(D_g)$ the index of D_g with respect to f (i.e. the winding number of f around D_g or still - the number of times that $f(x)$ winds around the origin as x moves around $D_g = g^{-1}(0) = \{x \in \mathbb{R}^2; g(x) = 0\}$. Set also $D_g^- = \{x \in \mathbb{R}^2; g(x) \leq 0\}$.

Theorem 5.2. Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be of class C^1 with $\dot{g}(x) \neq 0$ for all $x \in g^{-1}(0)$. Moreover, assume that D_g is a Jordan curve and that D_g^- is the bounded connected component of D_g (i.e. D_g^- is "the inside" of D_g [11, p.153]). If f is a (B-N) field on D_g^- without zeros on D_g , then $I_f(D_g)$ is one.

Proof. According to Corollary 5.1, the (B-N) tangency of f to D_g^- is equivalent to (5.6), where

$\dot{g}(x)(f(x))$ is the inner product of vectors $\dot{g}(x) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right)$ and $f(x) = (f_1(x), f_2(x))$, $x = (x_1, x_2)$.

Set $G(x) = \left(-\frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_1}\right)$. Then G is a continuous tangent vector field to D_g (in classical sense)

without zeros in D_g , so $I_G(D_g) = 1$ (see e.g. [7]). But $I_{\dot{g}}(D_g) = I_G(D_g) = 1$. Finally (5.6) yield $I_f(D_g) = I_{\dot{g}}(D_g) = 1$ which completes the proof.

Corollary 5.2. If f is a (B-N) field on D_g^- (with g as in Theorem 5.2), then f has a zero in D_g^- (with g as in Theorem 5.2), then f has a zero in D_g^- .

Proof. If f has no zeros on the boundary D_g of D_g^- , then $I_f(D_g) = 1$ so f has a zero in \dot{D}_g^- .

Remark 5.2. The conclusions of Theorem 5.2 and of Corollary 5.2 remain valid if the hypothesis “ f is a (B-N) field on D_g^- ” is replaced by “ f is normal to D_g^- ”. Corollary 5.2 includes the classical result that a periodic orbit of $u' = f(u)$ contains inside a zero of f [7].

We have to point out that Condition (5.6), i.e. the (B-N) tangency to D_g^- is a characteristic property of “velocity fields” on D_g^- . In general, if f is a force field on D_g^- under which D_g can be described [20, p. 123] then (5.6) holds in the strict sense, i.e.

$$\dot{g}(x)(f(x)) < 0, \quad \forall x \in D_g \quad (5.8)$$

This is the subject we want to study in what follows. Therefore, let us consider the “Newtonian equation of motion”.

$$u''(t) = f(u(t)), \quad u(0) = x \in D_g \text{ (i.e. } g(x) = 0), \quad u'(0) = y \quad (5.9)$$

Suppose that $u \rightarrow \dot{g}(x)(u)$ from a Banach space X into \mathbb{R}^n is surjective for $x \in D_g$ and g is twice Fréchet differentiable on an open subset A containing D_g (such that D_g is closed in A). It is easy to show that a necessary condition for the solution u of (5.9) to remain in D_g (i.e. to satisfy $g(u(t)) = 0, \forall t > 0$) is given by $(u(0), u'(0))$.

$$(u(0), u'(0)) = (x, y) \in M_{D_g}, \quad (u(t), u'(t)) \in M_{D_g} \quad \forall t > 0 \quad (5.10)$$

where

$$M_{D_g} = \{(x, y) \in A \times X, g(x) = 0, \dot{g}(x)(y) = 0, \ddot{g}(x)(y)(y) + \dot{g}(x)(f(x)) = 0\} \quad (5.11)$$

This fact can be proved by differentiating $g(u(t)) = 0, t \geq 0$.

Indeed, we get

$$\dot{g}(u(t))(u'(t)) = 0, \quad \ddot{g}(u(t))(u'(t))(u'(t)) + \dot{g}(u(t))(u''(t)) = 0 \quad t > 0 \quad (5.12)$$

Therefore, the following definition is natural.

Definition 5.1. D_g is said to be a flow-invariant set with respect to $u''=f(u)$ (in short: with respect to the force field f on A) if every solution u to (5.9) with $(u(0), u'(0)) \in M_{D_g}$ remains in D_g as long as it exists (i.e. $(u(t), u'(t)) \in M_{D_g}, t > 0$).

Suppose that D_g is flow-invariant with respect to f (we also say that the orbit D_g can be described under the action of the force field f). Then (5.12) holds. Substituting $u''(t)=f(u(t))$ in (5.12) and then differentiating with respect to t we get (for $t=0$, provided f is Fréchet differentiable on D_g)

$$\dot{g}'(x)(y^3) + 3\ddot{g}(x)(y)(f(x)) + \dot{g}(x)(\dot{f}(x)(y)) = 0 \quad (5.13)$$

for all $(x, y) \in M_{D_g}$, where $\ddot{g}(x)(y^3) = \ddot{g}(x)(y)(y)(y)$.

Therefore, if g is three times Fréchet differentiable, then (5.13) is a necessary condition for D_g to be flow-invariant with respect to the force field f on A . Thus, (5.13) is derived easier than in [23]. The main point is that:

$$\text{If } u \rightarrow (\dot{g}(x)(u), \ddot{g}(x)(y)(u)) \text{ from } X \text{ into } \mathbb{R}^n \times \mathbb{R}^n \text{ is surjective for each } (x, y) \in M_{D_g} \quad (5.14)$$

then (5.13) is necessary and sufficient for D_g to be flow-invariant with respect to f [20, ch.3]. Now let H be a real Hilbert space of inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and $g(x) = \frac{1}{2}(\|x\|^2 - 1)$, $x \in H$. Then $D_g = S$, $\ddot{g}(x)(y)(y) = \|y\|^2$. Thus

$M_{D_g} = M_S$ becomes

$$M_S = \{(x, y) \in A \times H; \|x\| = 0, \langle x, y \rangle = 0, \|y\|^2 + \langle x, f(x) \rangle = 0\} \quad (5.15)$$

and so (5.13) takes the form

$$3\langle y, f(x) \rangle + \langle x, \dot{f}(x)(y) \rangle = 0, \forall x \in S, y \in H, \langle x, y \rangle = 0 \quad (5.16)$$

First of all $(x, y) \in M_S$ implies $y \neq 0$ (the "initial speed of projection" cannot be zero). Therefore

Corollary 5.3. A force field f under which D_g can be described is a (B-N) field on A , in a strict sense (i.e. (5.8) holds, or still: f is nowhere tangent to D_g , $f(x)$ points inside D_g for each $x \in D_g$).

Let us observe that for $D_g = S = g^{-1}(0)$ the surjectivity condition (5.14) is fulfilled. In this case it becomes

$$u \rightarrow (\langle x, u \rangle, \langle y, u \rangle) = F(u), \quad \|x\| = 1, \langle x, y \rangle = 0 \quad (5.17)$$

from H into \mathbb{R}^2 is surjective.

Indeed, take $(a, b) \in \mathbb{R}^2$. Then for $u = ax + by / \|y\|^2$, $F(u) = (a, b)$ so $F: H \rightarrow \mathbb{R}^2$ is surjective.

Therefore we have

Theorem 5.3 Let H be a real Hilbert space and let $f: A \subset H \rightarrow H$ be a Frechet differentiable force field on A (an open neighborhood of S). Then a necessary and sufficient condition for S to be described under the action of f is given by (5.16).

Remark 5.3. On the basis of Theorem 5.3, Condition (5.16) is necessary and sufficient for the solution u to the Cauchy problem,

$$u'' = f(u), u(0) = x \in S, u'(0) = y, \langle x, y \rangle = 0, \|y\|^2 = -\langle x, f(x) \rangle \quad (5.18)$$

to satisfy $\|u(t)\| = 1, \forall t \geq 0$, for every $x \in S$ and $y \in S$ as in (5.18).

Of course, in the case $H = \mathbb{R}^3, A = \mathbb{R}^3 - \{0\}, S = S(R+h)$ (R -the radius of the Earth $r > 0$ sufficiently large, and $f_N(x) = -GMx/\|x\|^3, x \in A$ (the Newtonian gravitational field, GM -the power of the force centre), then y given by (5.18),

i.e. $\|y\| = \{GM/(R+r)\}^{1/2}$ is the magnitude of the well-known "first cosmic speed". In what follows we will derive (on the basis of (5.16) and on an optimum principle), a necessary and sufficient condition for f , in order for "the motion on S " to be uniform.

We say that the motion on S (under the action of the force field f) is uniform if for every x, y as in (5.18), the corresponding solution $u = u(t, 0, x)$ of (5.18) satisfies $\|u'(t)\| = C, \forall t > 0$ (where $C > 0$ depends on y and f but not on $x \in S$).

Therefore, if the motion on S under f is uniform, then necessarily

$$G(x) = \langle x, f(x) \rangle = -C^2 = \text{const for } x \in S, S \subset H \quad (5.19)$$

Now let us recall the following principle of optimum (given on a Banach manifold by Motreanu and the author (cf [17], or [20, ch.4]).

Theorem 5.4. Let A be an open subset of a normed space X and D a nonempty subset of A .

Suppose that $G: A \rightarrow \mathbb{R}$ is continuous and

$$\inf\{G(x); x \in D\} = G(x_0), x_0 \in D \quad (5.20)$$

If G is Frechet differentiable at x_0 , then

$$\dot{G}(x_0)(y) \geq 0, \text{ for all } y \in K_D(x_0) \quad (5.21)$$

[Indeed, if $y \in K_D(x_0)$, then $x_0 + hy + hp(h) \in D$ for $h > 0$, with some $p(h) \rightarrow 0$ as $h \downarrow 0$ (see (3.3)).

Hence $G(x_0 + hy + hp(h)) - G(x_0) \geq 0, \forall h > 0$ which yields (5.21)].

In the case $D = S$,

$$K_D(x) = \{y \in H; \langle x, y \rangle = 0\}, \forall x \in S \quad (5.22)$$

so $K_D(x)$ is a subspace of H . Thus, if G is the function defined by (5.19), we have (by (5.21))

$\dot{G}(x)(y) = 0$, for all $y \in H$ with $\langle y, x \rangle = 0$ and $x \in S$ i.e.

$$\langle y, x \rangle + \langle x, \dot{f}(x)(y) \rangle = 0, \forall x \in S, y \in H \text{ with } \langle x, y \rangle = 0 \quad (5.23)$$

Obviously, (5.16) and (5.23) yield

$$\langle y, f(x) \rangle = 0, \quad \forall x \in S, y \in H \text{ with } \langle x, y \rangle = 0 \quad (5.24)$$

We conclude that if the motion on S under f is uniform, then (5.24) is (necessarily) fulfilled and it implies $f(x) = \alpha x$, with $\alpha \in \mathbb{R}$ and $x \in S$. This in conjunction with (5.16) give

$$f(x) = -C^2 x, \quad \forall x \in S \quad (5.25)$$

Conversely, if f is given by (5.25), then (5.19) (which implies (5.23)) and (5.24) are satisfied. Or, (5.23) and (5.24) show that (5.16) holds true (i.e. (5.13) and (5.14) are fulfilled). Thus, (5.25) implies that S is invariant under f as in (5.25) and the motion on S is uniform.

Therefore we have given a new proof to

Theorem 5.5 The unit sphere S of a real Hilbert space H is invariant under a differentiable force field f on an open neighborhood A of S , and the motion on S under f is uniform if and only if f is an attractive central force field (i.e. f is given by (5.25)).

(A different proof is given in [20, pp. 135-136]).

Remark 5.4. The Newtonian gravitational field f_N above is a (B-N) field on $D_g^- - \{0\}$ with $g(x) = \frac{1}{2}(\|x\|^2 - 1)$ (hence $\dot{g}(x) = x$), so $\dot{g}(x)(f_N(x)) = -GM < 0$, $\forall \|x\| = 1$ (i.e. (5.8) holds). However, f_N is nowhere zero. This does not contradict the conclusion of Corollary 5.2, as f is undefined at $0 \in D_g^-$. Does the conclusion of Theorem 5.5 remain valid in every real (strictly convex) Banach space X ?

APPENDIX

Lemma 1. Let D be a compact of X and let f be a (B-N) field on D (i.e. (3.5) holds pointwise in D). Then (3.5) holds uniformly in D .

Proof. Take x arbitrarily in D . Let $u = u(t, 0, x)$ be corresponding solution to $u'(t) = f(u(t))$, $u(0) = x$, $u(t) \in D$ for t in the domain of u . Set $M = \sup\{\|f(z)\|; z \in D\}$. For $\epsilon > 0$, there is $\delta(\epsilon) > 0$, such that $\|f(z) - f(w)\| < \epsilon$ for all $z, w \in D$ with $\|z - w\| < \delta(\epsilon)$. But $\|u(s) - u(0)\| \leq Ms$ for $s > 0$ sufficiently small. It follows that for all $h \in]0, M^{-1}\delta(\epsilon)[$ and for all $x \in D$, we have

$$h^{-1}d(x + hf(x); D) \leq h^{-1}\|x + hf(x) - u(h)\| \leq h^{-1} \int_0^h \|f(u(s)) - f(u(0))\| ds < \epsilon, \text{ which completes the proof.}$$

Proof of Proposition 3.1. The only fact we have to prove is the implication

$$"f(x) \in K_D(x), \quad \forall x \in D" \Rightarrow "f(x) \in C_D(x), \quad \forall x \in D".$$

Therefore, let $x \in D$, $x_n \in D$, $x_n \rightarrow x$ and $h_n \downarrow 0$. In view of Lemma 1, (3.5) implies $h_n^{-1}d(x_n + h_n f(x_n); D) \rightarrow 0$ as $n \rightarrow \infty$, which is equivalent to $x_n + h_n(f(x_n) + a_n) \in D$, $n = 1, 2, \dots$, for some $a_n \in X$ with $a_n \rightarrow 0$ as $n \rightarrow \infty$. The proof is complete.

REFERENCES

1. E. Asplund, Averaged norms, *Israel J. Math.* 5(1967) 227-233.
2. S. Banach, *Théorie Des Opérations Linéaires*, New York (1955).
3. G. Bouligand, Sur les surfaces dépourvues des point hyperlimités. *Ann. Soc. Polon. Math.* 9(1930)32-41.
4. H. Brezis, On a characterization of flow-invariant sets. *Comm. Pure Appl. Math.* 23 (1970) 261-263.
5. L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, *Math. Ann.* 71, 97-115.
6. F. H. Clarke, *Optimization and Nonsmooth Analysis*, New York (1983).
7. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, New York, 1955.
8. M. G. Crandall, A generalization of Peano's existence theorem and flow-invariance. *Proc. Amer. Math. Soc.* 36(1971) 151-155.
9. M. M. Day, On the basis problem in normed spaces, *Proc. Amer. Math. Soc.* 13(1962), 655, 658.
10. J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* 1(1951), 353-367.
11. Th. W. Gamelin and R. E. Greene, *Introduction to Topology*, New York (1983).
12. N.G. Lloyd, *Degree Theory*, London 1978.
13. J. T. Marti, *Introduction to the Theory of Bases*, Springer-Verlag New York 1969.
14. R. H. Martin Jr., Differential equations on closed subsets of a Banach space, *Trans. Amer. Math. Soc.* 179(1973)339-414.
15. J. W. Milnor, *Topology from the Differentiable Viewpoint*, The Univ. Press of Virginia, Charlottesville, VA., 1965.
16. D. Motreanu and N. H. Pavel, Quasi-tangent vectors in flow-invariance and optimization problems on Banach manifolds, *J. Math. Anal. Appl.* 88 (1982) 116-132.
17. D. Motreanu and N. H. Pavel, Flot-invariance par rapport aux équations différentielles du second ordre sur variété. *C. R. Acad. Sci, Paris* 297(1983)157-160.
18. M. Nagumo, Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen. *Proc. Phys. Math. Soc. Japan* 24 (1942) 551-559.
19. N. H. Pavel, Semilinear equations with dissipative t -dependent domain perturbations. *Israel J. Math.* 46(1983)103-122.
20. N. H. Pavel, *Differential Equations, Flow-invariance and Applications*, Pitman Res. Notes in Math. 113(1984), London.
21. N. H. Pavel, *Nonlinear Evolution Operators and Semigroups. Applications to PDE*, Lecture Notes in Math. Springer Verlag 1260(1987).

22. N. H. Pavel, The Brouwer fixed point theorem and zeros of some vector fields. (to appear)
23. N. H. Pavel and C. Ursescu, Flow-invariant sets for autonomous second order differential equations and applications in Mechanics, *Nonlinear Anal.* 6(1982)35-74.
24. L. C. Piccinini, G. Stampachio and G. Vidossich, Ordinary Differential Equations in \mathbb{R}^n . Springer-Verlag, Appl. Math. Sco, 39(1984)
25. E. H. Rothe, Introduction to Various aspects of Degree Theory in Banach Spaces, *Math. Surveys and Monographs*, Vol. 23 (1986) AMS.
26. I. Singer, Basis in Banach Spaces. Springer-Verlag-Bucharest. Vol I(1970), Vol. II, 1981.
27. C. Ursescu, Tangent sets and differentiable functions, *Banach Center Publ.* 1(1974), 151-155.
28. J. A. Yorke, Invariance for ordinary differential equations. *Math. Systems Theory* 1(1967) 353-372.
29. M.A. Krasnosel' skii and P.P. Zabreiko, Geometrical Methods of Nonlinear Analysis, Springer-Verlag (1984).

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