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To my professor, Nicolae Teodorescu,
in gratitude on his 80th birthday

BASIC REPRESENTATIONS OF POLYANALYTIC FUNCTIONS

DAN PASCALI

By using the Pompeiu iterated operator, we establish a general and simple representation of polyanalytic functions. This formula is valid also in higher dimensional spaces and allows us to extend easily some analytic function principles. We relate this representation with new topics of nonanalytic complex functions, developed in the last decade in monographs [1], [8], [25] or at the conference [6] and originated in the contributions of the interwar Romanian mathematical school.

The concept of areolar derivative as deviation measure from the monogeneity of a complex function was introduced by D. Pompeiu in 1912 in two notes in "Rendinconti del Circolo Matematico di Palermo", ([18], pp.137-143; 173-177). The areolar derivative of a continuous function $U(z) = u(x,y) + iv(x,y)$ is

$$\left.\frac{\partial U}{\partial \bar z}\right|_{z=z_0} = \lim_{\gamma\to z_0} \frac{\dfrac{1}{2\pi i}\displaystyle\int_\gamma U(z)\,dz}{\dfrac{1}{\pi}\displaystyle\iint_\delta dx\,dy}, \qquad (z = x + iy) \qquad (1)$$

where the limit is taken on all closed simple rectificable curves γ, enclosing a domain ω, and continuously contractible to the point z_0. If $U \in C^1$ then

$$\frac{\partial U}{\partial \bar z} = \frac{1}{2}\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right] = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right].$$

However, both D. Pompeiu and other Romanian mathematicians as G. Călugăreanu, N. Ciorănescu, M. Ghermănescu, M. Nicolescu or foreign ones as P. Burgatti, M. Krasner consider only the C^1-function case, called also polygenic functions, as solutions of the nonhomogenuous Cauchy-Riemann system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 2\,A(x,y), \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\,B(x,y),$$

where $A(x,y)$ and $B(x,y)$ are continuous functions. We must emphasize N. Teodorescu's merit [21] for instituting in 1931 a systematic study of the areolar derivative independently of the existence of partial derivatives of functions $u(x,y)$ and $v(x,y)$ in the limit (1). The extended definition of the areolar derivative allows us to replace the equation $\frac{\partial U}{\partial \bar{z}} = V(z)$ with a continuous right hand side in a plane domain Ω by the functional equation

$$\frac{1}{2i} \int_{\gamma} U(z)\, dz \; - \; \iint_{\omega} V(z)\, dx\, dy = 0,$$

which holds for all pairs $(\gamma,\omega) \subset \Omega$ and provides generalized solutions equivalent with those in the distribution sense for the Cauchy-Riemann system [23].

Any continuous function of z, whose areolar derivative vanishes in every point of a domain Ω is holomorphic in Ω. A function with a continuous areolar derivative is called α-holomorphic. N. Teodorescu extended to α-holomorphic functions the representation

$$U(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{U(t)}{t - z}\, dt \; - \; \frac{1}{\pi} \iint_{\Omega} \frac{\partial U}{\partial \bar{t}}\, \frac{d\xi\, d\eta}{t - z}, \qquad (t = \xi + i\eta)$$

$\Gamma = \partial\Omega$ consisting of a finite number of simple closed and rectifiable curves. This formula was previously established by D. Pompeiu for polygenic functions.

Since 1952 I. N. Vekua resumed the theory of areolar derivative, starting from the linear system with first order partial derivatives

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = au + bv + f \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = cu + dv + g \end{cases} \tag{2}$$

Setting

$$A = \frac{1}{4}(a - d + ic + id), \quad B = \frac{1}{4}(a + d + ic - id), \quad F = \frac{1}{2}(f + ig),$$

I. N. Vekua rewrote system (2) in the form

$$\frac{\partial U}{\partial \bar{z}} - A\,U - B\,\bar{U} = F \tag{3}$$

and applied successfully functional analysis methods, envolving the theory of generalized analytic functions [24]. The solutions of (3) are sought in the class $C_{\bar{z}}(\Omega)$ which coincides with the α-holomorphic function space, [23]. A similar point of view was developed by L. Bers [3].

A particular attention is paid to the Pompeiu integral operator [18, pp. 150-162; 197-201]

$$Tf(z) = -\frac{1}{\pi} \iint_{\Omega} \frac{f(t)}{t - z}\, d\xi\, d\eta. \tag{4}$$

For a bounded measurable function f in Ω, N. Teodorescu [22] proved that Tf is continuous in the whole plane, holomorphic outside $\bar{\Omega}$, vanishes at infinity, and

$$\frac{\partial Tf}{\partial \bar{z}} = f(z).$$

Furthermore, I.N. Vekua [24,pp. 38] showed that T is a completely continuous linear operator from $L_p(\Omega)$, $p > 2$, into $C_\alpha(\bar\Omega)$, the Hölder function space with exponent $\alpha = \frac{p-2}{p}$.

One of the natural generalization of analytic functions are the solutions of the equation

$$\frac{\partial^{n+1} f}{\partial \bar z^{n+1}} = 0, \qquad f(z) = P(x,y) + i\, Q(x,y) \tag{5}$$

called polyanalytic functions or areolar polynomials of the n-the order. For $n = 0$ we have the complex form of the Cauchy-Riemann equations. The general study of polyanalytic functions was initiated by P. Burgatti and N. Teodorescu.

We assume henceforth that the domain Ω contains the origin. Since the kernel of the operator $\frac{\partial}{\partial \bar z}$ coincides with the holomorphic function space in Ω, we derive that a representation of the general solution of equation (4) is

$$f(z) = \phi_0(z) + \bar z\, \phi_1(z) + \bar z^2\, \phi_2(z) + \ldots + \bar z^n\, \phi_n(z), \tag{6}$$

where $\phi_k(z)$, $k = 0,1,\ldots,n$, are holomorphic functions in Ω. A more accurate development was established by N. Teodorescu [21]

$$f(z) = \sum_{k=0}^{n} \frac{1}{2\pi i k!} \int_\Gamma \frac{(\bar z - \bar t)^k}{t - z} \frac{\partial^k f(t)}{\partial \bar t^k}\, dt = \sum_{k=0}^{n} \frac{\bar z^k}{2\pi i k!} \int_\Gamma \frac{\phi_k(t)}{t - z}\, dt. \tag{7}$$

Form (6) allows us to draw nearer the representations of polyanalytic and polyharmonic functions in the plane. Taking into account that $\Delta = 2^2 \frac{\partial^2}{\partial z \partial \bar z}$, where $\frac{\partial}{\partial z} = \frac{1}{2}\left\{ \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right\}$ is the areolar co-derivative, we can easily prove by induction [13] that: Any (n+1)-th order polyharmonic function in a plane domain Ω can be written as the real or imaginary part of an n-th order polyanalytic function in Ω. Formula (6) for $n = 1$ is known as the Goursat representation of biharmonic functions. Moreover, the following unicity conditions can be stated:

Given a (n+1)-th order polyharmonic function $P(x,y) = Re\,[f(z)]$, the functions $\phi_k(z)$ in (6) are uniquely determined provided that:

$$\begin{cases} Im\,[\phi_0(0)] = 0; & \\ \phi_1(0) = 0; \quad Im\,[\phi_1'(0)] = 0; & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \phi_n(0) = 0; \quad \phi_n'(0) = 0; \ \ldots\ ; \ \phi_n^{(n-1)}(0) = 0; \quad Im\,[\phi_n(0)] = 0. & \end{cases}$$

Here Re [ ] and Im [ ] denote the real and the imaginary part of [ ]. For $n = 0$, we recognize the condition of the unique determination of the conjugate harmonic function.

Assuming the above unicity conditions fulfilled, we actually have

$$\phi_k(z) = z\,(a_k + a_{k+1} z + a_{k+2} z^2 + \ldots) = z^k (u_k + i\, v_k)$$

$k = 1,2,\dots,n$, where u_k and v_k are the harmonic components of the holomorphic function developed within the brackets. If we set $|z| = r$, the representation (6) yields

$$P = u_o + r^2u_1 + r^4u_2 + \dots + r^{2n}u_n, \qquad (8)$$

which is nothing but the Almansi expansion in a neighbourhood of the origin of the polyharmonic function $P(x,y)$, [12].

It is worth noting that to any real analytic function cooresponds a series of the form (8); this representation as an infinite order polyharmonic function was established by N. Ciorănescu in 1937, ([5], pp. 387-404). An infinite expansion of the form (6) is called an areolar series. The above interplay between polyharmonic and polyanalytic functions was extended to a similar interplay between analytic functions and areolar series in [15]. A comprehensive account can be found in [12] and [1] on polyharmonocity, while in [2] and [6, pp. 68-84] on polyanalitycity. M. Roşculeţ [20] gave a comparative description of elliptic, parabolic and hyperbolic analiticity.

We pointed out in [16] a new form for polyanalytic functions by means of the iterates of integral operator (4) which is similar to an expansion of polyharmonic functions using the iterated Green's function due to M. Nicolescu [12, pp. 326-335]. Denoting

$$K(\zeta - z) = -\frac{1}{\pi(\zeta - z)} \quad \text{and} \quad K^j(\zeta - z) = \iint_\Omega K^{j-1}(\zeta - t)\, K(t - z)\, d\xi d\eta,$$

we consider the Pompeiu iterated operators

$$T^j g(z) = \iint_\Omega K^j(t - z)\, g(t)\, d\xi d\eta, \qquad j = 2,3,\dots,n.$$

Since

$$T^j g(\zeta) = \iint_\Omega K(z - \zeta)\, dxdy \iint_\Omega K^{j-1}(t - z)\, g(t)\, d\xi d\eta = T\,(\,T^{j-1}g)(\zeta),$$

we derive

$$\frac{\partial T^n g(z)}{\partial \bar z} = T^{n-1}g(z), \quad \dots\,, \quad \frac{\partial^n T^n g(z)}{\partial \bar z^n} = g(z).$$

Moreover, if the function $g(z)$ is holomorphic in Ω, then

$$\frac{\partial^{n+1}\, T^n g(z)}{\partial \bar z^{n+1}} = 0.$$

Let us consider the formal polynomial

$$f(z) = \varphi_o(z) + T\varphi_1(z) + T^2\varphi_2(z) + \dots + T^n\varphi_n(z), \qquad (9)$$

where $\varphi_k(z)$, $k = 0,1,\dots,n$, are holomorphic functions in Ω. Taking succesively areolar derivatives, we get

$$\frac{\partial f(z)}{\partial \bar z} = \varphi_1(z) + T\varphi_2(z) + \dots + T^{n-1}\varphi_n(z)$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{\partial^n f(z)}{\partial \bar z^n} = \varphi_n(z) \quad \text{and} \quad \frac{\partial^{n+1} f(z)}{\partial \bar z^{n+1}} = 0,$$

and we infer that $f(z)$ is an n-th order polyanalytic function in Ω.

Moreover, since

$$\frac{\partial^k f(z)}{\partial \bar{z}^k} = \varphi_k + T(\varphi_{k+1}(z) + T\varphi_{k+2}(z) + \ldots + T^{n-k-1}\varphi_n(z)), \qquad (10)$$

the holomorphic functions φ_k are defined by the values of $\dfrac{\partial^k f}{\partial \bar{z}^k}$ on Γ by

$$\varphi_k(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\partial^k f(t)}{\partial \bar{t}^k}\,\frac{dt}{t-z}, \qquad k = 0,1,\ldots,n. \qquad (11)$$

In this way, we have obtained the following structural result:

THEOREM 1. Any n-th order polyanalytic function $f(z)$ in Ω has the form (9) where the holomorphic components are given by (11).

By using the Morera theorem and equation (10), we can easily prove:

THEOREM 2. Necessary and sufficient conditions that (n+1) continuous functions $\Phi_j(z)$ in Ω should represent an n-th order polyanalytic function $f(z) = \Phi_0(z)$ and its areolar derivatives $\dfrac{\partial^j f(z)}{\partial \bar{z}^j} = \Phi_j(z)$ are that

$$\int_\gamma (\Phi_{j-1}(t) - T\,\Phi_j(t))\,\frac{dt}{t-z} = 0, \qquad j = 1,\ldots,n,$$

for any closed simple rectificable curve γ in Ω, enclosing a domain ω, and any point z outside $\bar{\omega}$.

A version of expansion (9) for n-the order generalized analytic functions, when the derivative $\dfrac{\partial f}{\partial \bar{z}}$ is replaced by the left side hand of equation (3) is found in [17].

Later on, an analogous representation in the Euclidean space $\mathbb{R}^m$, $m \geq 3$, is pointed out. First, we briefly present elements of matrix analysis, describing an extension of the monogenic conditions or Cauchy-Riemann equations.

Let S be a closed hypersurface with an interior normal $\vec{n}_M$ at each point M. Denote by V^+ and V^- the interior and exterior domain separated by S, respectively. Let $\gamma = (\gamma_1,\ldots,\gamma_m)$ be a vector of constant square matrices of order 2s and let $\bar{\gamma} = (\bar{\gamma}_1,\ldots,\bar{\gamma}_m)$ be the vector of the transposed matrices. We confine ourselves to matrix fields $\Phi = \{\Phi_{jk}\} \subset C^1(V)$ and we introduce the linear operators

$$D\Phi = \sum_{j=1}^{m} \gamma_j \frac{\partial \Phi}{\partial x_j} \qquad \text{and} \qquad \bar{D}\Phi = \sum_{j=1}^{m} \bar{\gamma}_j \frac{\partial \Phi}{\partial x_j}$$

called the spatial derivative and spatial co-derivative, respectively. To obtain $\bar{D}D = D\bar{D} = I\Delta$, where I is the unit matrix, we impose the conditions

$$\gamma_j\bar{\gamma}_k + \bar{\gamma}_j\gamma_k = 2\,\delta_{jk}I, \qquad i,k = 1,2,\ldots,m.$$

A simple example of such a set of the fourth order matrices in $\mathbb{R}^3$ is given in [11]:

$$\gamma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \gamma_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \gamma_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Regarding the operator D, we have the Teodorescu-Moisil representation formula

$$\frac{1}{\sigma_m}\int_S \frac{(\overrightarrow{MP}\cdot\bar{\gamma})}{\overline{MP}^m}(n_M\cdot\gamma)\,\Phi(M)\,d\sigma_M + \frac{1}{\sigma_m}\int_{V^+} \frac{(\overrightarrow{QP}\cdot\bar{\gamma})}{\overline{QP}^m}\,D\Phi(Q)\,d\omega_Q = \begin{cases} \Phi(P) & \text{if} \quad P \in V^+, \\ 0 & \text{if} \quad P \in V^-, \end{cases} \tag{12}$$

where σ_m is the area of the unit hypersphere and $(\bullet\cdot\bullet)$ denotes the standard inner product in $\mathbb{R}^m$.

Solutions of the equation $D\Phi = 0$ are called (γ)-monogenic. For these solutions (12) is actually an extended Cauchy integral formula. A generalization of the Morera theorem for (γ)-monogenic functions follows easily from the Gauss-Ostrogradskii formula.

On the other hand, the Teodorescu-Moisil integral operator

$$\Pi\psi(P) = \frac{1}{\sigma_m}\int_{V^+} \frac{(\overrightarrow{QP}\cdot\bar{\gamma})}{\overline{QP}^m}\,\psi(Q)\,d\omega_Q \tag{13}$$

extends to $\mathbb{R}^m$ all the properties of the Pompeiu operator (4). Moreover, we can define the iteratives of Π and prove that (γ)-polymonogenic functions, i.e. solutions of the equation

$$D^{n+1}F(P) = 0$$

admit a representation in the form

$$F(P) = \Psi_0(P) + \Pi\Psi_1(P) + \Pi^2\Psi_2(P) + \ldots + \Pi^n\Psi_n(P)$$

where its (γ)-monogenic components are determined by

$$\Psi_k(P) = \frac{1}{\sigma_m}\int_S \frac{(\overrightarrow{MP}\cdot\bar{\gamma})}{\overline{MP}^m}(n_M\cdot\gamma)\,D^kF(M)\,d\omega_M\,, \quad k = 0,1,\ldots,n.$$

In a similar way, we prove a Morera theorem for (γ)-polymonogenic functions.

The matrix equation $D\Phi = \Psi$ includes Dirac's systems as well as other known systems in elasticity theory and hydrodinamics. Analogously with the areolar derivative theory, Gr. C. Moisil and N. Teodorescu studied, in the early thirties, the generalized solutions of this equation as solutions of a preceding integral identity. Using an elaborate technique, M. Nedelcu-Coroi [7] constructed some operators equivalent with $\frac{\bar{z}^n}{n!}$ and obtained an expansion of type (7) for (γ)-polymonogenic functions.

Functions of a hypercomplex variable provide an alternative basis for extending the basic concepts of one complex variable functions to higher dimensional spaces. This approach, initiated by Gr C. Moisil [10], was applied by the author to invert the singular integral equations and used by V.Iftimie [9] for a complete description of the integral operator corresponding to (13). A general theory of monogenic functions on commutative algebras is done by M.N.Roşculeţ [19].

The monograph [8] is a survey of the foreign contributions to the hyperanalytic functions. A special attention was paid to matrix systems

$$\frac{\partial w}{\partial \bar{z}} = A\, w + B\, \bar{w}$$

which were first studied by the author [14] for arbitrary n. A similarity principle for these systems establishes a representation $w = S\varphi$ where S is a nonsingular matrix continuous in the complex plane and φ is a vector analytic in the domain of w. This representation allow us to extend basic results of one complex variable functions to solutions of all elliptic systems in the plane,[4].

Finally, we distinguish two starting points in the study of the latter topic, the function theoretic methods [8] and the modern theory of partial differential equations [25]. Wendland's book is also a good guide for related numerical approaches.

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