

SOLUTIONS OF THE FULL POTENTIAL EQUATION
USING EXACT INTEGRAL METHOD

Z. Fang and I. Paraschivoiu

ABSTRACT

The full potential equation for inviscid compressible flows is solved using exact integral method. The volume integral is evaluated in a body-fitted grid generated by finite element discretization. With body fitted grid, the volume integral is evaluated only outside the body. Therefore, there is no extra boundary treatment required for evaluation of the volume integral. The source term is also evaluated in each finite integral volume and assumed to be constant in an finite integral volume. The volume integrals only need to be evaluated once and can be stored in computer memory for further usage.

INTRODUCTION

The exact integral method for solving partial differential equations is a very powerful tool in computational transonic flows. This method provides an alternative approach to the finite difference and finite element methods for solving transonic potential flows. Computational schemes based on integral formulation have been developed[1-6]. One of the advantages of the exact integral method is that solutions can be obtained by only using surface panels and solutions everywhere else can be represented by the solutions on surface panels.

In this work, the exact integral method is applied to solve the full potential equation for compressible flows. The linear doublet potential method is applied in this work[7]. The coefficients matrix is formulated on body grid points only. The nonlinear full potential equation is solved by Newton-Raphson iteration procedure. In each iteration the influence matrix does not change and is decomposed only once. Only the source term in the volume integral needs to be evaluated in each iteration. After potential values on the body have

been solved, potential values anywhere else can be calculated. Because only body grid points need to be formulated in this matrix, the requirement for computer CPU time is severely reduced. For transonic flows, a shock integral is calculated and its contribution is added to the right hand side of the discretized simultaneous equations. Our results show that the shock integral is necessary for catching a sharp shock jump.

In this work, the two dimensional flows over NACA0012 airfoil are calculated. Numerical results are compared with experimental data[8]. Good agreement between numerical results and experimental data shows that the exact integral method is a very useful tool for analytical and design purposes in aeronautical engineering. The extension of the exact integral method to three dimensional flows is under way.

GOVERNING EQUATION AND BOUNDARY CONDITIONS

In the Cartesian coordinates, the conservative form of the full potential equation is

$$\nabla \cdot (\rho \nabla \phi) = 0 \quad (1)$$

where ϕ is the full potential, ρ is the density, and ∇ is defined by

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \quad (2)$$

The full potential equation given by Eq. (1) is not well suitable for integral representation because the operator is not a Laplace operator. However, the full potential equation can be written in another form

$$\nabla^2 \phi = M^2 \phi_{ss} \quad (3)$$

where M is the Mach number, and ϕ_{ss} is the rate of velocity change along the stream line direction. The right hand side term is presented as σ and is called the source term.

In the exact integral methods, the solution of Equation (3) can be represented as

$$\phi = \phi_{\infty} + \phi_f \quad (4)$$

where ϕ_{∞} is the undisturbed potential and thus

$$\phi_{\infty} = V_{\infty} (x \cos \alpha + y \sin \alpha) \quad (5)$$

The surface boundary conditions for inviscid flows is the tangential flow conditions which is given by

$$(\nabla\phi) \cdot \mathbf{n} = 0 \quad (6)$$

where \mathbf{n} is the normal direction of the body surface.

Kutta condition is also implemented, which requires that the flow leaves the trailing edge smoothly.

INTEGRAL FORMULATION

Using Green's theorem, solution ϕ of Eq. (3) can be represented by the sum of boundary integrals and a volume integral as

$$\phi(x_p, y_p) = \oint \left[(\mathbf{n} \cdot \nabla\phi) \phi_L - \phi(\mathbf{n} \cdot \nabla\phi_L) \right] ds + \int_V \sigma \phi_L dV \quad (7)$$

where the surface integral is on boundaries of a single connected region and the volume integral is inside the region. $\phi(x_p, y_p)$ is the potential value and

$$\phi_L = \frac{1}{2\pi} \ln \left[(x-x_p)^2 + (y-y_p)^2 \right]^{0.5} \quad (8)$$

Thus, the disturbed solution can be expressed by

$$\phi_f(x_p, y_p) = \int_{B+C+SH} \left[(\mathbf{n} \cdot \nabla\phi) \phi_L - \phi(\mathbf{n} \cdot \nabla\phi_L) \right] ds + \int_V \sigma \phi_L dV \quad (9)$$

where B are body surfaces, C are wakes or cuts between region boundaries, and SH are shock surfaces. On body surfaces and wakes,

$$\int_{B+C} (\mathbf{n} \cdot \nabla\phi) \phi_L ds = 0 \quad (10)$$

Across shocks,

$$\int_{SH} \phi(\mathbf{n} \cdot \nabla\phi_L) ds = 0 \quad (11)$$

Thus, Eq. (9) becomes

$$\begin{aligned} \phi_f(x_p, y_p) = & - \int_B \phi(\mathbf{n} \cdot \nabla \phi_L) ds - \Gamma \int_C (\mathbf{n} \cdot \nabla \phi_L) ds \\ & + \int_{SH} (\mathbf{n} \cdot \nabla \phi) \phi_L ds + \int_V \sigma \phi_L dV \end{aligned} \quad (12)$$

where Γ is the circulation.

In Eq. (12), potential value everywhere is represented by boundary integrals and a volume integral. For incompressible flows, the source term σ is zero and there is no shock. Thus, the last two integrals on the right hand side of Eq. (12) disappear. The potential depends only on boundary integrals on body surfaces and wake. After potential values on body surface are determined, potential solutions everywhere can be determined by those on body surface. To determine potential solutions on body surface, Eq. (12) is applied to points on body surface. Therefore, the body surface needs to be cut into many pieces of small panels. Applying the surface integrals to each panel will result a group of simultaneous equations. Potential solutions on body surface are obtained from these simultaneous equations.

In the first integral on the right hand side of Eq. (12), a linear potential formulation on each panel is assumed. In the second integral, the circulation Γ is determined by the solution as

$$\Gamma = \phi_+ - \phi_- \quad (13)$$

where ϕ_+ and ϕ_- are potential values across the wake and are obtained from the solution.

For compressible flows, the source term must be evaluated throughout the volume. For transonic flows, the integral for shocks is needed because the normal derivatives of potential across shock wave is discontinuous. In this work, potential values across shock wave are assumed to be continuous.

The source term is obtained after potential solutions are known everywhere. Thus, the potential value represented by Eq. (12) must be solved iteratively.

TREATMENT OF VOLUME INTEGRAL

The volume integral in Eq. (12) is evaluated in a discretized field by a body-fitted grid which is shown in Fig. 1. In each element volume the source term inside the volume integral is assumed as a constant. Thus,

$$\int_V \sigma \phi_L dV = \sum_1^M \sigma_1 \int_{V_1} \phi_L dV \quad (14)$$

Integral $\int_{V_1} \phi_L dV$ is calculated analytically. The finite integral volume boundaries can be represented by

$$y = at + bt x \quad (15)$$

The limit function of the volume integral is

$$\begin{aligned} F(\bar{x}) = & (AT \bar{x} + 0.5 BT \bar{x}^2) \ln \left[(1 + BT^2)(\bar{x} + 2 B \bar{x} + C) \right] \\ & + \bar{x}^2 \tan^{-1} \frac{AT + BT \bar{x}}{\bar{x}} \\ & - \frac{BT(C + 2B^2) - 2 AT B D}{2} \ln(\bar{x}^2 + 2 B \bar{x} + C) \\ & + \frac{AT(C - 2 B^2)D + 2 BT B^3}{E} \tan^{-1} \frac{\bar{x} + B}{E} \\ & - 1.5 BT \bar{x}^2 + (BT B - 2 AT)\bar{x} - AT D \bar{x} \end{aligned} \quad (16)$$

where

$$\bar{x} = x - x_p,$$

$$AT = at - y_p + bt x_p$$

$$BT = bt$$

$$B = \frac{AT BT}{1 + BT^2}$$

$$C = \frac{AT^2}{1 + BT^2}$$

$$D = \frac{1 + 2 BT^2}{1 + BT^2}$$

$$E = \frac{AT}{1 + BT^2}$$

If $AT = 0$, the singularity location (x_p, y_p) is on the line represented by $y=at+bt_x$ as shown in Figure 2. The limit function becomes

$$F(\bar{x}) = 0.5 BT \bar{x}^2 \ln[(1 + BT^2) \bar{x}^2] - 1.5 BT \bar{x}^2 + \bar{x}^2 \tan^{-1}(BT) \quad (17)$$

If the singularity location (x_p, y_p) is on the node point of the finite integral volume, $x=x_p$ and $y=y_p$. Thus, the limit function value becomes zero.

The value of a volume integral is obtained by calling the limit functions. In Figure 3, a typical finite integral volume is shown. The integral region is divided into three regions. In region 1, the integral value is

$$I_1 = F_{AU}(\bar{x}_D) - F_{AD}(\bar{x}_D) - [F_{AU}(\bar{x}_A) - F_{AD}(\bar{x}_A)] \quad (18)$$

In region 2, we have

$$I_2 = F_{AU}(\bar{x}_U) - F_{DB}(\bar{x}_U) - [F_{AU}(\bar{x}_D) - F_{DB}(\bar{x}_D)] \quad (19)$$

and in region 3,

$$I_3 = F_{UB}(\bar{x}_B) - F_{DB}(\bar{x}_B) - [F_{UB}(\bar{x}_U) - F_{DB}(\bar{x}_U)] \quad (20)$$

The integral value in this integral volume is

$$I = I_1 + I_2 + I_3 \quad (21)$$

Special care should be taken, when the finite integral volume appears to be

different.

RESULTS AND DISCUSSION

The solution procedure discussed in above sections is applied to compressible flows over the NACA-0012 airfoil for a range of Mach number from subsonic to transonic flows. Solution results are compared with experimental data[8].

In Fig. 4 and Fig. 5, pressure coefficients for flows over NACA0012 airfoil with Mach number at the infinity 0.5 and 0.7, angle of attack 0.0 are presented. Good agreement between numerical results and experimental data in these cases are shown. In Fig. 6, pressure coefficient for flow over naca0012 airfoil with Mach number 0.5 at the infinity, angle of attack 2.0 is presented. Numerical results predicted lower pressure coefficients along the upper surface of the airfoil. Pressure coefficients for flow over NACA0012 with Mach number 0.8 at the infinity, angle of attack 0.0 is shown in Fig. 7. Good agreement is shown everywhere except across the shock. Necessary corrections is going under way.

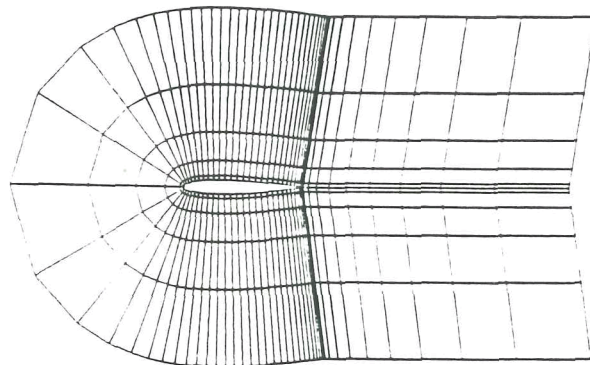


Figure 1. Grid for volume integral.

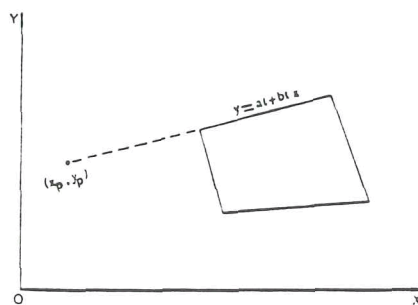


Figure 2. Demonstration for AT = 0.

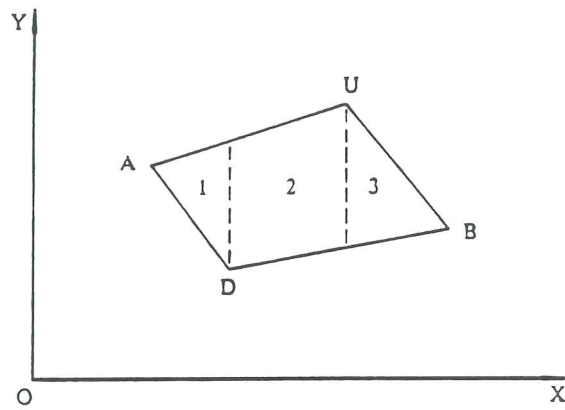


Figure 3. A typical integral volume.

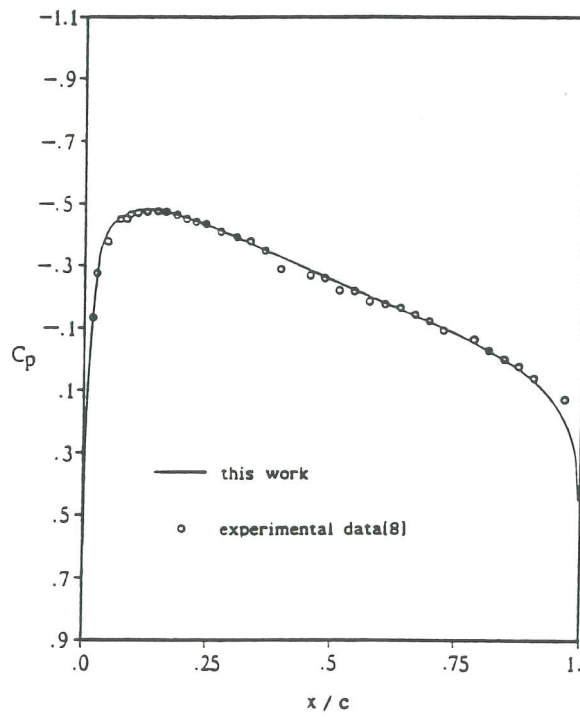


Figure 4. Pressure coefficient for NACA0012 airfoil
 $M_\infty = 0.5, \alpha = 0.0$

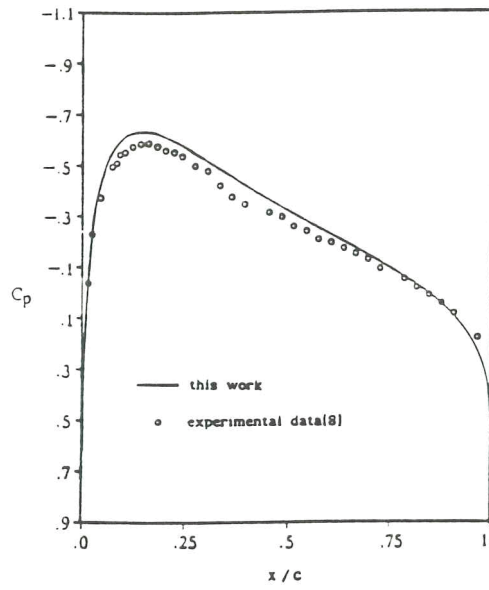


Figure 5. Pressure coefficient for NACA0012 airfoil
 $M_{\infty} = 0.7, \alpha = 0.0$

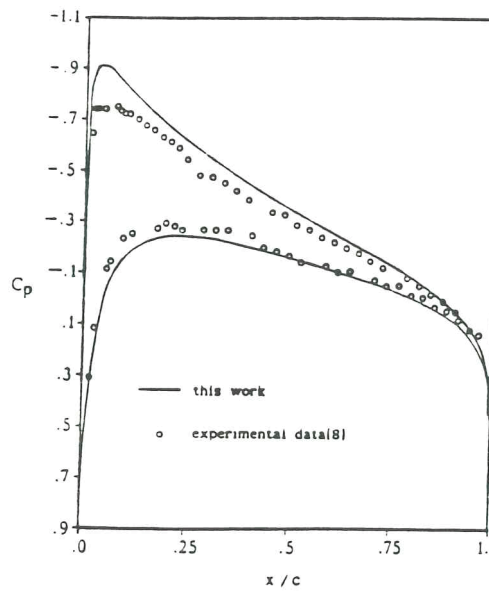


Figure 6. Pressure coefficient for NACA0012 airfoil
 $M_{\infty} = 0.5, \alpha = 2.0$

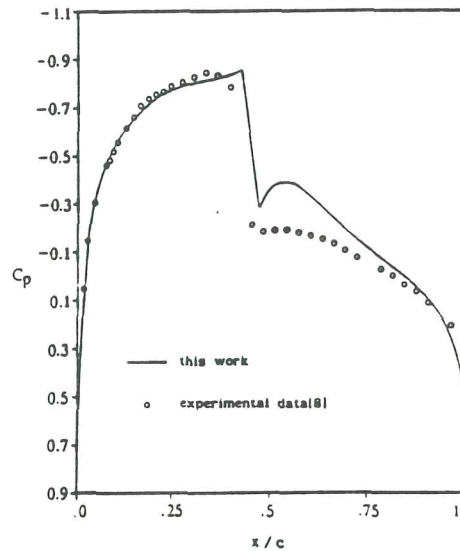


Figure 7. Pressure coefficient for NACA0012 airfoil
 $M_{\infty} = 0.8$, $\alpha = 0.0$

REFERENCES

1. Sinclair, P. M., "An Exact Integral (Field Panel) Method for the Calculation of Two-Dimensional Transonic Potential Flow Around Complex Configurations," *Aeronautical Journal*, June/July, pp. 227-236, 1986.
2. Oskam, B., "Transonic Panel Method for the Full Potential Equation Applied to Multicomponent Airfoils," *AIAA Journal*, Vol. 23, No. 9, pp. 1327-1334, 1985.
3. Crown, J. C., "Calculation of Transonic Flow Over Thick Airfoils by Integral Methods," *AIAA Journal*, Vol. 6, No. 3, pp. 413-423, 1968.
4. Kandil, O. A., Hu, H., "Full-Potential Integral Solution for Transonic Flows with and without Embedded Euler Domains," *AIAA Journal*, Vol. 26, No. 9, pp. 1079-1086, 1988.
5. Tseng, K., Morino, L., "Nonlinear Green's Function Methods for Unsteady Transonic Flows," *Transonic Aerodynamics*, edited by D. Nixon, AIAA, New York, pp. 565-603, 1982.
6. Masson, C., "Panel Method for Transonic Flows," MSc. Thesis, Ecole Polytechnique of Montreal, August, 1989.
7. Moran, J., *Theoretical and Computational Aerodynamics*, John Wiley & Sons, 1984.
8. Thibert, J. J., Grandjacques, M., Ohman, L. H., "NACA-0012 Airfoil, An Experimental Data Base for Computer Program Assessment," AGARD-AR-138, pp. A1.1-A1.10, May, 1979.