LIBERTAS MATHEMATICA , VOL. 9 (1989)

NONEXISTENCE RESULTS FOR THE ABSTRACT CAUCHY PROBLEM

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Abstract

We obtain some non -existence results for the first order classical or ultra-weak abstract Cauchy problem: $u'(t) = Au(t)$, $o \le t \le T$, $u(o) = u_0$, in separable Hilbert spaces.

Introduction

This work is a development of some ideas and remarks in the Section 2 of [1].Extending a situation apparently well-known for the backward heat equation we consider a first order abstract differential equation $u'(t) = Au(t)$ with non-negative operator A, and look for solution to the initial value problem, $u(0) = u_0$ given in advance. We establish a necessary condition for the existence of a continuous ultra-weak solution and then apply it, in a few special cases, in order to obtain non-existence of solutions, in both classical or ultra-weak sense, for a suitable choice of the initial data. Let us also note that only elementary functional analysis is involved in the present paper; we refer to [2] for the basic definitions and results.

1. In the (real or complex) Hilbert space H we consider a linear operator A, with domain $D(A) \subseteq H$ and range in H too. We assume that A is hermitian (that is $(Ah, k) = (h,Ak)$ $\forall h,k \in D(A)$ and non-negative (that is $(Ah,h) \geq 0$ $\forall h \in D(A)$). We assume furthermore that A possesses a complete (in H) sequence of eigen-vectors $(e_n)_1^\infty$ corresponding to non-negative eigen-values $(\lambda_n)_1^\infty$ (thus $e_n \in D(A)$ $\forall n = 1,2,\ldots$, and $D(A)$ is dense in H).

We define - as usual, see for ex. [3] - the concept of continuous ultra-weak solution of the first order differential equation $u'(t) = Au(t)$ in the finite interval $[0,T]$.

Note first that the Hilbert adjoint of A, denoted with A^* is a well-defined operator, and $A \subset A^*$. Then, the class of vector-valued test-functions $K_{A^*}(0,T)$ is composed of all the functions $\varphi(t), 0 < t < T \rightarrow D(A^*)$, such that $\varphi(t) \in C_0^1(]o,T[;H)$, and $A^* \varphi \in C(]o,T[;H)$.;

Next, let us consider continuous functions $u(t)$, $o \le t \le T \rightarrow H$ verifying the integral relation

$$\int_0^T (u(t), \varphi'(t) + (A^*\varphi)(t))\,dt = 0 \quad , \quad \forall \varphi \in K_{A^*}(0,T) \ . \qquad 1.1)$$

In this case we say that $u(t)$ is a continuous ultra-weak solution of the (abstract differential) equation

$$u'(t) - Au(t) = \theta \quad , \quad 0 \le t \le T. \qquad 1.2)$$

If, in addition we ask that $u(0) = u_0 \in H$, where u_0 is given in advance, we have a solution of the ultra-weak Cauchy problem. At this stage we are ready for the statement of the folowing

Theorem 1. Under the above stated hypothesis, if $u(t) \in C([0,T];H)$ is a solution of 1.1) with $u(o) = u_o$,it follows that the infinite numerical series

$$\sum_{n=1}^{\infty} e^{2\lambda_n T} | (u_o, e_n) |^2 \tag{1.3}$$

is convergent.

Proof. We shall take in 1.1) functions $\varphi(t)$ of the special form: $\varphi(t) = w(t)\, e_k$, where $w(t)$ is a real-valued function in $C_o^1(o,T)$ while e_k is an eigen-vector of A. We obtain the relation

$$\int_o^T (u(t), w'(t)\, e_k + w(t)\, A^* e_k)\, dt = o \qquad \forall w \in C_o^1(o,T) \tag{1.4}$$

Using the fact that $A \subset A^*$, we obtain that $A^* e_k = Ae_k = \lambda_k e_k$ and we get

$$\int_o^T \{ w'(t)\, (u(t), e_k) + w(t)\, (u(t), \lambda_k e_k) \}\, dt = 0 \tag{1.5}$$

If we denote with $u_k(t)$ the Fourier coefficient $(u(t), e_k)$ we find

$$\int_o^T u_k(t) w'(t)\, dt = - \int_o^T \lambda_k\, u_k(t) w(t) dt \quad , \quad \forall w \in C_o^{\infty}(o,T) \tag{1.6}$$

which means that

$$\frac{d}{dt} u_k(t) = \lambda_k u_k(t) \tag{1.7}$$

in the sense of distributions on (o,T). Due to the continuity of $u_k(t)$ we deduce that 1.7) holds also in the usual (classical) sense. It follows that

$$u_k(t) = e^{\lambda_k (t-\epsilon)} u_k(\epsilon), \quad \text{for } 0 < \epsilon < t < T \qquad 1.8)$$

Again, from the continuity of u and u_k on the closed interval $[0,T]$ we find that

$$u_k(t) = e^{\lambda_k t} (u_0, e_k), \quad 0 \leq t \leq T \qquad 1.9)$$

On the other hand, from completeness of the sequence $(e_k)_1^\infty$ in H we obtain

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\, e_k = \sum_{k=1}^{\infty} e^{\lambda_k t} (u_0, e_k)\, e_k \qquad 1.10)$$

with strong H-convergence for all $t \in [0,T]$. We also have Parseval's equality

$\| u(t) \|^2 = \sum_1^\infty | u_k(t) |^2 < \infty \quad \forall\, t \in [0,T]$, in particular for $t = T$. Thus we obtain

$$\sum_1^\infty | (u_0, e_k) |^2 e^{2\lambda_k T} < +\infty \qquad 1.11)$$

q.e.d.

It appears therefore that 1.11) is a necessary condition for the existence of the continuous ultra-weak solution $u(t)$ on $[0,T]$ with u_0 as initial (Cauchy) value.

2. In this section we shall first prove the following result:

Theorem 2. In the Hilbert space H is given the hermitian operator A verifying the conditions in Section 1 with eigenvalues $\lambda_n = n$, or $\lambda_n = \log n$, $n=1, 2 \ldots$. *Then, if, respectively*

$$u_o = \sum_{n=1}^{\infty} \frac{1}{n} e_n \text{ or } u_o = \sum_{n=1}^{\infty} \frac{1}{n^{T+1/2}} e_n ,$$

there is no ultra-weak continuous solution $u(t)$ of 1.2) with $u(o) = u_o$.

Proof

If (in the case where $\lambda_n = n$) , we consider the vector in H given by

$$u_o = \sum_{n=1}^{\infty} \frac{1}{n} e_n ,$$

from existence of a continuous ultra-weak solution $u(t)$ with $u(o) = u_o$ it would follow the convergence of the series

$$\sum_{1}^{\infty} e^{2nT} \frac{1}{n^2} ,$$

which is false, as $\lim_{n\to\infty} 1/n^2 e^{2nT} = +\infty$. If (in the case where $\lambda_n = \log n$), we take

$$u_o = \sum_{k=1}^{\infty} \frac{1}{k^{T+1/2}} e_k , \text{ we first note that } \sum_{k=1}^{\infty} \frac{1}{k^{2T+1}} < +\infty , \text{ as } T > o ;$$

thus u_o is a well-defined element of H. Again, from existence of the ultra-weak solution with u_o as initial datum, it would follow that

$$\sum_{k=1}^{\infty} e^{2(\log k)T} \frac{1}{k^{2T+1}} < +\infty .$$

Actually, this is the series

$$\sum_{k=1}^{\infty} k^{2T} \; k^{-2T-1} = \sum_{k=1}^{\infty} \frac{1}{k} ,$$

which is not convergent.

We now consider the case of classical (regular) solutions of the equation $u'(t) - Au(t) = \theta$, that is $u(t)$, $0 \leq t \leq T \rightarrow D(A)$, $u(t) \in C^1([o,T];H)$, such that $Au \in C([o,T];H)$. As regular solutions are ultra-weak in the above specified sense, the necessary condition 1.11) still holds true. However, to check non-existence, we must investigate only $u_o \in D(A)$, the case $u_o \in H/D(A)$ being trivial. We have now

Theorem 3. Under the hypothesis of Theorem 1, with $\lambda_k = k$ and A a closed operator, the element

$$u_o = \sum_{k=1}^{\infty} \frac{1}{k^2} e_k$$

belongs to $D(A)$ and no regular solution of the equation $u'(t) - Au(t) = \theta$ on $[o, T]$ with $u(o) = u_o$ will exist, at all.

Proof We need a very simple

Lemma. Let A be an operator as in Section 1 which is also closed, and consider an element $(\xi_n)_1^{\infty}$ of ℓ^2 such that $(\lambda_n \xi_n)_1^{\infty}$ is also in ℓ^2. Then the element $\Sigma_1^{\infty} \xi_n e_n$ of H belongs to $D(A)$ too.

In fact, each finite sum:

$$\sum_{1}^{n} \xi_p e_p$$

belongs to $D(A)$ and the equality

$$A\left(\sum_1^n \xi_p e_p\right) = \sum_1^n \xi_p \lambda_p e_p$$

holds. If $n \to \infty$, it follows that

$$\sum_1^n \xi_p e_p \to u_o = \sum_1^\infty \xi_p e_p \in H ,$$

while

$$\sum_1^n \xi_p \lambda_p e_p \to \sum_1^\infty \xi_p \lambda_p e_p = v_o \in H .$$

Due to closedness of A it follows that $u_o \in D(A)$ (and $Au_o = v_o$).

Continuing the proof of the Theorem we now see that

$$\sum_1^\infty \frac{1}{k^2} e_k \in D(A).$$

Then, for existence of a regular (hence ultra-weak) continuous solution with $u(o) = u_o$ it is

necessary that

$$\sum_1^\infty e^{2kT} \frac{1}{k^4}$$

be convergent, which is not true.

Our last result (Th. 4 below) examines the case of eigen-values $\lambda_k = \log k$. We state in fact

Theorem 4. Under the hypothesis of Theorem 1, with $\lambda_k = \log k$ and A a closed operator, the vector u_o given by the series

$$\sum_{k=1}^\infty \frac{1}{k^\alpha} e_k \quad ,$$

where

$$\frac{1+\varepsilon}{2} < \alpha \leq \frac{1}{2} + T \quad , \ o < \varepsilon < 2T$$

<u>is an element of</u> D (A) <u>for which there is no classical solution on</u> [o,T] <u>with</u> $u(o) = u_o$.

<u>Proof</u>. As $2\alpha > 1$, we see that

$$\left(\frac{1}{k^\alpha}\right)_1^\infty \in \ell^2 .$$

Next, the sequence

$$\left(\frac{\log k}{k^\alpha}\right)_1^\infty$$

also belongs to ℓ^2 (this is because for large k and any $\varepsilon > o$ fixed, $(\log k)^2 < k^\varepsilon$; hence

$$\left(\frac{\log k}{k^\alpha}\right)^2 < k^{\varepsilon - 2\alpha} \quad , \ k > k_o .$$

As $2\alpha - \varepsilon > 1$ we get the desired convergence).

Hence $u_o \in D(A)$

Again, if a solution with initial datum u_o could exist, it would follow that

$$\sum_{k=1}^{\infty} \frac{e^{(\log k)\, 2T}}{k^{2\alpha}} < \infty \quad .$$

The general term here is $k^{2T-2\alpha}$ and the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\alpha-2T}}$$

is here <u>divergent</u> due to the condition $2\alpha - 2T \leq 1$. This proves Th. 4.

References

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This paper was written while the author was a part time visitor at the "Centre de Recherches Mathématiques" of the Université de Montréal.