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A NOTE ON DUBINSKII TYPE INEQUALITY

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ABSTRACT. In the present note we establish a new integral inequality of the Dubinskii type , involving functions of several independent variables and their first order partial derivatives . The inequalities of this type have significant applications in the Theory of Partial Differential Equations.

1. INTRODUCTION

The integral inequality

$$(1)\quad \int_G |u|^{\alpha_0+\alpha_1} dx \leq K\left[\int_\Gamma |u|^{\alpha_0+\alpha_1} dS + \int_G |u|^{\alpha_0} \left|\frac{\partial u}{\partial x_i}\right|^{\alpha_1} dx\right],$$

valid for $i=1,\dots,n$, $-\infty < \alpha_0 < +\infty$, $\alpha_1 \geq 1$; $u(x)$, $|u(x)|^{\alpha_0+\alpha_1} \in C^1(G)$, where G is a bounded region in E_n (the n-dimensional Euclidean space) with the boundary Γ ; $C^1(G)$ is the space of functions $u(x)$ with bounded first order derivatives in $\bar{G}$ (the closure of G) and the constant K depends on α_0, α_1 and G is established in 1964 by Dubinskiĭ [1]. A number of inequalities of the type (1) which can be used in the study of partial differential equations are also given by Dubinskiĭ in [1]. The aim of the present note is to establish an integral inequality similar to the inequality (1) which in the special cases yield the slight variant of the inequality (1) and also the inequality given by the present author in [9, Theorem 1]. The proof given here is elementary and based on the application of Gauss integral theorem.

2. STATEMENT OF RESULTS

Let E be an n-dimensional Euclidean space and let $x = (x_1,\dots,x_n)$ denote a variable point in E. An open, simply connected, bounded set of points in E is said to be a normal domain B if B admits the application of the following Gauss integral theorem (see, [3, p.49]).

On the set of boundary points $x \in \partial B$ *with* $\bar{B} = B + \partial B$ (*union of* B *and* ∂B) *there is a real-valued vector field* $z(x) = (z_1(x),\dots,z_n(x))$ *with* $|z| = \left\{\sum_{i=1}^{n} z_i^2(x)\right\}^{\frac{1}{2}} = 1$ *such that for all complex valued functions* $w(x) = w(x_1,\dots,x_n) \in C^1(\bar{B})$

$$(2) \qquad \int_B \frac{\partial}{\partial x_i} w(x)dx = \int_{\partial B} w(x)z_i(x)dS\,, \qquad i = 1,\dots,n,$$

where $dx = dx_1\dots dx_n$ *is the volume element and* dS *is the surface element corresponding to* ∂B.

Our main result is embodied in the following theorem.

THEOREM 1. *Let* B *be a normal domain in* E *with boundary* ∂B *and* $\bar{B} = B + \partial B$. *Let* $u(x)$ *and* $v(x)$ *be real-valued functions belonging to* $C^1(\bar{B})$ *and* $p,\ q \geq 2$ *are real constants. Then*

$$(3) \qquad \int_B |u(x)|^p |v(x)|^q dx \leq L\Big[\int_{\partial B} |u(x)|^p |v(x)|^q dS + \sum_{i=1}^{n} \int_B \Big[|u(x)|^p \left|\frac{\partial}{\partial x_i} v(x)\right|^q + |v(x)|^q \left|\frac{\partial}{\partial x_i} u(x)\right|^p\Big]dx\Big],$$

where

$$L = \max\left\{ 2\delta,\ \frac{2\delta}{n[4\delta(p-1)]^{p-1}},\ \frac{2\delta}{n[4\delta(q-1)]^{q-1}} \right\},$$

and $\delta = \max\{ |x_1|, \ldots, |x_n| \}$.

As an immediate consequence of Theorem 1 we have the following

THEOREM 2. *Assume that in the hypotheses of Theorem 1 we have* $v(x) = u(x)$. *Then*

$$(4)\quad \int_B |u(x)|^{p+q}\, dx \le M\left[\int_{\partial B} |u(x)|^{p+q}\, dS + \sum_{i=1}^{n} \int_B |u(x)|^p \left|\frac{\partial}{\partial x_i} u(x)\right|^q dx \right],$$

where $M = \max\{ L, 2L \}$.

Remark 1. We note that the inequality (3) is a further generalization of the inequality recently established by the present author in [9, Theorem 1]. The inequality (4) is a slight variant of the inequality given in (1). If we take $p = q = 2$ in (4), we get the following interesting variant of the well known Sobolev's inequality (see, [7,p.1])

$$(5)\quad \int_B |u(x)|^4\, dx \le M\left[\int_{\partial B} |u(x)|^4\, dS + \int_B |u(x)|^2 |\operatorname{grad} u(x)|^2\, dx \right].$$

For different versions of the inequalities of the forms (3) - (5), see [1,2,5-10] and some of the references given therein.

3. PROOFS OF THEOREMS 1-2

If we set $w(x) = x_i |u(x)|^p |v(x)|^q$ in Gauss integral formula (2), then we have

(6) $$\int_B |u(x)|^p |v(x)|^q dx = \int_{\partial B} x_i |u(x)|^p |v(x)|^q z_i(x) dS - \int_B x_i \Big[|u(x)|^p q |v(x)|^{q-1} \frac{\partial v(x)}{\partial x_i} \operatorname{sign} v(x) + |v(x)|^q p |u(x)|^{p-1} \frac{\partial u(x)}{\partial x_i} \operatorname{sign} u(x) \Big] dx$$

for $i=1,\ldots,n$. From (6) and using Schwarz inequality for sum we observe that

(7) $$n \int_B |u(x)|^p |v(x)|^q dx = \int_{\partial B} \Big\{ |u(x)|^p |v(x)|^q \sum_{i=1}^{n} x_i z_i(x) \Big\} dS - \sum_{i=1}^{n} \int_B x_i \Big[q |u(x)|^p |v(x)|^{q-1} \frac{\partial v(x)}{\partial x_i} \operatorname{sign} v(x) + p |v(x)|^q |u(x)|^{p-1} \frac{\partial u(x)}{\partial x_i} \operatorname{sign} u(x) \Big] dx$$
$$\le n\delta \int_{\partial B} |u(x)|^p |v(x)|^q dS + \sum_{i=1}^{n} \int_B |x_i| \Big[q |u(x)|^p |v(x)|^{q-1} \Big|\frac{\partial v(x)}{\partial x_i}\Big| + p |v(x)|^q |u(x)|^{p-1} \Big|\frac{\partial u(x)}{\partial x_i}\Big| \Big] dx.$$

Using the following versions of the Young's inequality

$$ab \le \frac{1}{q} \epsilon_1^q a^q + \left(\frac{q-1}{q}\right) \epsilon_1^{-\frac{q}{q-1}} b^{\frac{q}{q-1}} ,$$

and

$$ab \leq \frac{1}{p}\ \epsilon_2^p\, a^p + \left(\frac{p-1}{p}\right) \epsilon_2^{\frac{p}{p-1}}\, b^{\frac{p}{p-1}} ,$$

where $a, b \geq 0$, $p, q \geq 2$, $\epsilon_1 > 0$, $\epsilon_2 > 0$ and setting

$$\epsilon_1 = \frac{1}{[\,4\delta(q-1)]^{\frac{q-1}{q}}} , \quad \epsilon_2 = \frac{1}{[\,4\delta(p-1)\,]^{\frac{p-1}{p}}} ,$$

we observe that

$$(8)\quad |x_i|\left[\, q|u(x)|^p |v(x)|^{q-1} \left|\frac{\partial v(x)}{\partial x_i}\right| + p|v(x)|^q |u(x)|^{p-1} \left|\frac{\partial u(x)}{\partial x_i}\right| \,\right]$$

$$\leq \delta\left[\, q|u(x)|^p \left\{ \frac{1}{q}\ \epsilon_1^q \left|\frac{\partial v(x)}{\partial x_i}\right|^q + \left(\frac{q-1}{q}\right) \epsilon_1^{\frac{q}{q-1}} |v(x)|^q \right\}\right.$$

$$\left. + p|v(x)|^q \left\{ \frac{1}{p}\ \epsilon_2^p \left|\frac{\partial u(x)}{\partial x_i}\right|^p + \left(\frac{p-1}{p}\right) \epsilon_2^{\frac{p}{p-1}} |u(x)|^p \right\} \,\right]$$

$$= \frac{1}{2}\ |u(x)|^p\ |v(x)|^q + \frac{\delta}{[\,4\delta(q-1)]^{q-1}}\ |u(x)|^p \left|\frac{\partial v(x)}{\partial x_i}\right|^q$$

$$+ \frac{\delta}{[\,4\delta(p-1)\,]^{p-1}}\ |v(x)|^q \left|\frac{\partial u(x)}{\partial x_i}\right|^p .$$

From (7), (8) we obtain

$$(9)\quad n\int_B |u(x)|^p |v(x)|^q \, dx \le n\delta \int_{\partial B} |u(x)|^p |v(x)|^q \, dS + \frac{1}{2} n \int_B |u(x)|^p |v(x)|^q \, dx$$

$$+ \frac{\delta}{[4\delta(q-1)]^{q-1}} \sum_{i=1}^{n} \int_B |u(x)|^p \left|\frac{\partial v(x)}{\partial x_i}\right|^q dx$$

$$+ \frac{\delta}{[4\delta(p-1)]^{p-1}} \sum_{i=1}^{n} \int_B |v(x)|^q \left|\frac{\partial u(x)}{\partial x_i}\right|^p dx.$$

From (9) and the definition of L, the desired inequality in (3) follows. This completes the proof of Theorem 1.

The proof of Theorem 2 is obvious by taking $v(x) = u(x)$ in inequality (3).

4. TRACE TYPE INEQUALITY

In 1979 C.O. Horgan [4] established the trace inequality of the form

$$(10)\qquad \int_{\partial\Omega} u^2 \, dS \le c \int_{\Omega} \{ u^2 + u_{,i}\, u_{,i} \} \, dx,$$

where $u(x)$ is sufficiently regular function defined on a bounded domain Ω in R^n $(n > 1)$ with smooth boundary $\partial\Omega$. We next establish the inequality of the type (10) which yields in the special case the trace type inequality recently established by the present author in [9, Theirem 2].

THEOREM 3. *Let* B *be a normal domain in* E *with sufficiently smooth boundary* ∂B *and* $\bar{B} = B + \partial B$. *Let* $u(x)$ *and* $v(x)$ *be real-valued functions belonging to* $C^1(\bar{B})$ *and* $p, q \geq 2$ *are real constants. Then*

$$(11)\quad \int_{\partial B} |u(x)|^p |v(x)|^q \, dS \leq N\Big[\int_B |u(x)|^p |v(x)|^q \, dx + \sum_{i=1}^{n} \int_B \Big[|u(x)|^p \Big|\frac{\partial v(x)}{\partial x_i}\Big|^q + |v(x)|^q \Big|\frac{\partial u(x)}{\partial x_i}\Big|^p \Big] dx \Big],$$

where

$$N = \max\Big\{ \Big[c_0 + c_1(q-1)c^{\frac{q}{q-1}} + c_1(p-1)c^{\frac{p}{p-1}}\Big],\ c_1 c^q n^{q-1},\ c_1 c^p n^{p-1} \Big\},$$

in which $c > 0$ *is arbitrary constant and for auxiliary functions* $\alpha_i(x)$ *belonging to* $C^1(\bar{B})$, *we set*

$$c_0 = \sup_{x \in B} \Big| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \alpha_i(x) \Big|, \quad c_1 = \sup_{i=1,\dots,n} \Big\{ \sup_{x \in B} |\alpha_i(x)| \Big\} .$$

As a consequence of Theorem 3 we have the following

THEOREM 4. *Assume that in the hypotheses of Theorem* 3 *we have* $v(x) = u(x)$. *Then*

$$(12) \quad \int_{\partial B} |u(x)|^{p+q} \, d\varsigma \leq Q\left[\int_{B} |u(x)|^{p+q} dx + \sum_{i=1}^{n} \int_{B} |u(x)|^{p} \left|\frac{\partial u(x)}{\partial x_i}\right|^{q} dx \right]$$

where $Q = \max \{ N, 2N \}$.

The proof of Theorem 3 is very close to that of the proof of Theorem 2 given in [9] with suitable modifications in view of the proof of our Theorem 1 given above. In particular, to get the desired inequality in (11) one needs to use the Young's inequalities involved in the proof of Theorem 1 with ϵ_1 and ϵ_2 replaced by an arbitrary constant $c > 0$. We omit the details. The proof of Theorem 4 is immediate from Theorem 3.

Remark 2. If we take $p = q = 2$, the inequality (3) reduces to the inequality of the form given by the present author in [9, Theorem 2]. In the special case when $p = q = 2$, the inequality (12) reduces to an interesting variant of the trace inequality given in (10).

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